A variational framework for a set of hyperelastic-viscoplastic isotropic models

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Abstract

The rate tensor of plastic deformation in elasto-viscoplastic models is usually decomposed in direction and magnitude. In the case of small-strain J2 models, this leads to the well known radial return mapping algorithms in which the flow direction is known a priori at each load step. In the context of the variational large-strain elasto-viscoplastic models, this decomposition between constitutive and cinematic aspects is accomplished by the choice of logarithmic strain measures, exponential integration algorithm and quadratic (Hencky) elastic potentials. The aim of this paper is to show that the mentioned decoupling properties can be extended to a wide set of hyperelastic-plastic isotropic models not restricted to Hencky-type elastic behavior by relaxing the classical decomposition amplitude/direction into the sum of spectral quantities.

1 Introduction

A variational formulation of irreversible (i.e. dissipative) constitutive models was initially proposed in [1, 2], in an isothermal context, and later extended to a fully coupled thermo-mechanical context in [3]. One of the most relevant aspects of variational approaches is that they provide appropriate mathematical basis for error estimation and mesh adaptation [2]. Applications to non-linear finite viscoelasticity were studied in [4] where an spectral decomposition of elastic/inelastic strain/strain-rates quantities is proposed in order to allow the inclusion of arbitrary elastic and inelastic (isotropic) potentials within the same formalism.

In elasto-plasticity and elasto-viscoplasticity, the rate tensor of plastic deformation (or its incremental value), is usually decomposed in direction, related to the gradient of a yield potential, and magnitude. In the case of von Mises (J2) type flows and small strain range, this decomposition provides a complete separation between kinematic and constitutive aspects which reduces the problem to the determination of the plastification in a known radial direction. Analogous results are obtained with

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classical hyperelastic based models by using appropriate logarithmic strain measures and exponential integration algorithm [5–8]. In the variational approach, both internal quantities are determined by a local minimization process. Moreover, It is shown in [1] that the kinematic and constitutive decoupling is, once again, achieved by the choice of logarithmic strain measures, exponential integration algorithm and quadratic elastic potential (Hencky model).

The aim of this paper is to show that the mentioned decoupling properties may be extended to a wide set of simple hyperelastic-viscoplastic isotropic models not restricted to quadratic elastic behavior. This is performed by relaxing the classical decomposition amplitude/direction in a similar way as done in [4], by using spectral quantities. Moreover, this approach allows for a natural combination of viscoelastic and viscoplastic dissipative mechanisms appropriate for a group of thermoplastic polymers, which is the final goal of this research.

The paper is organized as follows. Section 2 briefly presents the variational approach for irreversible constitutive problems. The application of this approach to elasto-viscoplastic materials is stated in Section 3 where the main results of this work are shown. Section 4 shows two particular material models within the present context while Section 5 presents a couple of numerical examples. Final remarks are shown in Section 6.

2 Incremental formulation for inelastic constitutive behavior

The main assumption in Hyperelasticity is the existence of a potential function Ψ which depends on the value of strains only and whose derivative provides the state of stress of a material point, i.e.,

$$\mathbf{P} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \tag{1}$$

In (1) $\mathbf{F} = \nabla_0 \mathbf{x}$ denotes the gradient of deformations, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the Cauchy strain tensor and $\mathbf{P}$ the first Piola Kirchhoff stress tensor. Assuming the satisfaction of compatibility and constitutive equations, the equilibrium problem may be defined by the minimization of the Potential Energy

$$\min_{\mathbf{x}\in K} \mathcal{H}(\mathbf{x})$$

$$\mathcal{H}(\mathbf{x}) = \int_{\Omega_0} \Psi(\mathbf{F}(\mathbf{x}))\, d\Omega_0 - \left[\int_{\Omega_0} \mathbf{b}_0 \cdot \mathbf{x}\, d\Omega_0 + \int_{\Gamma_0} \mathbf{f}_0 \cdot \mathbf{x}\, d\Gamma_0\right] \tag{2}$$

where K is the set of admissible deformations.

On the other hand, the state of stress of an inelastic path dependent dissipative phenomenon cannot be obtained just from the value of final strains and it is not any longer possible to define a potential function with the property (1). The history of the process is usually is described incrementally with the aid of internal (dissipative) variables. However it is shown in [1, 2], that a wide set of dissipative materials can be modelled by the aid of pseudo potentials that behave like hyperelastic (in the sense that satisfy (1)) within the interval of a load increment, i.e.,

$$\mathbf{P}_{n+1} = \frac{\partial \Psi(\mathbf{F}_{n+1}; \mathcal{E}_n)}{\partial \mathbf{F}_{n+1}} = 2\mathbf{F}_{n+1}\frac{\partial \Psi(\mathbf{C}_{n+1}; \mathcal{E}_n)}{\partial \mathbf{C}_{n+1}} \tag{3}$$

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where $\mathcal{E}$ denotes a set of external and internal variables:

$$\mathcal{E} = \{\mathbf{F}, \mathbf{F}^i, \mathbf{Q}\} \qquad \mathbf{F} = \mathbf{F}^e\mathbf{F}^i \tag{4}$$

The tensors $\mathbf{F}^e$ and $\mathbf{F}^i$are the elastic and inelastic parts of the gradient of deformations while the quantity $\mathbf{Q}$ contains all the remaining internal variables used to describe the process. The subindexes n and $n+1$ indicates the beginning and end of the load increment and it is supposed that all quantities at time n are known.

For a quite general set of inelastic problems the potential $\Psi(\mathbf{F}_{n+1}; \mathcal{E}_n)$ takes the form (see [1], [2] for a detailed construction):

$$\Psi(\mathbf{F}_{n+1}; \mathcal{E}_n) = \min_{\substack{\mathbf{F}^i_{n+1}\\ \mathbf{Q}_{n+1}}} \left\{ W(\mathcal{E}_{n+1}) - W(\mathcal{E}_n) + \Delta t \psi \left(\mathring{\mathbf{F}}^i, \mathring{\mathbf{Q}}; \mathcal{E}_n \right) \right\} \tag{5}$$

$$W(\mathcal{E}) = \omega(\mathbf{F}) + \varphi^e(\mathbf{F}\mathbf{F}^{i-1}) + \varphi^i(\mathbf{F}^i, \mathbf{Q}) \tag{6}$$

where $\mathring{\mathbf{F}}\left(\mathbf{F}_{n+1}, \mathcal{E}_n\right)$, $\mathring{\mathbf{F}}^i\left(\mathbf{F}^i_{n+1}, \mathcal{E}_n\right)$ and $\mathring{\mathbf{Q}}\left(\mathbf{Q}_{n+1}, \mathcal{E}_n\right)$ are suitable incremental approximations of the rate variables $\dot{\mathbf{F}}, \dot{\mathbf{F}}^i$ and $\dot{\mathbf{Q}}$ respectively. The potentials W, and ψ inside (5) may assume different expressions depending on the particular model needed. The expression of $W(\mathcal{E})$ in (6) assumes that the free energy can be additively decomposed in potentials depending on $\mathbf{F}$, $\mathbf{F}^e$ and $\mathbf{F}^i, \mathbf{Q}$ respectively. The minimization in (5) with respect to the internal variables $\mathbf{F}^i_{n+1}$ and $\mathbf{Q}_{n+1}$ provides an evolution path of these variables within the time step and eliminates them from the potential Ψ enforcing it to be dependent only on the gradient of deformation $\mathbf{F}_{n+1}$.

3 A set of hyperelastic-viscoplastic isotropic models

In this section, we focus an elastic-viscoplastic models[1]. Consider a free energy function which $\omega(\mathcal{E})$ accounts for the separation of $\mathbf{F}$ in volumetric and isochoric parts:

$$\omega(\mathbf{F}) = \varphi(\hat{\mathbf{F}}) + U(J) \tag{7}$$

$$\hat{\mathbf{F}} = \frac{1}{J^{1/3}}\mathbf{F}, \qquad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}}, \qquad J = \det(\mathbf{F}) \tag{8}$$

Thus, we rewrite $W(\mathcal{E})$ as

$$W(\mathcal{E}) = \varphi(\hat{\mathbf{F}}) + \varphi^e(\hat{\mathbf{F}}\mathbf{F}^{p-1}) + \varphi^p(\mathbf{F}^p, \mathbf{Q}) + U(J) \tag{9}$$

where

$$\hat{\mathbf{F}} = \hat{\mathbf{F}}^e\mathbf{F}^p \Longrightarrow \hat{\mathbf{F}}^e = \hat{\mathbf{F}}\left(\mathbf{F}^p\right)^{-1}, \qquad \det \mathbf{F}^p = 1 \tag{10}$$

$$\hat{\mathbf{C}}^e = \hat{\mathbf{F}}^{eT}\hat{\mathbf{F}}^e = \sum_{j=1}^{3} c^e_j \mathbf{E}^e_j \tag{11}$$

[1] Just for facility of notation we change the superscript i (inelastic) for p (plastic).

A rate of plastic deformation (or plastic stretching) $\mathbf{D}^p$ is defined as

$$\mathbf{D}^p = \text{sym}(\mathbf{L}^p) = \mathbf{L}^p = \dot{\mathbf{F}}^p \left(\mathbf{F}^p\right)^{-1} \tag{12}$$

i.e., $\mathbf{L}^p$ is assumed to be symmetric. This means that no plastic spin $W^p = \left(\mathbf{L}^p - \mathbf{L}^{pT}\right)/2$ is considered.

If a von Mises' flow type is assumed, The tensor $\mathbf{D}^p$ may be decomposed as follows:

$$\mathbf{D}^p = \dot{q}\mathbf{M}, \tag{13}$$

$$\dot{q} \in \mathbb{R}^+, \tag{14}$$

$$\mathbf{M} \in K_M = \{\mathbf{N} \in Sym : \mathbf{N}\cdot\mathbf{N} = \frac{3}{2}, \quad \mathbf{N}\cdot\mathbf{I} = 0\}. \tag{15}$$

where the non-negative scalar $\dot{q}$ accounts for the *amplitude* of $\mathbf{D}^p$ while the normalized traceless tensor $\mathbf{M}$ provides the *direction* of $\mathbf{D}^p$. It is shown in [1] that, when this separation is combined with logarithmic strains and quadratic hyperelastic (Henky type) potentials, a complete separation of kinematic aspects (provided by $\mathbf{M}$) and constitutive aspects (provided by $\dot{q}$) is reached and constitutive expressions similar to those of infinitesimal plasticity theory are obtained. This result is also verified in more classical hyperelastic-plastic and viscoplastic approaches [8]. In order to extend these facilities to more general hyperelastic laws other than Hencky, a spectral decomposition of $\mathbf{D}^p$ is here used, following the ideas proposed in [4]:

$$\mathbf{D}^p = \mathbf{M} = \dot{q}\sum_{j=1}^{3} q_j \mathbf{M}_j \tag{16}$$

$$\dot{q} \in \mathbb{R}^+, \quad q_j \in K_Q = \left\{p_j \in \mathbb{R} : \textstyle\sum_{j=1}^{3} p_j = 0;\ \sum_{j=1}^{3} p_j^2 = 3/2\right\} \tag{17}$$

$$\mathbf{M}_j \in K_M = \{\mathbf{M}_j \in Sym : \mathbf{M}_j\cdot\mathbf{M}_j = 1, \quad \mathbf{M}_i\cdot\mathbf{M}_j = 0, \quad i \neq j\} \tag{18}$$

The set K_Q enforces the traceless form of $\mathbf{M}$ with fixed norm, while the set K_M accounts for usual properties of eigenprojections.

Equation (16) has a special meaning: it defines a flow rule for $\dot{\mathbf{F}}^p$ and establishes a constraint between $\mathbf{F}^p$ and q through the flow directions $q_j\mathbf{M}_j$. Due to this constraint, $\mathbf{F}^p$ becomes a internal variable dependent of the (independent) internal variables $\{q, q_j, \mathbf{M}_j\}$

An incremental approximation of $\mathbf{D}^p$ is obtained by the exponential mapping:

$$\Delta\mathbf{F}^p = \mathbf{F}^p_{n+1}\left(\mathbf{F}^p_n\right)^{-1} = \exp[\Delta t\mathbf{D}^p] \tag{19}$$

$$\Delta\mathbf{C}^p = \left(\Delta\mathbf{F}^p\right)^T \Delta\mathbf{F}^p = \mathbf{F}^{p-T}_n \mathbf{C}^p_{n+1}\left(\mathbf{F}^p_n\right)^{-1} = \exp[\Delta t\mathbf{D}^p]^2 \tag{20}$$

$$\Rightarrow \Delta\epsilon^p = \Delta t\mathbf{D}^p = \frac{1}{2}\ln\left(\Delta\mathbf{C}^p\right) = \Delta t\ \dot{q}\sum_{j=1}^{3} q_j\mathbf{M}_j = \Delta q\sum_{j=1}^{3} q_j\mathbf{M}_j \tag{21}$$

$$\Rightarrow \mathbf{F}^p_{n+1} = \exp\left[\Delta q\sum_{j=1}^{3} q_j\mathbf{M}_j\right]\mathbf{F}^p_n \tag{22}$$

The elastic potential φ^e is assumed to be an isotropic function of the elastic deformation depending on its eigenvalues:

$$\varphi^e(\hat{\mathbf{C}}^e) = \varphi^e(c_1^e, c_2^e, c_3^e), \quad \hat{\mathbf{C}}^e = \hat{\mathbf{F}}^{eT}\hat{\mathbf{F}}^e = \sum_{j=1}^{3} c_j^e \mathbf{E}_j^e \tag{23}$$

This function is also described in terms of the natural strain $\epsilon^e = \frac{1}{2}\ln\hat{\mathbf{C}}^e$ which has the same eigenprojections of $\hat{\mathbf{C}}^e$ and eigenvalues given by $\epsilon_j^e = \frac{1}{2}\ln c_j^e$:

$$\varphi^e(\epsilon^e) = \varphi^e(\epsilon_1^e, \epsilon_2^e, \epsilon_3^e)$$

The plastic potential φ^p accounts for hardening phenomena. In this case we consider an isotropic hardening depending on the internal variable q :

$$\varphi^p = \varphi^p(q) \tag{24}$$

$$q(t) = \int_0^t \dot{q}\; dt \qquad q_{n+1} = q_n + \Delta t\;\dot{q} = q_n + \Delta q \tag{25}$$

The dissipative isotropic plastic potential ψ depends on $\mathbf{D}^p$ through $\dot{q}$:

$$\psi(\mathbf{D}^p) = \psi\left(\frac{\Delta q}{\Delta t}\right) = \psi(\dot{q}) = \begin{cases} \bar{\psi}(\dot{q}) & \text{if } \dot{q} \geq 0 \\ +\infty & \text{if } \dot{q} < 0 \end{cases} \tag{26}$$

This definition of ψ has the objective of incorporating an exact penalization for negative values of $\dot{q}$.

The potential $\varphi(\hat{\mathbf{F}})$ is considered null for the present set of proposed models. The internal variables are reduced to Δq and the spectral quantities $\{q_j, \mathbf{M}_j\}$ that substitute the minimizing variables $\mathbf{Q}_{n+1}, \mathbf{F}_{n+1}^p$ in (5). The incremental potential Ψ is thus re-written as

$$\Psi(\mathbf{F}_{n+1};\mathcal{E}_n) = \Psi(\mathbf{C}_{n+1};\mathcal{E}_n) = \Delta U(J_{n+1}) + \min_{\Delta q, \mathbf{M}_j, q_j}\left\{\Delta\varphi^e(\hat{\mathbf{C}}_{n+1}^e) + \Delta\varphi^p(q_{n+1}) + \Delta t\bar{\psi}\left(\tfrac{\Delta q}{\Delta t}\right)\right\} \tag{27}$$

$$\Delta\varphi^e(\hat{\mathbf{C}}_{n+1}^e) = \varphi^e(\hat{\mathbf{C}}_{n+1}^e) - \varphi^e(\hat{\mathbf{C}}_n^e) \tag{28}$$

$$\Delta\varphi^p(q_{n+1}) = \varphi^p(q_{n+1}) - \varphi^p(q_n) \tag{29}$$

$$\Delta U(J_{n+1}) = U(J_{n+1}) - U(J_n) \tag{30}$$

where the minimization operation is constrained by the conditions

$$q_j \in K_Q = \left\{p_j \in \mathbb{R} : \textstyle\sum_{j=1}^3 p_j = 0;\; \sum_{j=1}^3 p_j m_j = 3/2\right\} \tag{31}$$

$$\mathbf{M}_j \in K_M = \{\mathbf{N}_j \in Sym : \mathbf{N}_j \cdot \mathbf{N}_j = 1, \quad \mathbf{N}_i \cdot \mathbf{N}_j = 0, \quad i \neq j\} \tag{32}$$

$$\Delta q \geq 0 \tag{33}$$

First order necessary optimality conditions of the minimization problem (27) take into account the derivatives of potential Ψ as well as the derivatives of constraints (31), (32) and (33). The minimization along the eigenprojections $\mathbf{M}_j$ can be calculated analytically. In order to perform this operation, a relevant relation between elastic and plastic deformations is shown:

$$\hat{\mathbf{F}}^e_{n+1} = \hat{\mathbf{F}}_{n+1}\left(\mathbf{F}^p_{n+1}\right)^{-1} = \hat{\mathbf{F}}^{pr}\left(\exp[\Delta t \mathbf{D}^p]\right)^{-1}, \quad \hat{\mathbf{F}}^{pr} = \hat{\mathbf{F}}_{n+1}\left(\mathbf{F}^p_n\right)^{-1} \tag{34}$$

$$\hat{\mathbf{C}}^e_{n+1} = \hat{\mathbf{F}}^{eT}_{n+1}\hat{\mathbf{F}}^e_{n+1} = \hat{\mathbf{C}}^{pr}\left(\exp[\Delta t \mathbf{D}^p]\right)^{-2}, \quad \hat{\mathbf{C}}^{pr} = \mathbf{F}^{p-T}_n \hat{\mathbf{C}}_{n+1}\left(\mathbf{F}^p_n\right)^{-1} \tag{35}$$

$$\epsilon^e_{n+1} = \frac{1}{2}\ln \hat{\mathbf{C}}^e_{n+1} = \epsilon^{pr} - \Delta t \mathbf{D}^p, \quad \epsilon^{pr} = \frac{1}{2}\ln \hat{\mathbf{C}}^{pr} \tag{36}$$

Equation (35) is only valid if co-linearity between $\hat{\mathbf{C}}^{pr}$ and $\mathbf{D}^p$ is assumed in order to allow permutation between both tensors. Using this assumption, it is possible to show that the minimization with respect to $\mathbf{M}_j$ is achieved when the tensors $\hat{\mathbf{C}}^e_{n+1}$, $\hat{\mathbf{C}}^{pr}$ and $\mathbf{D}^p$ share the same eigenprojections: $\mathbf{E}^e_j = \mathbf{E}^{pr}_j = \mathbf{M}_j$. This means collinearity between $\hat{\mathbf{C}}^{pr}$ and $\mathbf{D}^p$ which corroborates the permutability assumption made in (35). Moreover, the optimality conditions for q_j and Δq take the form

$$r_i = -\frac{\partial \Delta\varphi^e}{\partial \epsilon^e_i}\Delta q + \lambda + 2\beta q_i = 0, \quad i = 1,2,3 \tag{37}$$

$$r_4 = -\sum_{j=1}^{3}\frac{\partial \Delta\varphi^e}{\partial \epsilon^e_j} q_j + \frac{\partial \Delta\varphi^p}{\partial \Delta q} + \frac{\partial \psi}{\partial \dot{q}} = 0 \tag{38}$$

$$r_5 = \sum_{j=1}^{3} q_j = 0 \tag{39}$$

$$r_6 = \textstyle\sum_{j=1}^{3} q_j^2 = 3/2 \tag{40}$$

Once the minimization is performed, the derivative of Ψ with respect to $\hat{\mathbf{C}}_{n+1}$ and J_{n+1} should be calculated in order to obtain the Piola-Kirchhoff stress tensor. Due to the separation of potential Ψ in isochoric and volumetric contributions, the stress tensor $\mathbf{P}$ is re-written as

$$\mathbf{P}_{n+1} = 2\mathbf{F}_{n+1}\frac{\partial \Psi(\mathbf{C}_{n+1};\mathcal{E}_n)}{\partial \mathbf{C}_{n+1}} = \mathbf{F}_{n+1}\left[J^{-2/3}_{n+1}\mathrm{DEV}\left(2\frac{\partial \varphi^e}{\partial \hat{\mathbf{C}}_{n+1}}\right) + \frac{\partial U}{\partial J_{n+1}} J_{n+1}\mathbf{C}^{-1}_{n+1}\right] \tag{41}$$

where:

$$\frac{\partial \varphi^e}{\partial \hat{\mathbf{C}}_{n+1}} = \left(\sum_{j=1}^{3}\frac{\partial \varphi^e}{\partial c^{pr}_j}\frac{\partial c^{pr}_j}{\partial \hat{\mathbf{C}}^{pr}}\right)\frac{\partial \hat{\mathbf{C}}^{pr}}{\partial \hat{\mathbf{C}}_{n+1}} = \left(\mathbf{F}^p_n\right)^{-1}\left(\sum_{j=1}^{3}\frac{\partial \varphi^e}{\partial \epsilon^e_j}\frac{1}{2c^{pr}_j}\mathbf{E}^{pr}_j\right)\left(\mathbf{F}^p_n\right)^{-T}. \tag{42}$$

4 Material tensors

An important aspect form the numerical implementation point of view is the determination of the tangent matrix, consistent with the constitutive incremental update algorithm. The contribution to the tangent matrix from geometric terms is common to any hyperelastic model. Thus, we focus here on the expression of the second derivative of the present incremental material update. We use here the notation $\frac{d(\cdot)}{d\hat{\mathbf{C}}_{n+1}}$ as the total derivative of the argument with respect $\hat{\mathbf{C}}_{n+1}$. We define thus the tensor $\mathcal{C}$:

$$\mathcal{C} = \frac{d}{d\hat{\mathbf{C}}_{n+1}}\left(\frac{\partial\Psi}{\partial\hat{\mathbf{C}}_{n+1}}\right) = \frac{d}{d\hat{\mathbf{C}}_{n+1}}\left(\frac{\partial\varphi^e}{\partial\hat{\mathbf{C}}_{n+1}}\right) \tag{43}$$

Considering $\hat{\mathbf{C}}^{pr} = \left(\mathbf{F}^p_n\right)^{-T}\hat{\mathbf{C}}_{n+1}\left(\mathbf{F}^p_n\right)^{-1}$, calling $\mathbf{f}^{\mathbf{p}n} = \left(\mathbf{F}^p_n\right)^{-1}$ and dropping index $n+1$, we have:

$$\mathcal{C}_{ijkl} = \sum_{m,t,p,q=1}^{3} \mathbf{f}^{\mathbf{p}n}_{im}\mathbf{f}^{\mathbf{p}n}_{jt}\frac{d}{d\hat{\mathbf{C}}^{\mathbf{pr}}_{pq}}\left(\frac{\partial\varphi^e}{\partial\hat{\mathbf{C}}^{\mathbf{pr}}_{mt}}\right)\mathbf{f}^{\mathbf{p}n}_{kp}\mathbf{f}^{\mathbf{p}n}_{lq} = \mathcal{C}^{\varphi^e}_{klij} = \mathcal{C}^{\varphi^e}_{jikl}.$$

The critical point is the obtention of the derivatives of φ^e with respect to $\hat{\mathbf{C}}^{pr} = c^{pr}_j\mathbf{E}^{pr}_j$. In spectral coordinates this requires the computation of the following functions:

$$y_i = \frac{\partial\varphi^e}{\partial c^{pr}_i} = \frac{\partial\varphi^e}{\partial\epsilon^e_i}\frac{1}{2c^{pr}_i}, \tag{44}$$

$$y_{i,j} = \frac{d}{dc^{pr}_j}\left(\frac{\partial\varphi^e}{\partial\epsilon^e_i}\frac{1}{2c^{pr}_i}\right) = \frac{\partial^2\varphi^e}{\partial\epsilon^e_i\partial\epsilon^e_i}\frac{d\epsilon^e_i}{d\epsilon^{pr}_j}\frac{1}{4c^{pr}_i c^{pr}_j} - \frac{\partial\varphi^e}{\partial\epsilon^e_i}\frac{\delta_{ij}}{2\left(c^{pr}_i\right)^2}. \tag{45}$$

The terms $\frac{\partial\varphi^e}{\partial\epsilon^e_k}$ and $\frac{\partial^2\varphi^e}{\partial\epsilon^e_k\partial\epsilon^e_l}$ are straightforward. On other hand, the relation $\epsilon^e_k(\epsilon^{pr}_1, \epsilon^{pr}_2, \epsilon^{pr}_3)$ is defined by the derivation of the nonlinear system (37,40) which provides the terms $\frac{d\epsilon^e_i}{d\epsilon^{pr}_j}$.

5 Examples

In this section, some expressions for the potentials considered are stated in order to obtain different models.

The dissipative term ψ play an important role: it defines the threshold among the elastic and inelastic regions and its definition is non differentiable. Its expression also defines the rate dependence or independence of the plastic deformation. The Perzina law is achieved for

$$\psi(\dot{q}) = \begin{cases} \dfrac{mY_0\dot{q}_0}{m+1}\left(\dfrac{\dot{q}}{\dot{q}_0}\right)^{\frac{m+1}{m}} & \text{if } \dot{q} \geq 0 \\ +\infty \quad \text{if } \dot{q} < 0 \end{cases} \tag{46}$$

When $m \to \infty$ the potential becomes

$$\psi(\dot{q}) = \begin{cases} Y_0\dot{q} & \text{if } \dot{q} \geq 0 \\ +\infty & \text{if } \dot{q} < 0 \end{cases} \tag{47}$$

providing a rate independent plastic deformation.

A quadratic expression for the plastic potential express a linear hardening rule:

$$\varphi^p(q) = \Sigma_0 q + \frac{h}{2}\left(q\right)^2 \tag{48}$$

5.1 Hencky-based model

In this section we analyze the case when the elastic potential is based on quadratic form of the logarithmic strain tensor (Hencky-type potentials [8, 9]):

$$\varphi^e = \mu^e \sum_{j=1}^{3} \left(\epsilon_j^e\right)^2$$

The potentials ψ and φ^p take the forms (47) and (48) respectively. Thus,

$$\frac{\partial \varphi^e}{\partial \epsilon_j^e} = 2\mu^e \epsilon_j^e = 2\mu^e(\epsilon_j^{pr} - \Delta q \; q_j)$$

$$\frac{\partial \psi}{\partial \dot{q}} = \quad \text{if } \dot{q} \geq 0$$

$$\frac{\partial \varphi^p}{\partial \Delta q} = \Sigma_0 + hq_{n+1}$$

In this case, the conditions (37) take the particular form

$$r_i = -2\mu^e(\epsilon_i^{pr} - \Delta q \; q_i)\Delta q + \lambda + 2\beta q_i = 0, \quad i = 1, 2, 3 \tag{49}$$

$$r_4 = -\sum_{j=1}^{3} 2\mu^e(\epsilon_j^{pr} - \Delta q \; q_j)q_j + \Sigma_0 + hq_{n+1} + Y_0 = 0 \tag{50}$$

$$r_5 = \sum_{j=1}^{3} q_j = 0 \tag{51}$$

$$r_6 = \textstyle\sum_{j=1}^{3} q_j^2 = 3/2 \tag{52}$$

After some algebra , the following expressions may be obtained:

$$\Delta q = \frac{\sqrt{\frac{3}{2}} \, \|\mathbf{s}^{pr}\| - (Y_0 + \Sigma_0 + hq_n)}{3\mu^e + h} \quad \text{if } \Delta q \geq 0 \tag{53}$$

$$q_i = \frac{3\mu^e \epsilon_i^{pr}}{a\Delta q + b} \tag{54}$$

which is the usual expression for elastoplastic radial return von-Mises model. Finally, (54) allow the computation of $\epsilon_i^e = 2\mu^e(\epsilon_i^{pr} - \Delta q \; q_i)$ needed for the elastic potential.

5.2 Ogden-based model

In the previous case, the quadratic function of the logarithmic strains is particularly convenient to obtain an explicit expression for for the minimizing argument Δq_j^p. In spite of this advantage, it is well known that this type of hyperelastic potential do not fit very well other materials like polymers.

For that case, a more adequate choice may be the Ogden hyperelastic model ([8–10]) which has also the property of generalizing others models like neo-Hookean and Mooney-Rivlin:

$$\varphi^e = \sum_{i=1}^{3}\sum_{p=1}^{N}\frac{\mu_p^e}{\alpha_p}\left([\exp(\epsilon_i^e)]^{\alpha_p} - 1\right) \tag{55}$$

$$\frac{\partial\varphi^e}{\partial\epsilon_i^e} = \sum_{p=1}^{N}\mu_p^e\left[\exp(\epsilon_i)\right]^{\alpha_p} \tag{56}$$

In this case, the conditions (37) take the particular form

$$r_i = -\sum_{p=1}^{N}\mu_p^e\left[\exp(\epsilon_i)\right]^{\alpha_p}\Delta q + \lambda + 2\beta q_i = 0, \quad i = 1,2,3 \tag{57}$$

$$r_4 = -\sum_{p=1}^{N}\mu_p^e\left[\exp(\epsilon_i)\right]^{\alpha_p} q_j + \Sigma_0 + hq_{n+1} + Y_0 = 0 \tag{58}$$

$$r_5 = \sum_{j=1}^{3} q_j = 0 \tag{59}$$

$$r_6 = \textstyle\sum_{j=1}^{3} q_j^2 = 3/2 \tag{60}$$

whose solution provides $\Delta q, q_j, \lambda$ and β.

6 Numerical example. Uniaxial traction test

This simple example illustrates the behavior of this approach for the case of a traction test submitted to a constant strain rate. Two material models were tested: Hencky and Ogden. Both materials were considered incompressible through a convenient penalization value of K. The Ogden model used $N = 3$, i.e., $p = 1,2,3$. The corresponding parameters are listed in Table 1. For both cases, potentials (46) and (48) were used with parameters $Y_0 = 1$, $m = 0.8$, $\dot{q}_0 = 0.1$, $\Sigma_0 = 20$, $h = 20$. It is important to remark that this used values are merely illustrative, with no relation to a specific material. Both specimens were elongated up to $\lambda = 3$ ($\varepsilon = \ln\lambda = 1.0986$) with constant strain rates of 1, 0.5 and 0.1 and unloaded at the same strain rate. The results for the Hencky model are shown in Figure 1 where the expected dependence of the strain rate for the permanent deformation region is found. In Figure 2 the results for the Ogden model are shown. Classical Ogden-type elastic response is obtained, followed by a rate dependent plastic behavior.

7 Concluding remarks

A general set of hyperelastic-viscoplastic material models was proposed in this paper. This approach, imbedded within an unified variational approach of inelastic material models [1], is characterized by a spectral decomposition of viscoplastic strains and by generic hyperelastic and viscoplastic isotropic

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Table 1: Material parameters for cyclic shear test.

Potential	Ogden			Hencky
μ_i	-94.22	140.42	35.21	$\mu = 30.0$
α_i	3.0559	1.3328	3.8812	

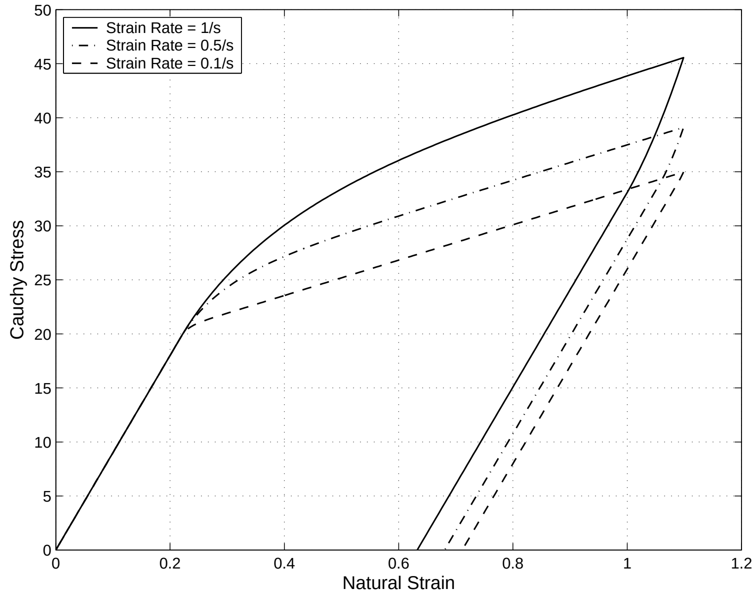

Figure 1: Uniaxial traction test. Hencky model.

potential functions. Different material models may be obtained just by changing the expression of these potentials and their derivatives. The stress and internal variables updates require (in the general case) the solution of a six nonlinear equation system to determine the eigenvalues of viscoplastic increments. This nonlinear system has an analytically invertible tangent matrix which provides computationally inexpensive Newton solutions and the symmetric material tensor for the tangent matrix calculations.

All these characteristics will allow us to combine this formulation with that already obtained for nonlinear viscoelastic behavior [4] in order to propose consistent models for polymeric materials subject to combined viscoelastic and viscoplastic strains, which is the subject of future works.

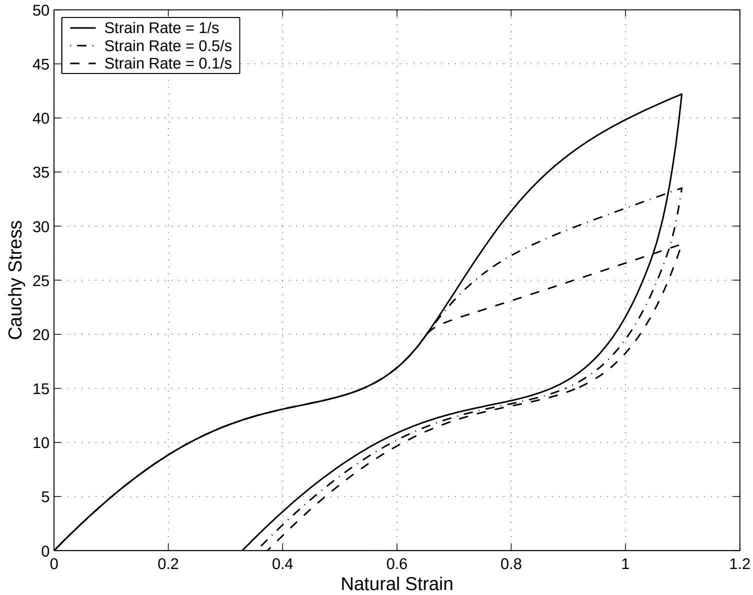

Figure 2: Uniaxial traction test. Ogden model.

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