Shear bands: formulation, analysis and numerical simulations

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Abstract

This work deals with the formulation, analysis and computational simulation of non-local and rate dependent plasticity models in one dimensional set. In the formulation, described within the continuum mechanics scheme, the plastic deformation is viewed as an additional degree of freedom. The basic laws are: the principle of virtual power and inequality of dissipation. A thermodynamically consistent theory is developed, in which non-local and rate effects are accounted for by allowing constitutive dependence on the gradient and rate of plastic deformation. The governing equations are obtained after combining the basic balances with the constitutive theory. In particular, theories of plasticity with and without elastic region are obtained. A simplified version of the theory with elastic region is subject to a linear stability analysis in order to obtain instability criterion, a qualitative analysis showing that shear bands are associated with homoclinic (heteroclinic) orbit. A computational simulation based on the finite element method and an Euler implicit scheme and staggered algorithm is presented. The numerical results illustrate the accumulation of plastic deformation in narrow bands.

Keywords: non-local plasticity, shear bands, rate dependent plasticity, continuum mechanics.

1 Introduction

Nowadays there is a renewed interest in the study of the phenomenon of plasticity. This occurs because plastic deformation is a high dissipative problem and still remains not fully understood, and moreover, because of its importance in the study of the mechanical behavior of structural components.

Mechanics of Solids in Brazil 2009, H.S. da Costa Mattos & Marcílio Alves (Editors)
Brazilian Society of Mechanical Sciences and Engineering, ISBN 978-85-85769-43-7

There are an increasing interest in models which take into account effects not considered by the conventional theories, such as microstructure, non-local effects and rate dependence. These important issues are essential in the development of capable theories to represent the most diverse kinds of plastic instability, a phenomenon with a rich phenomenology classified as transients (momentary) or permanent (persistent), stationary or propagative. The most well known phenomena of instability are the shear bands, Lüders' bands and the effect of Portevin-Le Chatelier (PLC effect). The shear bands occur after a certain increase of flow plastic, the material exhibits a softening and may occur strain concentration in narrows bands, and once they take place they persist in the initial positions, and hence it is classified as stationary instability permanent [1, 2]. The Lüders' bands occur at a level of constant stress and are classified as propagative and transient instabilities [3, 4]. The PLC effect is associated with a negative strain rate, when a strain plastic is concentrated in narrow bands which propagate through the sample in pico-second or few hours. This phenomenon refers to instability in a form of repeated stress drops followed by periods of reloading, observed when tensile specimens are deformed in a certain range of strain rates and temperatures. It is classified as propagative and permanent instability [5–7]. There is an extensive literature about the diversity aspect of plasticity see for example Aifantis [8, 9], Gurtin [10], Anand et al. [11], Borst [4], Sikora et al. [12], Wang [1], Estrin [13], Kubin and Estrin [6].

To study the phenomenon of plastic instability in accordance with the formalism of modern continuum mechanics [14], we elaborate the framework by introducing the basic balances, the free energy imbalance and a theory constitutive. The plastic deformation is considered as an additional degree of freedom, and by introducing the plastic gradient deformation, we take into account non-local effects. We deduce the governing equations by considering microstructure processes and scale effect, within the scheme given by Duda and Souza [15]. In addition the forces pattern, micro forces were introduced with the assumptions of additive decomposition of the deformation into its elastic and plastic parts.

We consider as basic laws the principle of virtual work and the dissipation inequality, and combined with thermodynamic consistent constitutive theory, we obtain the equations which govern the plasticity theory. The balance of micro forces has resulted in two groups of kinetic laws and, with appropriate choice, some classical theories and different phenomenon of plasticity could be boarded. In the second part we carry out a qualitative analysis of the stationary problem to obtain an instability criterion, the size of localization zones and the propagation velocity of plastic waves. We look for special solutions, such as orbits which bound the points of saddle (homoclinic, heteroclinic), because they represent the localization zones of deformation connecting two constant states of different deformations. Finally we present the numerical results and illustrate the accumulations of plastic deformation in narrows bands.

2 Theoretical framework

2.1 Preliminaries

Let be B a one-dimensional body identified with the fixed interval $[0, L]$. We denote x an element or particle of B, and t a time instant. We denote $D := [x_1, x_2]$ one typical part of B. A generic quantity

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of interest f is a regular function of x and t, with spatial and temporal derivates denoted by $f_x = \frac{\partial f}{\partial x}$ and $\dot{f} = \frac{\partial f}{\partial t}$ respectively.

The notation is showed in Table 1.

Table 1: Notation

t	time	ε	deformation	σ_{ext}	external stress
x	position	ε_e	elastic deformation	s_{int}	micro internal force
u	displacement	ε_p	plastic deformation	s_{ext}	micro external force
y	placement	σ	stress	σ_c	stress contact
v	macroscopic velocity	ξ	micro stress	ρ	the mass density per unit length

The longitudinal motion of B is described by a mapping:

$$y(x,t) := x + u(x,t) \tag{1}$$

In the micro processes, the micro kinematics is described by the elastic and plastic deformation with additive decomposition:

$$\varepsilon(x,t) = \varepsilon_e(x,t) + \varepsilon_p(x,t) \tag{2}$$

In the rate form: $\dot{\varepsilon}(x,t) = \dot{\varepsilon}_e(x,t) + \dot{\varepsilon}_p(x,t)$, where: $\dot{\varepsilon}(x,t) = \frac{\partial v(x,t)}{\partial x}$

By considering Eqs. (1) and (2) we conclude that two of the three fields y,ε_e and ε_p are independents. We consider y and ε_p as independent kinetically descriptor. We define $V := (v, v_p)$ the velocity field, **V** the virtual velocity space and $\bar{V} := (\bar{v}, \bar{v}_p)$ a generic element of **V,** where $\bar{v}$ and $\bar{v}_p$ are the virtual velocities associated to display and plastic deformation respectively.

2.2 Basic laws

The principle of virtual power is used to generate the field equations and the boundary conditions, corresponding to the basic force balances. For any part $D \subset B$ and within a first-gradient theory, we adopt the following prescriptions for the virtual power of the external and internal forces expended on an arbitrary virtual velocity $\bar{v}$:

- Virtual power of the external forces:

$$P_{ext}(D,\bar{V}) := \int_{x_1}^{x_2} (\sigma_{ext}\bar{v} + s_{ext}\bar{v}_p)dx + \sum_{i=1}^{2} (\sigma_c(x_i)\bar{v}(x_i) + q(x_i)\bar{v}_p(x_i)) \tag{3}$$

where the smooth field σ_c and σ_{ext} describe contact and external forces; s_{ext} and q describe the external and contact micro force.

- Virtual power of the internal forces:

$$P_{\text{int}}(D,\bar{V}) := -\int\limits_{x_1}^{x_2}\Big(\sigma\frac{\partial\bar{v}}{\partial x}+\xi\frac{\partial\bar{v}_p}{\partial x}+s_{\text{int}}\bar{v}_p\Big)dx \tag{4}$$

where the smooth field σ, ξ and s_{int} describe internal interactions.

In Eqs. (3) and (4) we introduce the system for macroscopic and microscopic forces. The macro force system, associated with y, is formed by the stress σ and contact force σ_c. The micro force system, associated with the microstructure (plastic deformation), is formed by the micro-stress ξ, internal and external micro forces (s_{ext} and s_{int}), and the contact micro force q.

The principle of virtual power [16] considers that for the each fixed time and part D:

$$P_{ext}(D,\bar{V})+P_{\text{int}}(D,\bar{V})=0 \tag{5}$$

By considering Eq. (3), (4) and (5), we have:

$$\int\limits_{x_1}^{x_2}\Big(-\sigma\frac{\partial\bar{v}}{\partial x}-\xi\frac{\partial\bar{v}_p}{\partial x}-s_{\text{int}}\bar{v}_p+\sigma_{ext}\bar{v}+s_{ext}\bar{v}_p\Big)dx+\sum_{i=1}^{2}\left(\sigma_c\left(x_i\right)\bar{v}\left(x_i\right)+q\left(x_i\right)\bar{v}_p\left(x_i\right)\right)=0 \tag{6}$$

In the local form:

$$\begin{cases}\dfrac{\partial\sigma}{\partial x}+\sigma_{ext}=0\\[2mm] \dfrac{\partial\xi}{\partial x}+s_{ext}-s_{\text{int}}=0\end{cases}\qquad\qquad \begin{cases}\sigma(x_2)=\sigma_c(x_2), & \sigma(x_1)=-\sigma_c(x_1)\\ \xi(x_2)=q(x_2), & \xi(x_1)=-q(x_1)\end{cases} \tag{7}$$

We also consider as basic a mechanical version of the second law thermodynamics. Namely ψ the free energy per unit of length, we have:

$$\frac{d}{dt}\int\limits_{D}\psi dx\leqslant P_{ext}(D,V) \tag{8}$$

for each part $D\subset B$. After using the principle virtual power, this version localizes into the dissipation inequality:

$$\dot{\psi}\leqslant\sigma\dot{\varepsilon}+\boldsymbol{s}_{\text{int}}\dot{\varepsilon}_p+\xi\dot{\boldsymbol{p}}=\sigma\dot{\varepsilon}_e+\pi\dot{\varepsilon}_p+\xi\dot{\boldsymbol{p}} \tag{9}$$

where $\boldsymbol{p}:=\frac{\partial\varepsilon_p}{\partial x}$ and $\pi:=\boldsymbol{s}_{\text{int}}+\sigma$

2.3 Constitutive theory

In the absence of internal links, the inequality of dissipation Eq. (9) suggests that constitutive prescriptions should be given for ψ, σ, ξ and π. We consider the list $(\varepsilon_e,\varepsilon_p,\dot{\varepsilon}_p,\boldsymbol{p})$ as the independent variables, i.e.:

$$\psi = \hat{\psi}(\mathbf{e}, \dot{\varepsilon}_p), \quad \sigma = \hat{\sigma}(\boldsymbol{e}, \dot{\varepsilon}_p), \quad \xi = \hat{\xi}(\boldsymbol{e}, \dot{\varepsilon}_p), \quad \pi = \hat{\pi}(\boldsymbol{e}, \dot{\varepsilon}_p) \tag{10}$$

where $\boldsymbol{e} := (\varepsilon_e, \varepsilon_p, \boldsymbol{p})$. These response functions are considered smooth, excepting $\hat{\pi}$. In the following case 1, $\hat{\pi}$ is smooth for parts (discontinuous in $\dot{\varepsilon}_p = 0$), and in the case 2, $\hat{\pi}$ is smooth everywhere (continuous on $\dot{\varepsilon}_p = 0$), and then, π is not constitutively determined when $\dot{\varepsilon}_p = 0$.

Using the procedure of Coleman-Noll [17] in the inequality (9), we get the following restrictions:

$$\frac{\partial \hat{\psi}}{\partial \dot{\varepsilon}_p} = 0, \quad \hat{\sigma} = \frac{\partial \hat{\psi}}{\partial \varepsilon_e}, \quad \hat{\xi} = \frac{\partial \hat{\psi}}{\partial p} \tag{11}$$

And, the response function $\hat{\pi}$ satisfies the reduced inequality:

$$\left(\hat{\pi} - \frac{\partial \hat{\psi}}{\partial \varepsilon_p} \right) \dot{\varepsilon}_p \geqslant 0 \tag{12}$$

Therefore, from Eq. (11), we have: $\psi = \hat{\psi}(\mathbf{e}), \sigma = \hat{\sigma}(\mathbf{e}), \xi = \hat{\xi}(\mathbf{e})$

By denoting $\hat{\pi}_d := \hat{\pi} - \frac{\partial \hat{\psi}}{\partial \varepsilon_p}$ in Eq. (12), the reduced inequality can be written as:

$$\hat{\pi}_d(\mathbf{e}, \dot{\varepsilon}_p).\dot{\varepsilon}_p \geqslant 0 \rightarrow \begin{cases} \hat{\pi}_d(\mathbf{e}, \dot{\varepsilon}_p) \geqslant 0 & \text{if} \quad \dot{\varepsilon}_p > 0 \\ \hat{\pi}_d(\mathbf{e}, \dot{\varepsilon}_p) \leqslant 0 & \text{if} \quad \dot{\varepsilon}_p < 0 \end{cases} \tag{13}$$

By considering the micro forces balance and the constitutive equations, we obtain the kinetic equation for the plastic deformation. In general we have:

$$\hat{\pi}_d(\mathbf{e}, \dot{\varepsilon}_p) = \hat{\boldsymbol{a}}(\mathbf{e}, \dot{\varepsilon}_p) + \hat{\boldsymbol{b}}(\mathbf{e}, \dot{\varepsilon}_p) \tag{14}$$

where:

$$\hat{\boldsymbol{a}}(\mathbf{e}, \dot{\varepsilon}_p) = \begin{cases} \boldsymbol{a}^+(\mathbf{e}) & \text{if} \quad \dot{\varepsilon}_p > 0 \\ \boldsymbol{a}^-(\mathbf{e}) & \text{if} \quad \dot{\varepsilon}_p < 0 \end{cases} \quad \hat{\boldsymbol{b}}(\mathbf{e}, \dot{\varepsilon}_p) = \begin{cases} \boldsymbol{b}^+(\mathbf{e}, \dot{\varepsilon}_p) & \text{if} \quad \dot{\varepsilon}_p > 0 \\ \boldsymbol{b}^-(\mathbf{e}, \dot{\varepsilon}_p) & \text{if} \quad \dot{\varepsilon}_p < 0 \end{cases} \tag{15}$$

The signals ($\pm$) are dependent of the $\dot{\varepsilon}_p$ signal. We consider two cases: the case 1, with elastic region, and the case 2 without.

Case 1: With elastic region ($\hat{\boldsymbol{a}} \neq 0$).

This case is separated in two sub-cases:

Sub-case 1a: We consider a theory of plasticity rate independent, and then the kinetic equation is given by:

$$\hat{\sigma}(\mathbf{e}) - \left(\hat{\boldsymbol{a}}(\mathbf{e}, \dot{\varepsilon}_p) - \frac{\partial^2 \hat{\psi}}{\partial \boldsymbol{x} \partial \mathrm{p}} + \frac{\partial \hat{\psi}}{\partial \varepsilon_p} \right) = 0 \tag{16}$$

Sub-case 1b: We consider a theory of plasticity rate dependent, and then the kinetic equation is given by:

$$\hat{\boldsymbol{b}}(\mathbf{e},\dot{\varepsilon}_p) := \hat{\sigma}(\mathbf{e}) - \left(\hat{\boldsymbol{a}}(\mathbf{e},\dot{\varepsilon}_p) - \frac{\partial^2\hat{\psi}}{\partial\boldsymbol{x}\partial\mathbf{p}} + \frac{\partial\hat{\psi}}{\partial\varepsilon_p}\right), \quad \text{if } \dot{\varepsilon}_p \neq 0 \tag{17}$$

Case 2: Rate dependent, without elastic region ($\hat{\boldsymbol{a}} = 0$).

In this case the dissipative response function is smooth and thermodynamically ensures that $\hat{\pi}_d$ is zero when $\dot{\varepsilon}_p = 0$. In this case the micro forces balance provides directly the equation that governs the plastic deformation, without additional constitutive assumptions, thus:

$$\hat{\boldsymbol{b}}(\mathbf{e},\dot{\varepsilon}_p) := \sigma(\mathbf{e}) - \left(\frac{\partial\hat{\psi}}{\partial\varepsilon_p} - \frac{\partial^2\hat{\psi}}{\partial\boldsymbol{x}\partial\mathbf{p}}\right), \quad \text{if } \dot{\varepsilon}_p \neq 0 \tag{18}$$

Observation: By choosing adequately, the response functions $\hat{a}$ and $\hat{b}$ can represent some of the classic theories of plasticity, as for instance, different behavior in traction and compression, rate independence, perfectly plastic material, isotropic hardening, Perzynas' model, viscous-plastic regularization, effect of Baushinger, shear bands, effect of PLC and others.

2.4 Simplified model

In this simplified model we are interested in the **sub-case 1b**, when there is plastic deformation by traction, i.e., $\dot{\varepsilon}_p > 0$. Let us consider a bar, fixed at one end and tensioned at the other.

We consider the following free energy:

$$\psi(\mathbf{e}) = \frac{\boldsymbol{E}\varepsilon_e^2}{2} + \frac{\boldsymbol{c}}{2}\mathbf{p}^2 \tag{19}$$

where E is the Young module, and $\boldsymbol{c}$ is the diffusion coefficient related with the micro-scale displacement. The first term represents the strain energy, and the second term is the interface energy. In Eq. (17) we define $\hat{\boldsymbol{a}}(\mathbf{e}) = \boldsymbol{g}_o(\varepsilon_p), \hat{\boldsymbol{b}}(\mathbf{e},\dot{\varepsilon}_p) = \boldsymbol{g}_1(\dot{\varepsilon}_p), \boldsymbol{g}_1(0) = 0$, where $\boldsymbol{g}_o$ and $\boldsymbol{g}_1$ are smooth functions and differentiable in almost everywhere.

With these assumptions we have:

$$\begin{cases} \sigma = \boldsymbol{E}\varepsilon_e \\ \xi = \boldsymbol{c}\frac{\partial\varepsilon_p}{\partial\boldsymbol{x}} \end{cases} \tag{20}$$

Therefore the micro force balance is:

$$\boldsymbol{c}\frac{\partial^2\varepsilon_p}{\partial\boldsymbol{x}^2} + \sigma - \boldsymbol{g}_o(\varepsilon_p) - \boldsymbol{g}_1(\dot{\varepsilon}_p) = 0 \tag{21}$$

By considering the body force as inertial, the external force takes the form $\sigma_{ext} = -\rho\ddot{\boldsymbol{u}}$, with $\rho \geqslant 0$. From the balance laws and the above assumptions we have the following relations:

$$\begin{cases} \sigma = \boldsymbol{E}\varepsilon_e = \boldsymbol{E}(\varepsilon - \varepsilon_p) \\ \varepsilon = \varepsilon_e + \varepsilon_p \\ \dfrac{\partial \sigma}{\partial \boldsymbol{x}} = \rho \ddot{\boldsymbol{u}} \\ \boldsymbol{c}\dfrac{\partial^2 \varepsilon_p}{\partial x^2} - \boldsymbol{g}_o(\varepsilon_p) - \boldsymbol{g}_1(\dot{\varepsilon}_p) + \sigma = 0 \end{cases} \tag{22}$$

By taking the derivative of Eq. (22) $_1$, we have the following equations system:

$$\begin{cases} \boldsymbol{E}\left(\dfrac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} - \dfrac{\partial \varepsilon_p}{\partial \boldsymbol{x}}\right) = \rho \ddot{\boldsymbol{u}} \\ \boldsymbol{c}\dfrac{\partial^2 \varepsilon_p}{\partial \boldsymbol{x}^2} - \boldsymbol{g}_o(\varepsilon_p) - \boldsymbol{g}_1(\dot{\varepsilon}_p) + \boldsymbol{E}\left(\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \varepsilon_p\right) = 0 \end{cases} \tag{23}$$

2.4.1 Qualitative analysis of the traction problem

Motivated by the work of Kubin et al. [6], Estrin [13], Brechet et al. [7], Coleman et al. [2] and Anand et al. [11], we consider a qualitative analysis of Eq. (23) in the quasi-static case. We start finding some trivial solutions. In this case, Eq. (23) is reduced to the nonlinear differential equation:

$$c\frac{\partial^2 \varepsilon_p}{\partial x^2} - g_o(\varepsilon_p) - g_1(\dot{\varepsilon}_p) + \sigma(t) = 0 \tag{24}$$

Trivial solutions

(a) There is a stationary uniform solution [2, 7, 14] of Eq. (24). Thus, the solution satisfies:

$$\boldsymbol{c}\frac{d^2 \varepsilon_{p_s}(\boldsymbol{x})}{d\boldsymbol{x}^2} - \boldsymbol{g}_o(\varepsilon_{p_s}(\boldsymbol{x})) + \sigma_0 = 0 \tag{25}$$

where σ_o is a constant prescribed stress.

(b) There is a steady state homogeneous solution [1, 6, 18] of Eq. (24). Thus, the solution satisfies:

$$\boldsymbol{g}_o(\varepsilon_{p_s}(\boldsymbol{t})) + \boldsymbol{g}_1(\dot{\varepsilon}_{p_s}(\boldsymbol{t})) - \sigma(\boldsymbol{t}) = 0 \tag{26}$$

(c) There is a stationary homogenous and uniform solution [1, 12] of Eq. (24) satisfying the relationship:

$$\boldsymbol{g}_o(\varepsilon_{p_s}) - \sigma_0 = 0 \tag{27}$$

2.4.2 Stability analysis

We analyze the plastic instability by linearizing Eq. (24) around a homogeneous state $\varepsilon_{p_s}(\boldsymbol{t})$. We consider solutions of the form: $\varepsilon_p(\boldsymbol{x}, \boldsymbol{t}) = \varepsilon_{p_s}(\boldsymbol{t}) + \delta\varepsilon_p$, where $\delta\varepsilon_p(\boldsymbol{x}, \boldsymbol{t}) = \delta\varepsilon_o \boldsymbol{e}^{\boldsymbol{ikx}+\omega \boldsymbol{t}}$ is a small perturbation, $\delta\varepsilon_o$ is the initial perturbation, $\boldsymbol{k}$ is the wave number and ω is the amplification rate or attenuation, that determines the growth or decline of the disturbance. By substituting this solution in Eq. (24), we have:

$$\left(\boldsymbol{s}\omega + \boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2\right)\delta\varepsilon_p = 0, \quad \text{then } \omega = -\frac{\boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2}{\boldsymbol{s}} \tag{28}$$

where: $\boldsymbol{h} := \frac{\partial \boldsymbol{g}_o(\varepsilon_{p_s})}{\partial \varepsilon_p}$ and $\boldsymbol{s} := \frac{\partial \boldsymbol{g}_1(\dot{\varepsilon}_{p_s})}{\partial \dot{\varepsilon}_p}$.

The linear stability is guaranteed when $\omega < 0$, otherwise we have instability, which occurs when:

$$\begin{cases} \boldsymbol{s} < 0 \quad \text{and} \quad \boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2 \geq 0 \\ \boldsymbol{s} > 0 \quad \text{and} \quad \boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2 \leq 0 \end{cases} \tag{29}$$

We get the following types of instability:

- **Type H**: $\boldsymbol{s} < 0, \boldsymbol{h} < 0, \boldsymbol{c} > 0, \boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2 \leq 0$
- **Type S**: $\boldsymbol{s} < 0, \boldsymbol{h} > 0, \boldsymbol{c} < 0, \boldsymbol{h} + \boldsymbol{c}\boldsymbol{k}^2 \geq 0$ or just when $\boldsymbol{s} < 0$.

Observations:

(a) The value of $\boldsymbol{h}$ is not necessarily positive in the inequality (29).

(b) It is possible to determine the length of the critical wave $\boldsymbol{L}_c = \frac{2\pi}{\boldsymbol{k}_c}$ when $\boldsymbol{c}$ and $\boldsymbol{h}$ have opposite signs. The parameter $\boldsymbol{k}_c := \sqrt{\frac{-\boldsymbol{h}}{\boldsymbol{c}}}$ is called critical wave number because can occur a bifurcation or the deformation can become heterogeneous. The length $\boldsymbol{L}_c$, called *internal characteristic length scale*, appears in different studies of wave dispersion and traveling waves.

2.5 Shear bands

After a certain increase of plastic flow, the material exhibits deformation by softening and may occur a concentration of strain in narrow bands [7, 9]. In Eq. (23) we consider $\boldsymbol{g}_1(\dot{\varepsilon}_p) = \boldsymbol{s}\dot{\varepsilon}_p$ and $\boldsymbol{g}_o$ a continuous function, differentiable and not monotonous, so that $\boldsymbol{g}_o(0) = \sigma_y$, (σ_y is yield stress), and $\boldsymbol{s}$ a positive parameter. Therefore we get the shear bands model:

$$\boldsymbol{c}\frac{\partial^2 \varepsilon_p}{\partial \boldsymbol{x}^2} - \boldsymbol{g}_o(\varepsilon_p) - \boldsymbol{s}\dot{\varepsilon}_p + \boldsymbol{E}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \varepsilon_p\right) = 0 \tag{30}$$

The evolution of the bands occurs when $\boldsymbol{g}_o'(\varepsilon_p) < 0$ [12]. This type of deformation can be seen in soft steel, polymers and various metals, such as: $\boldsymbol{Cu}$ 4% $\boldsymbol{Si}$, $\boldsymbol{Al}$ 0.7% $\boldsymbol{Li}$, etc.

Motivated by the work of Coleman et al. [2], Anand et al. [11] and Sikora et al. [12], we are interested in the stationary uniform solution of Eq. (30), assuming that the diffusion coefficient $\boldsymbol{c} > 0$. The Eq. (30) is equivalent to the first order system:

$$\begin{cases} \dfrac{d\varepsilon_p}{d\boldsymbol{x}} = \boldsymbol{w} \\ \dfrac{d\boldsymbol{w}}{d\boldsymbol{x}} = \dfrac{1}{\boldsymbol{c}}\left(\boldsymbol{g}_o(\varepsilon_p) - \sigma_o\right) \end{cases} \tag{31}$$

We say that an equilibrium point is *homogenous* if it is constant related with x, i.e., when $\boldsymbol{w} = 0, \boldsymbol{g}_o(\varepsilon_p) = \sigma_o$. Therefore, the equilibrium points are $(\varepsilon_{p_s}, 0)$. The first integral of the system is given by:

$$\boldsymbol{K} = \frac{\boldsymbol{w}^2}{2} + \boldsymbol{V}(\varepsilon_p) \tag{32}$$

where $\boldsymbol{K} = \boldsymbol{K}(\varepsilon_p, w)$ is constant through any solution, and the function $\boldsymbol{V}$ is obtained through the integral:

$$\boldsymbol{V}(\varepsilon_p) := \frac{1}{\boldsymbol{c}} \int_{\varepsilon_{p_o}}^{\varepsilon_p} (\sigma_o - \boldsymbol{g}_o(\varsigma_p))\, \boldsymbol{d}\varsigma_p \tag{33}$$

The system (31) is the Hamiltonian type. The first integral corresponds to *total energy function*, where the *kinetic* energy is given by $\boldsymbol{E}_c := \frac{\boldsymbol{w}^2}{2}$, and the *potential energy* is given by $\boldsymbol{V}$. From Eqs. (32) and (33) we get:

$$\frac{\boldsymbol{d}\varepsilon_p}{\boldsymbol{dx}} = \boldsymbol{w} = \pm\sqrt{2(\boldsymbol{K} - \boldsymbol{V}(\varepsilon_p))} \tag{34}$$

Then:

$$\boldsymbol{x} = \int_{\varepsilon_p(0)}^{\varepsilon_p(x)} \frac{\boldsymbol{d}\zeta_p}{\sqrt{2(\boldsymbol{K} - \boldsymbol{V}(\zeta_p))}} \tag{35}$$

As the kinetic energy is not negative $\boldsymbol{K} > \boldsymbol{V}(\varepsilon_p)$, for different values of $\boldsymbol{K}$ is possible to draw the trajectories in the phase-plane (level curves).
Comments: The point $(\varepsilon_{p_s}, 0)$ is an equilibrium point if and only if $\boldsymbol{g}(\varepsilon_{p_s}) = \sigma_o$ (or $\boldsymbol{V}'(\varepsilon_p) = 0$). These points represent the equilibrium homogeneous stationary solutions, where the system stops moving in the phase-space.

- If $\boldsymbol{K} = \min \boldsymbol{V}(\varepsilon_p)$, the solution of Eq. (31) corresponds to an equilibrium point.
- If $\min \boldsymbol{V}(\varepsilon_p) < \boldsymbol{K} < \max \boldsymbol{V}(\varepsilon_p)$, the system (31) admits periodic orbits with characteristic length $\boldsymbol{L}_c = 2\pi\sqrt{\frac{\boldsymbol{c}}{-\boldsymbol{h}}}$.
- If $\boldsymbol{K} = \max \boldsymbol{V}(\varepsilon_p)$, the solution of Eq. (31) is an invariant manifold that corresponds to homoclinic (heteroclinic) orbits. Each homoclinic orbit represents a homoclinic shear band that starts and ends at the some value to slip. And each heteroclinic orbit represents a shear band connecting two homogeneous stationary states and uniforms for different deformations.

2.6 Numerical results

In this section we describe some numerical simulations of the shear bands Eq. (30). We consider a bar of 100 mm. length, with cross section 1.00 mm^2, fixed at one end. We simulate the softening by

prescribing a monotonous displacement at free end ($\boldsymbol{u}_L(\boldsymbol{t}) = \boldsymbol{a}$) and plastic deformation in both ends ($\varepsilon_p(0, \boldsymbol{t}) = \varepsilon_p(\boldsymbol{L}, \boldsymbol{t}) = 0$). We consider the initial conditions: $\boldsymbol{u}(\boldsymbol{x}, 0) = 0$, $\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) > 0$, $\varepsilon_p(\boldsymbol{x}, 0) = 0$.
The used parameters are showed in Table 2.

Table 2: Parameters

Elasticity modulus (Pa)	$\boldsymbol{E} = 20 \cdot 10^9$	Yield stress (Pa)	$\sigma_y = 2 \cdot 10^6$
Softening modulus (Pa)	$\boldsymbol{h} = -2 \cdot 10^9$	Kinematic modulus (1/s)	$\boldsymbol{s} = 2.10^3$
Diffusion coefficient (N)	$\boldsymbol{c} = 5 \cdot 10^4$	Loading parameter (mm)	$\boldsymbol{a} = 0.125$

Figures 1-6 display the results for different values of diffusion coefficient $\boldsymbol{c}$. The Figures 1 and 2 show the plastic deformation field and the stress strain diagram, respectively, for the boundary conditions described previously. In the Figure 1 we can observe a concentration of plastic deformation at the middle of bar, and then the Figure 2 exhibits the stress strain diagram at this point, where we can see the softening effect. Figures 3 and 4 show the stress evolution at the middle of the bar and the displacement field, respectively. It is interesting to observe that the diffusion coefficient affects basically the localization of plastic deformation.

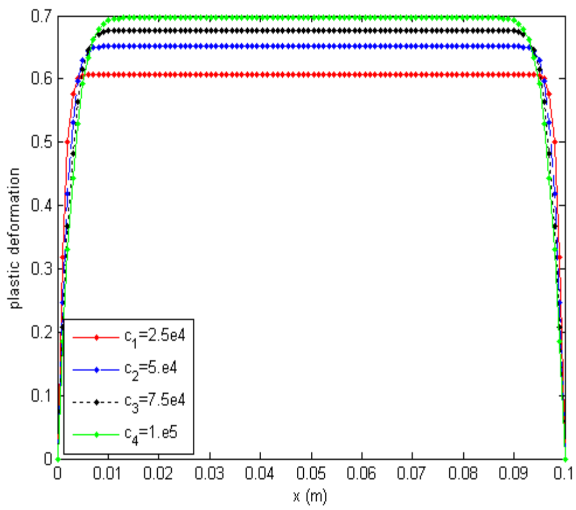

Figure 1: Plastic deformation field.

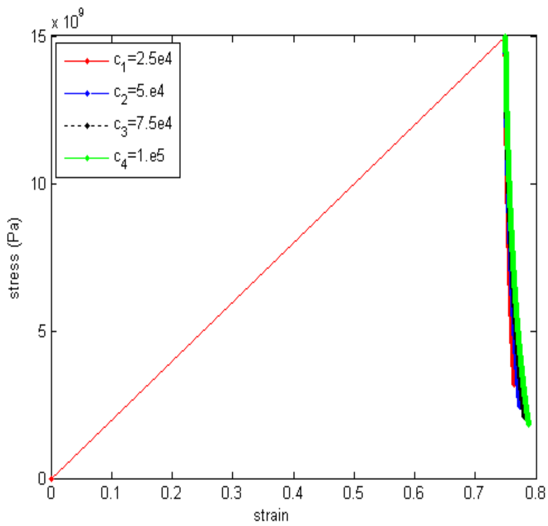

Figure 2: Stress strain diagram at the middle of the bar.

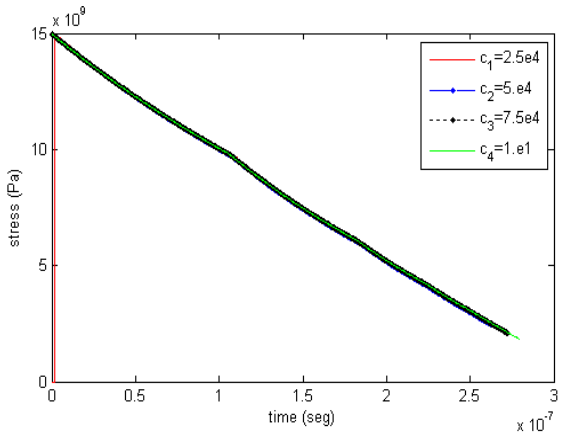

Figure 3: Stress evolution at the middle of the bar.

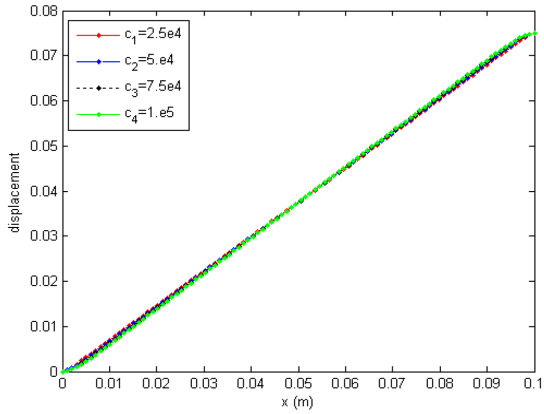

Figure 4: Displacement field.

We consider in Figures 5 and 6 as initial condition a plastic deformation of 0.09 prescribed in the middle bar. Figure 5 shows the plastic deformation field with this initial prescribed value, whereas the Figure 6 show the stress strain diagram at the middle of the bar.

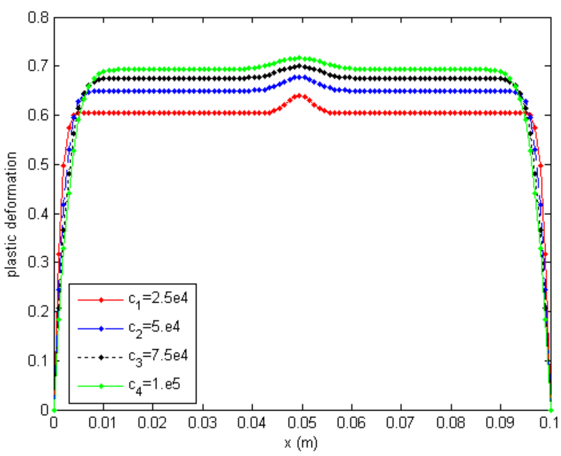

Figure 5: Plastic deformation field.

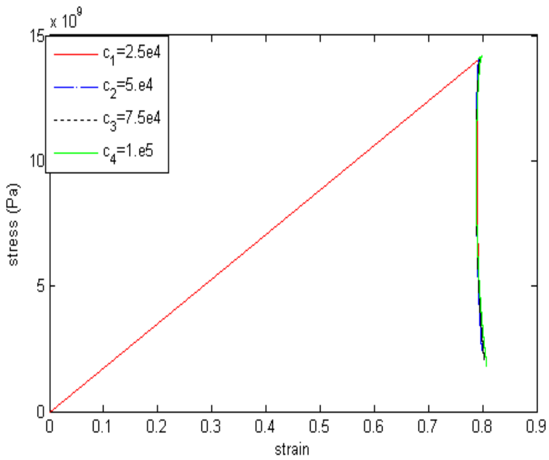

Figure 6: Stress strain diagram at the middle of the bar.

The Figures 7 and 8 display different curves related with different values of prescribed displacement at the free end $\boldsymbol{a}$. As the latter case, a plastic deformation of 0.09 was prescribed in the middle bar as initial condition. Since the diffusion coefficient is the same for all curves, we do not observe different effects in the localization of plastic deformation in Figure 7, only the effect of the prescribed displacement. However, in contrast with the Figure 6, Figure 8 displays distinct curves in consequence of different prescribed displacements.

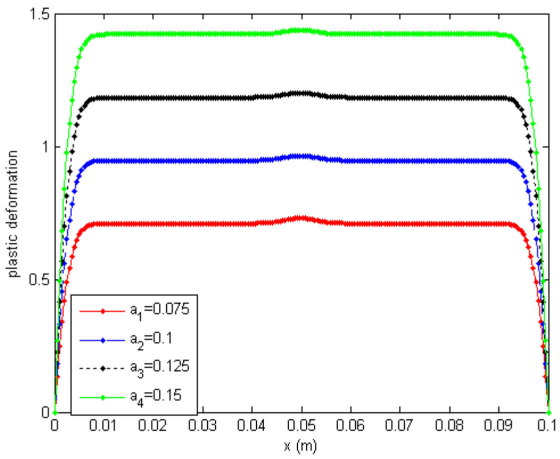

Figure 7: Plastic deformation field.

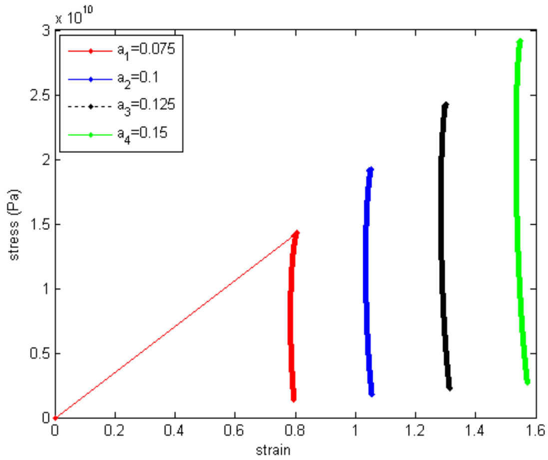

Figure 8: Stress strain diagram at the middle of the bar.

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