LINEAR PROGRAMMING FOR THE OPTIMAL CONTROL OF A ONE-DEGREE-OF-FREEDOM OVERHEAD CRANE SYSTEM

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Abstract. *This work discusses the use of the Linear Programming as an alternative for solving optimal control problems of time variant linear dynamic systems described in the discrete time state space where the elements of the control vector are the design free variables. Since the state vector at any sampling time can be written as a linear combination of control vector and initial state vector, a standard Linear Programming problem results. The cost function can be adapted to the particular optimization problem to be solved. One possible goal in optimal control problems is minimizing the sum of the absolute values of the control. Another possibility is to maximize the mean speed to indirectly solve a minimum time problem. These and other cost functions are easily formulated in the standard Linear Programming framework, i.e., as a linear combination of the control vector. A cart-pendulum lab system was considered to illustrate this alternative approach. The minimum control effort has been addressed explicitly in the cost function and the minimum time problem has been achieved indirectly by solving a series of minimum-effort problems with decreasing final time until feasibility could no longer be achieved. Restrictions of null angle and angular velocity at the extremes were incorporate in the design specification as well as other physical constraints. Results obtained illustrate that, in fact, the technique is simple, powerful and conclusive as the numerical solution of Linear Programming problems guarantees convergence to a global optimum provided there is a feasible solution.*

Keywords: *Linear Programming, Optimal Control, Time Control Low, Anti-oscillatory Control.*

1. INTRODUCTION

The problem of optimal control of overhead cranes has been receiving attention from the scientific community because of undeniable practical relevance. Several approaches Cheng and Chen (1996), Auerning (1987), Cruz *et al.* (2009), Chen *et al.* (2007), Nassif *et al.* (2010), Garrido *et al.* (2008) and Lee (2005) have been proposed to address the problem of minimum time and differ, for example, with respect to the model utilized, the constraints imposed and the performance index optimized.

In load transfer operations by a crane, a major problem is optimizing the movement from origin to destination, satisfying constraints related to the equipment such as the kinematics of the movement. The carrier may be considered primarily as a cart-pendulum system, where the length of the pendulum is usually variable, representing the lifting.

One difficulty in solving optimal control problems such as the optimal load transfer by a crane, is the necessity of solving a two-point boundary problem with constraints on the initial and final states. For example, a linear quadratic regulator generates an optimal control law but the final state can not be pre-determined. This limitation is discussed e.g. by Bemporad *et al.* (2001) and Blanchini (1994).

The objective function can be adapted to the particular optimization problem to be solved. A possible objective in optimal control problems is minimizing the sum of the absolute control value. Another possibility is maximizing the average speed to indirectly solve the minimum time problem. These and other objective functions are easily written in the standard form of a linear programming problem, i.e., as a linear combination of the control vector.

This paper discusses the use of Linear Programming as an alternative for solving this type of optimal control problem, assuming that the system dynamics are linear, time variant and expressed in the discrete time domain. In this scenario the discrete values of the control vector are the free design variables and the state vector at any sampling time may be written as a linear combination of the control vector and the initial conditions. This results in the standard structure of a Linear Programming (LP) problem.

A cart-pendulum lab system was considered to illustrate the approach. The movement cycle begins and ends at given positions and the load is at rest in both, the beginning and end of cycle. Moreover, in the application considered here, the lifting takes place at the beginning and end of the cycle with the cart stationary, i.e., there is no lift during cart movement. A more efficient strategy is obviously to perform lifting and cart translation simultaneously, but this work intended to show the potential of the methodology, applying it to a lab-scale system that has no motorized lifting.

2. METHOD

The optimal control problem of a dynamical linear system in the discrete time state space can be written in the form of a standard LP problem.

2.1 Linear Dynamics as LP Constraints

Consider the time-variant dynamic system in discrete time with a constant sampling time T and described in the state space.

$$x_{(k+1)} = A_{(k)}x_{(k)} + B_{(k)}u_{(k)} \ , \qquad k = 0, 1, 2... \tag{1}$$

For any sampling time nT we can write

$$\begin{array}{rcl} x_{(n)} & = & \left(A_{(n-1)}.A_{(n-2)} \cdots A_{(2)}.A_{(1)}.A_{(0)}\right) x_{(0)}+ \\ & + & \left(A_{(n-1)}.A_{(n-2)} \cdots A_{(2)}.A_{(1)}\right) B_{(0)}u_{(0)}+ \\ & + & \left(A_{(n-1)}.A_{(n-2)} \cdots A_{(2)}\right) B_{(1)}u_{(1)} + \cdots \\ \cdots + & & \left(A_{(n-1)}\right) B_{(n-2)}u_{(n-2)}+ \\ & + & B_{(n-1)}u_{(n-1)} \end{array}$$
$$x_{(n)} = Fx_{(0)} + GU \tag{2}$$

where

$$\begin{array}{l} F = A^{(n)} \\ G = \left[A^{(n-1)}A^{(n-2)} \cdots A^{(1)}A^{(0)}\right] .diag\left(\left[B_{(0)}B_{(1)} \cdots B_{(n-2)}B_{(n-1)}\right]\right) \\ U = \left[u_{(0)}u_{(1)} \cdots u_{(n-2)}u_{(n-1)}\right]^T \end{array} \tag{3}$$

where the following definitions apply

$$\begin{array}{rcl} A^{(n)} & \triangleq & A_{(n-1)}.A_{(n-2)} \ldots A_{(2)}.A_{(1)}.A_{(0)} \\ A^{(n-1)} & \triangleq & A_{(n-1)}.A_{(n-2)} \ldots A_{(2)}.A_{(1)} \\ A^{(n-2)} & \triangleq & A_{(n-1)}.A_{(n-2)} \ldots A_{(2)} \\ & \vdots & \\ A^{(2)} & \triangleq & A_{(n-1)}.A_{(n-2)} \\ A^{(1)} & \triangleq & A_{(n-1)} \\ A^{(0)} & \triangleq & I \end{array} \tag{4}$$

If the system is time invariant, the formulae above still hold but a standard matrix power must be used instead with the single system matrix $A, i.e., A^j = A.A \ldots A.A$.
Note that it is possible to represent the dynamic model in the form of constraints in the form of $\mathbf{A}X = \mathbf{B}$, which may include the initial conditions $x_{(0)}$ and the final conditions $x_{(n)}$ at the nT instant

$$\begin{array}{l} GU = x_{(n)} - Fx_{(0)} \\ \mathbf{A}X = \mathbf{B} \end{array} \tag{5}$$

where

$$\begin{array}{l} \mathbf{A} = G \\ X = U \\ \mathbf{B} = x_{(n)} - F\, x_{(0)} \end{array} \tag{6}$$

Note that the system dynamics was represented by linear constraints on the control vector. For state constraints in the form of inequalities of type $x(m) \geq \eta$, we can write:

$$\begin{array}{l} x_{(m)} = F_1\, x_{(0)} + G_1\, U_1 \\ F_1\, x_{(0)} + G_1\, U_1 \geq \eta \\ G_1 U_1 \geq \eta - F_1\, x_{(0)} \\ A_1 X_1 \geq B_1 \end{array} \tag{7}$$

To completely define the LP problem it remains to define a cost function which is linear on states and control. The choice of cost function depends on the optimization problem to be solved. For example, one can maximize the average speed or minimize the fuel consumption to travel a given distance.

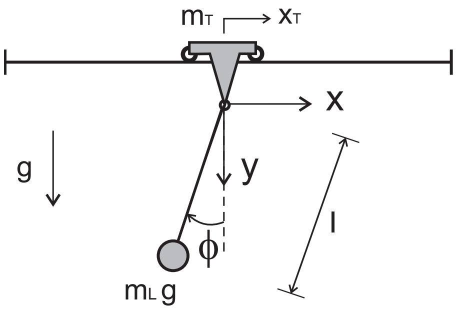

Figure 1. Cart-pendulum scheme.

2.2 Mechanical Model

A scheme of a cart-pendulum system is shown in Fig. 1
where m_T = is the cart mass, m_L = the load mass, x_T = the cart position and ϕ = the load angle.
The equations of motion describing the dynamics of the cart-pendulum model were derived using the Newton-Euler formalism as described in Schiehlen (1997) yielding

$$-\ddot{x}_T \cos\phi + \ddot{\phi} l = -g\,sen\,\phi - \frac{\bar{c}\dot{\phi}}{m_L}, \tag{8}$$

where g is the gravity acceleration and $\bar{c}$ a damping constant.
In handling anti-oscillatory problems, it is expected that, the maximum oscillation angle be small ($< 10°$). This condition leads to the approximations, $sen\phi \approx \phi$ and $\cos\phi \approx 1$. These approximations simplify the equations of motion to

$$-\ddot{x}_T + \ddot{\phi} l = -g\phi - \frac{\bar{c}\dot{\phi}}{m_L}. \tag{9}$$

2.3 Optimal Control

The objective function chosen here is the control effort in the form of the sum of the absolute control values $|u_0| + |u_1| + \cdots + |u_{n-1}|$. The LP admits a single objective function but we used an approach to minimize both the control effort and the time of operation. The minimum time is obtained by solving a series of minimal-effort LP problems with decreasing final time until contraints can no longer be satisfied. From equation (9) and defining $c = \frac{\bar{c}}{m_L}$ and $u = \ddot{x}_T$ as the control variable, we get

$$u = l\ddot{\phi} + c\dot{\phi} + g\phi. \tag{10}$$

In order to impose boundary conditions on the initial and final position and speed of the cart we create two new states x_3 and x_4 , which describe the position and speed of the cart yielding the following state equations

$$\begin{array}{llll} x_1 = & \phi & \dot{x}_1 = & x_2 \\ x_2 = & \dot{\phi} & \dot{x}_2 = & \frac{u - gx_1 - cx_2}{l} \\ x_3 = & x_T & \dot{x}_3 = & x_4 \\ x_4 = & \dot{x}_T & \dot{x}_4 = & u \end{array} \tag{11}$$

With the initial condition $x_{(t_0)}$ and final condition $x_{(t_f)}$, the boundary conditions are

$$\begin{array}{llll} x_1(t_0) = & 0 & x_1(t_f) = & 0 \\ x_2(t_0) = & 0 & x_2(t_f) = & 0 \\ x_3(t_0) = & 0 & x_3(t_f) = & 0.25m \\ x_4(t_0) = & 0 & x_4(t_f) = & 0 \end{array} \tag{12}$$

Based on the limitations of the real plant we imposed

$$\begin{array}{l} x_{4,\max} = 2\,m/s \\ u_{\max} = 0.9\,m/s^2 \end{array} \tag{13}$$

Rewriting Eq. 11

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l} & -\frac{c}{l} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{l} \\ 0 \\ 1 \end{bmatrix} u \tag{14}$$

2.4 Testing Apparatus

In order to validate the numerical results and implement the proposed control law, we used a lab equipment that allows inverted pendulum or simple pendulum experiments. The schematic diagram of the equipment and its photo is shown in Fig. 2.

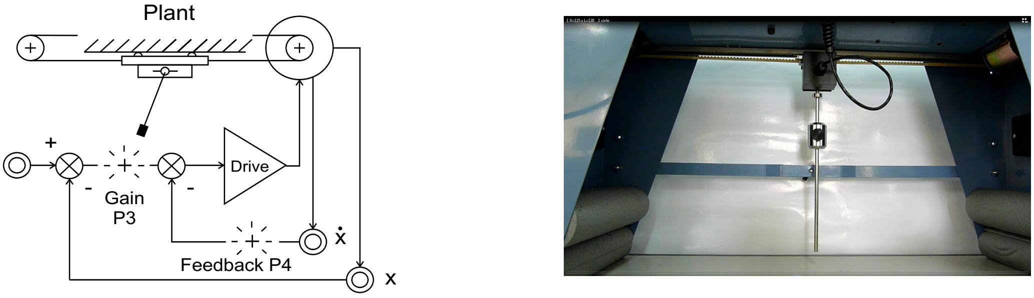

Figure 2. Schematic diagram of the equipment and its photo.

The pendulum consist of a $0.215kg$ mass connected to the cart by a rod. The mass can be fixed on the rod at different distances from the cart. The cart driver has a position control system with a speed compensation. In this loop there is access to the reference signal and the cart position and speed signals. There is also access to the pendulum angular position signal (not shown in the figure). Since the implementation details of this internal control system are not documented we chose to consider this as part of the cart sub-system and its transfer function was experimentally identified via step excitation and we selected a second order transfer function

$$P_{(s)} = \frac{K_n \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and found, $K_n = 0.025$, $\omega_n = 31rd/s$, $\zeta = 0.35$. We also determined the gain values for the speed and cart position sensors, $Ktaco = 4V/m/s$ and $Kpot = 40V/m$, respectively

3. RESULTS

3.1 Simulation

The state equation was discretized with a sampling time $T = 15\,ms$ and the optimal control vector, was obtained for N sampling periods.
The rod length was set to $l = 0.24m$ and after simple binary search, we found $N = 74$ as the minimum number of sampling periods that still leads to a feasible solution. The results are presented in Fig. 3.
From the results we concluded that the maximum angle os oscillation was $\phi = 10°$ and therefore the max oscillation amplitude constraint was satisfied. The minimal response time was $t = T(N) = 0.015(74) = 1.11\,s$.
Within the Matlab environment, the SimMechanics, allows studying a nonlinear mechanical system behavior, without the necessity of obtaining the corresponding differential equations. Figure 4 shows the model used in SimMechanics, as well as the nonlinear (8) and linear (9) models, all within the Simulink environment. The optimal controle solution $u_{(t)}$ was used as input to each of these models. The results of the integration of each of these models are presented in Fig. 5 and Fig. 6 for the angle and angular velocity, respectively.

3.2 Experimental Results

The optimal control problem was formulated considering the cart acceleration as the manipulated variable. So we need a way to impose the cart kinematics in the presence of modeling errors and disturbances. This was achieved through a state feedback control system with a feed-forward action for the acceleration as illustrated in the block diagram of Fig. 7. Note that, although there is a control loop to assure cart kinematics follows the prescribed trajectories, the optimal control itself is open loop, since there is no feedback for the angle trajectory. From $P_{(s)} = \frac{x_{(s)}}{Vdrive_{(s)}}$ and imposing $Vdrive_{(t)} = \frac{1}{(K_n\omega_n^2)}\left(2\zeta\omega_n\dot{x}_{(t)} + \omega_n^2 x_{(t)} + v_{(t)}\right)$, where $v_{(t)}$ is the new manipulated variable, we form a plant model $\bar{P}_{(s)}$ as a double integrator, or $v_{(t)} = \ddot{x}_{(t)}$.
The control system shown in Fig. 7 has three references that are consistent with each other - position, velocity and acceleration of the cart. The position and velocity are states and, therefore, their references apply to the loop, while the desired acceleration enters as a feed-forward action through a block that contains the inverse plant model. The gains

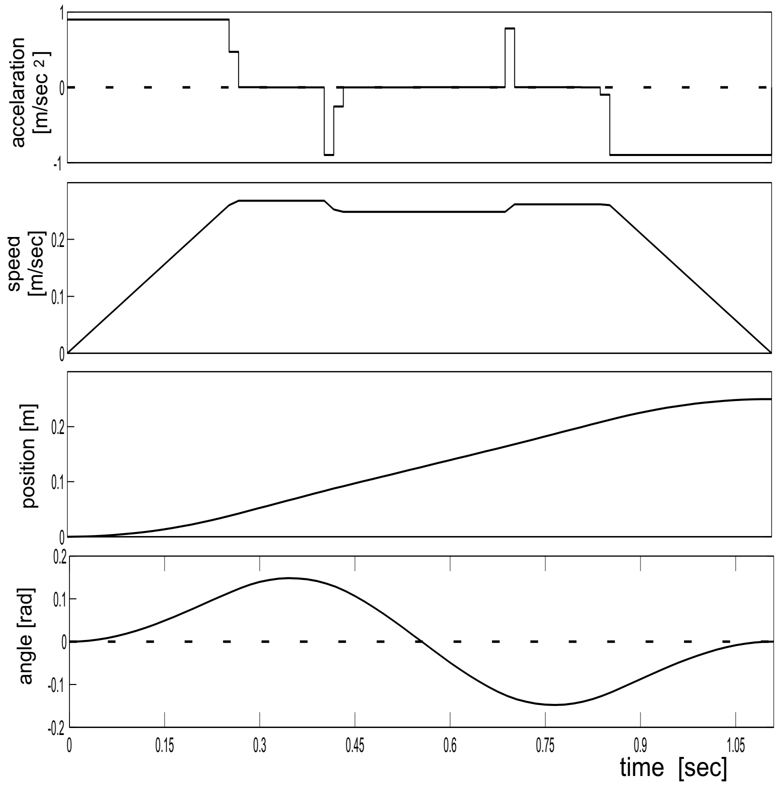

Figure 3. Simulation with 74 sampling periods.

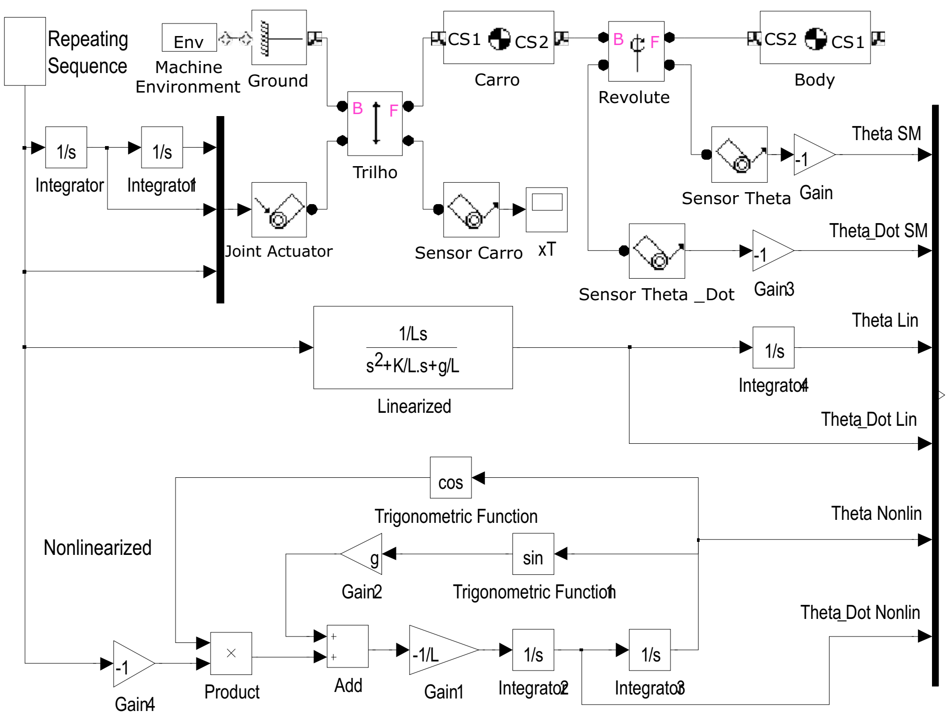

Figure 4. Block diagram of the model.

$K_1 = 30992$ and $K_2 = 1223$ are the gains from state feedback and were tuned interactively in order to obtain good tracking of the reference signals and for disturbances rejection. Because this control loop can not be perfect, the optimal control problem will contain errors because the cart acceleration will never be imposed with a infinite precision. The control system of Fig. 7, along with the trajectory generation and data aquisition, were implemented in Simulink in real time through a data acquisition board and Matlab Windows Target. Figs. 8, 9 and 10 show a comparison between the computers and measured position, speed and pendulum angle, respectively. Thin lines were used for generated signals

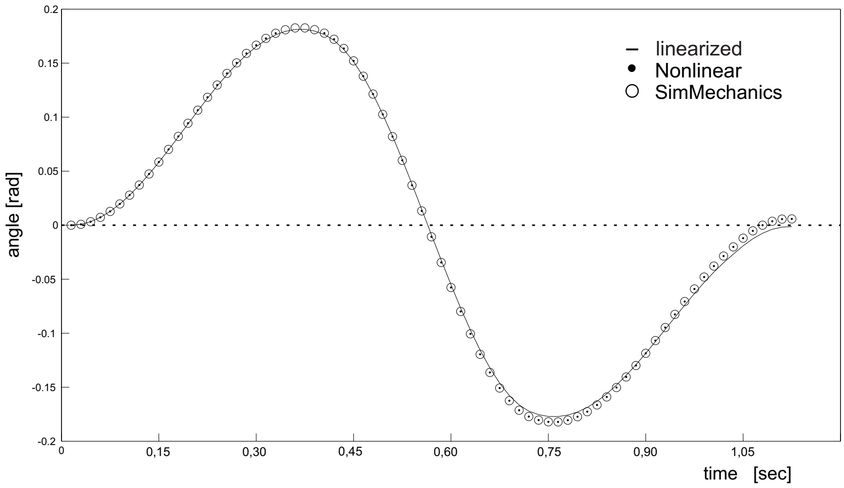

Figure 5. Comparison of the pendulum angle among SimMechanics, linear and nonlinear equations.

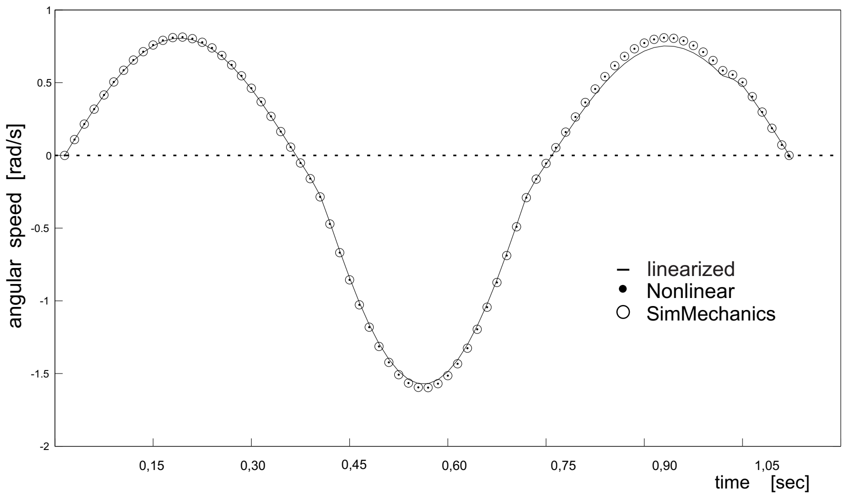

Figure 6. Comparison of the angular speed among SimMechanics, linear and nonlinear equations

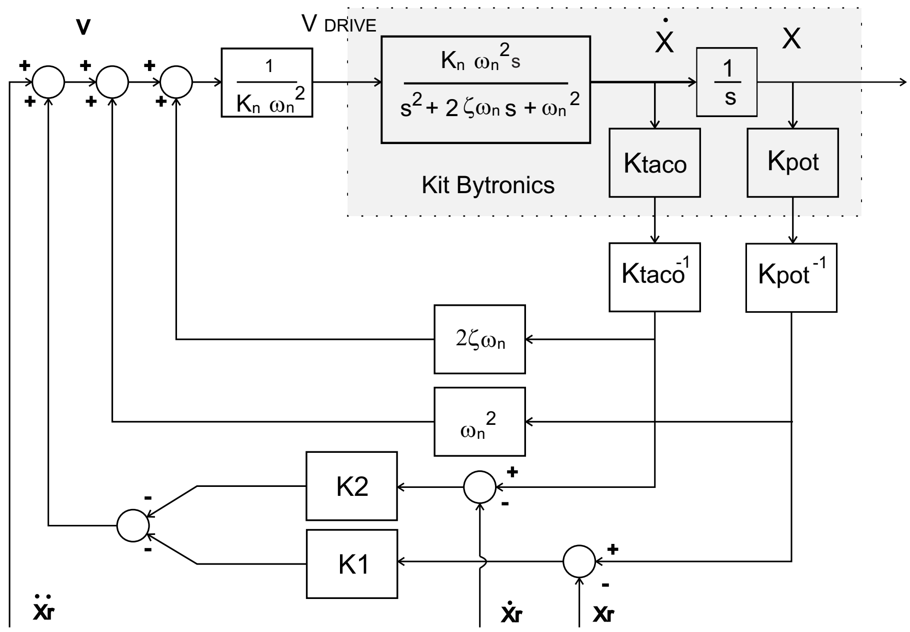

Figure 7. Cart control system.

and thick lines for the measured signals. Since the experimental apparatus does not have an acceleration sensor for the cart, the real acceleration is not compared with the optimum computed acceleration in none of the following cases.

The optimal trajectory of the cart-pendulum system is illustrated in Fig. 11, which shows a sequence of snap shots, at

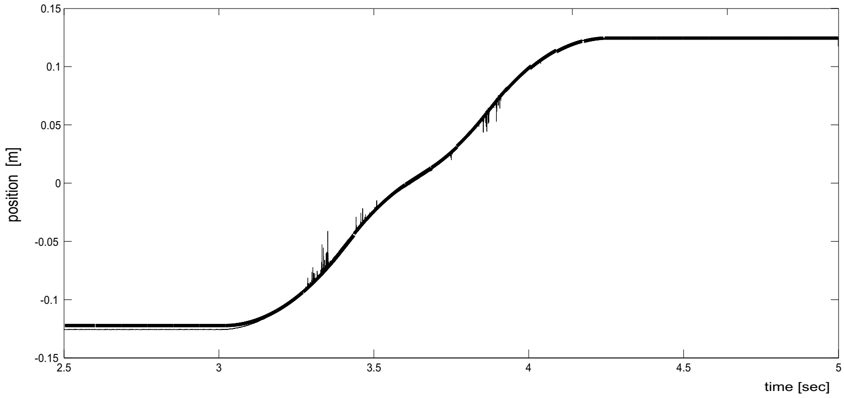

Figure 8. Reference signal and actual cart position.

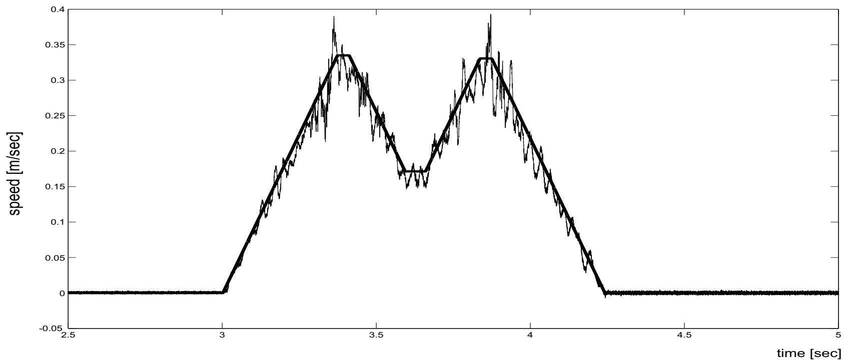

Figure 9. Reference signal and actual cart speed.

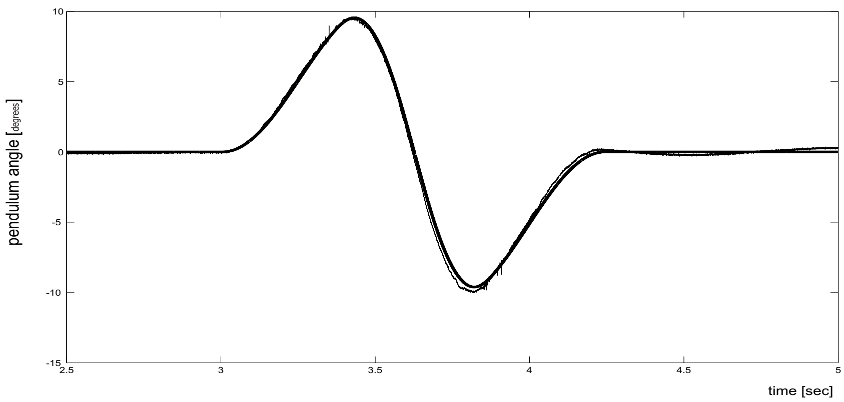

Figure 10. Reference signal and actual pendulum angle.

intervals of 0.1 of total time.

3.3 Sensitivity

A change imposed on the plant model, such as imposing a new mass position, modifies its response. Using the same optimal control vector obtained for $l = 0.24\,m$ we evaluate the sensitity of the system response to variations in rod length l. Starting from $l = 0.24m$ changes of length on $\pm 15\,mm$ and $\pm 30\,mm$ in the mass position were performed and the results compared to the optimal trajectory. Figs. 12 to 15 show the effect of changing the rod length on pendulum angle trajectory.

The results suggest that this optimal control problem is sensitive to modeling errors. That is, if the plant model is not well

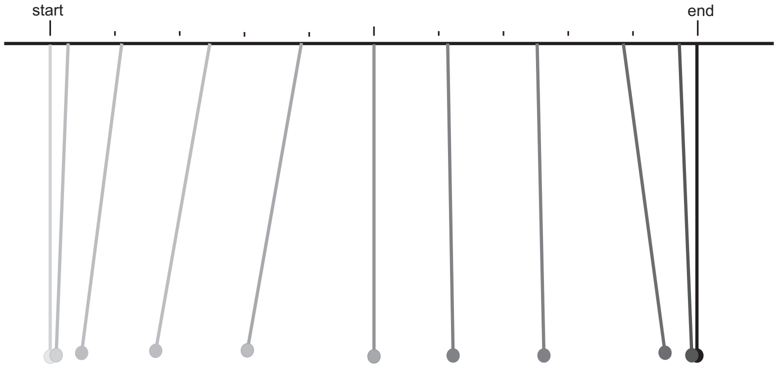

Figure 11. The optimal trajectory snap shots for $l = 0.24m$.

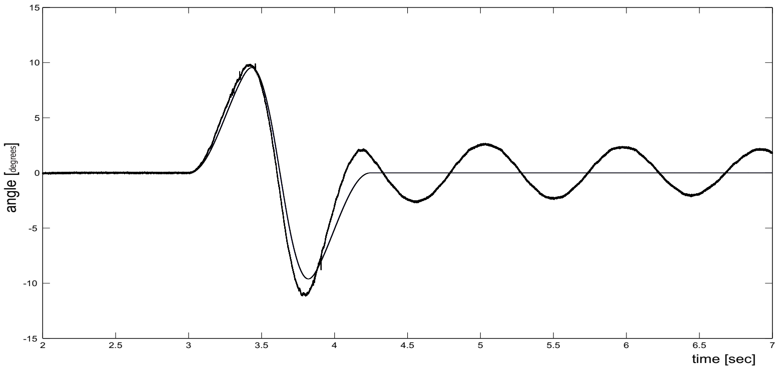

Figure 12. Optimal response for $l = 0.24m$ and measured response $l = 0.21m$.

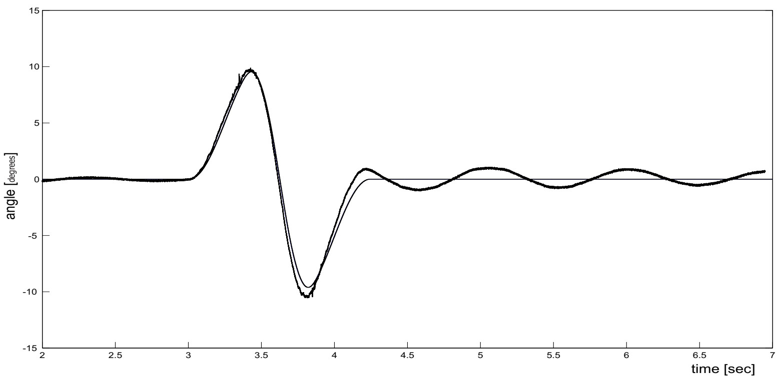

Figure 13. Optimal response for $l = 0.24m$ and measured response $l = 0.225m$.

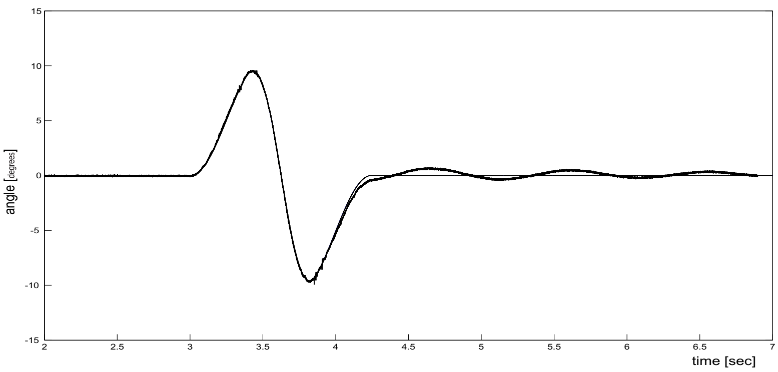

Figure 14. Optimal response for $l = 0.24m$ and measured response $l = 0.255m$.

known the optimal trajectory will not be assured. This sensitivity is due to the system is running with no feedback on angle.

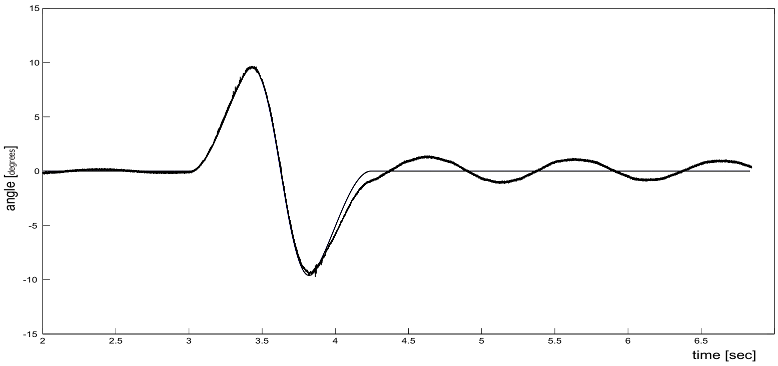

Figure 15. Optimal response for $l = 0.24m$ and measured response $l = 0.27m$.

4. DISCUSSION AND CONCLUSIONS

This paper discusses the use of Linear Programming for the solution of optimal control problems for linear systems whose dynamics is expressed in the discrete time domain. The method is general enough to accommodate time-variant, linear systems and a variety of boundary conditions. In particular, a strategy to solve simultaneously the minimal effort and the minimal time problems was proposed and applied to a crane system. Numerical as well as experimental investigations were performed.

The numerical results show that, for the particular problem formulation in this paper, a linear model is a good approximation of the nonlinear crane system. Because the optimal control problem of crane motion was written in a Linear Programming (LP) framework, the problem could be solved very efficiently. Furthermore, in opposition to Nonlinear Programming (NLP), LP guarantees that the global optimum is obtained. The minimal effort and minimum time optimal control strategy satisfied all the imposed boundary conditions and constraints.

Remarkably, the experimental results performed with a cart-pendulum lab equipment, agreed very well with the computational predictions. In fact, the optimal control vector determined by solving the optimal control problem was applied to the real cart-pendulum plant and the predicted behavior was reproduced within a small tolerance. This observation indicates both, that the linear model utilized is a good approximation of the plant behavior and that the optimal control vector computed indeed guarantees that the plant satisfies the imposed constraints, namely, resting condition at the beginning and end of the trajectory and maximal amplitude of pendulum oscillation.

The sensitivity of the response to model uncertainties was investigated by varying the distance between the mass and the cart. This analysis revealed a substantial sensitivity suggesting that an open-loop implementation of the optimal control would be prone to severe performance deterioration in the presence of model uncertainties and errors. A closed-loop control strategy could be devised to guarantee fulfillment of boundary conditions and trajectory constraints in the presence of disturbances and model uncertainties.

A different alternative to remediable the sensitivity to model errors and uncertainties is using a model matching closed-loop control strategy Jonckheere and R. (1999). This approach makes the system transfer function approach the plant nominal model even in the presence of modeling errors, i.e. using a robust model matching control structure. Under certain conditions, this strategy would have the desirable property of preserving the optimality of the open-loop optimal control solution. The general control structure of a model-matching control strategy is represented in Fig. 16 .

In the figure, $N_{(s)}$, $G_{(s)}$ and $K_{(s)}$ are the transfer functions of the reference model, plant and controller, respectively.

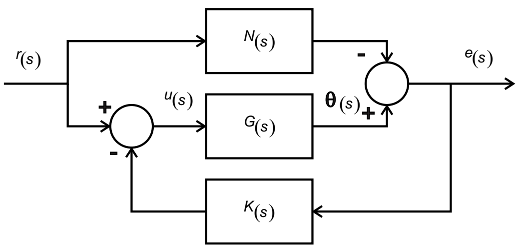

Figure 16. Model tracking Structure.

Note that $e_{(s)}$ is used for the feedback. Consequently, the reference model $N_{(s)}$ is an explicit part of the control system and can be changed even after the project. The model matching problem can be stated as searching for $K_{(s)}$ such that the magnitude of the transfer function $r_{(j\omega)}$ to $e_{(j\omega)}$ be below a certain prescribed value at a given frequency range and in the presence of modeling errors.

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