ABCM Symposium Series in Mechatronics - Vol. 1 - pp.243-253
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SPATIAL H_∞ CONTROL OF A FLEXIBLE BEAM CONTAINING PIEZOELECTRIC SENSORS AND ACTUATORS

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Abstract. *The aim of this paper is to design and evaluate the performance of a feedback H_∞ controller to suppress vibration of a flexible cantilever beam provided with strain actuator and sensor. The infinite-dimensional model is used to represent the flexible beam structure and it is controlled using a collocated piezoelectric PZT actuator-PVDF sensor pair bonded to the beam surface. For control design purposes, the infinite-dimensional model is simplified by removing higher modes which lie out of the bandwidth of interest. The truncation can considerably perturb the zeros of the truncated model. For outline this problem, this paper applies the method of minimizing the effect of removed higher order modes on the low-frequency dynamics of the truncated model by adding a zero frequency term to the low order model of the system. This corrected model is then used to design the H_∞ robust controller in a way that minimizes the effect of disturbances over the entire beam. The recent approach for the control of flexible structures - spatial H_∞ control framework - is used to construct the controller that guarantees a level of disturbance rejection over the entire structure. The final process of this approach is reduced to an ordinary H_∞ control problem that, in turn, is solved using standard software (Robust Matlab® Toolbox). The final controller is numerically implemented and the results illustrate the robust control performance of a cantilever beam type structure.*

***Keywords**. Spatial H_∞, control framework, Flexible beam, Piezoelectric elements, Vibration control.*

1. Introduction

A single piezoelectric element called a self-sensing actuator combines actuator and sensing capabilities for collocated control (Dosh et al, 1992). The use of piezoelectric actuator/sensor pairs has been widely used in many vibration control applications of flexible structures (Dimitriadis et al, 1991; Fuller et al, 1996).

Piezoelectricity was discovered in 1880 by the French scientists Pierre and Paul-Jacques Curie. The piezoelectric effect is observed in many crystalline materials, which strain when exposed to a voltage and produce a voltage when strained. In other words, these materials are capable of transforming mechanical energy into electrical energy and vice versa. Piezoelectric actuators and sensors are bonded to flexible structures such as beams and plates, forming the so-called smart structures.

Different methodologies have been used for modeling smart structures, however the most promising technique is the finite element method, as first presented for piezo-mechanical systems by Allik and Hughes (1970). Later this formulation was extended for different structural elements. Crawley and de Luis (1987) studied the modeling of one-dimensional piezoelectric patches embedded into the body of beams and formulated the moment generated by a voltage applied to the piezoceramics. Tzou and Tzeng (1990) presented a piezoelectric finite element approach aiming at applications devoted to distributed dynamic measurement and control of advanced structures. The assumed modes approach or infinite-dimensional model has been extensively used throughout the literature to model the dynamics of distributed systems such as flexible beams and plates (Meirovitch, 1997). This approach is used in this paper for modeling the flexible beam.

In control design problems, one is often interested in designing a controller for a particular frequency range. In such situations, it is common practice to remove the modes that correspond to frequencies that lie out of the bandwidth of interest and keep only the modes that directly contribute to the low frequency dynamics of the system. Moheimani (1999) suggested that the effect of higher frequency modes on the low frequency dynamics of the system can be captured by adding a *zero frequency term* to the truncated model to account for the compliance of the ignored modes. Such model can then be used in designing *Spatial Controllers* as noted in Halim and Moheimani (2002).

The concept of *Spatial Control* is concerned with using the spatial information embedded in the dynamical models of structures. If a controller is designed with a view to minimizing structural vibrations at a limited number of points, it could have negative effects on the vibration profile of the rest of the structure. In this paper, a recent design methodology (Hallim and Moheimani, 2002), the so-called *Spatial H_∞ Control* approach is presented. The idea behind this technique is to first formulate the problem as a spatial H_∞ minimization problem and then design a controller that minimizes cost functions similar to the ones arising in the standard H_∞ control problems. However, in this design, the elastic deflection or the total energy of the *entire structure* is to be minimized. The resultant *spatial* controller is of the same dimensions as that of the dynamical system.

To demonstrate the proposed controller, an *SISO* (*S*ingle *I*nput, *S*ingle *O*utput) *Spatial H_∞* controller is designed for a piezoelectric laminate cantilever beam to suppress the vibration of the first three five modes of the structure.

2. Electro-mechanical model of a beam containing piezoelectric sensors and actuators

In this section, a model for a piezoelectric laminate beam with an actuator-sensor pair is obtained by using modal analysis techniques.

Figure 1 (a) shows a flexible structure with an attached piezoelectric actuator-sensor pair. The piezoelectric patch on the upper side of the beam is used as an actuator, while the film on the bottom side serves as a sensor. The voltage that is applied to the actuator patch is represented by $V_a(t)$ and the voltage that is induced to the sensor film is represented by $V_s(t)$ (see Fig. 1 - b).

Consider a homogeneous Euler-Bernoulli beam (dimensions: L_b, b_b, h_b) as shown in Fig. 1 (a)-(b). The piezoelectric actuator and sensor have dimensions of $(x_{a2} - x_{a1}) \times b_a \times h_a$ and $(x_{s2} - x_{s1}) \times b_s \times h_s$, respectively (see Fig. 1 - b). The beam transverse deflection at point x and time t is denoted by $w(x,t)$, assuming the beam as a one-dimensional system.

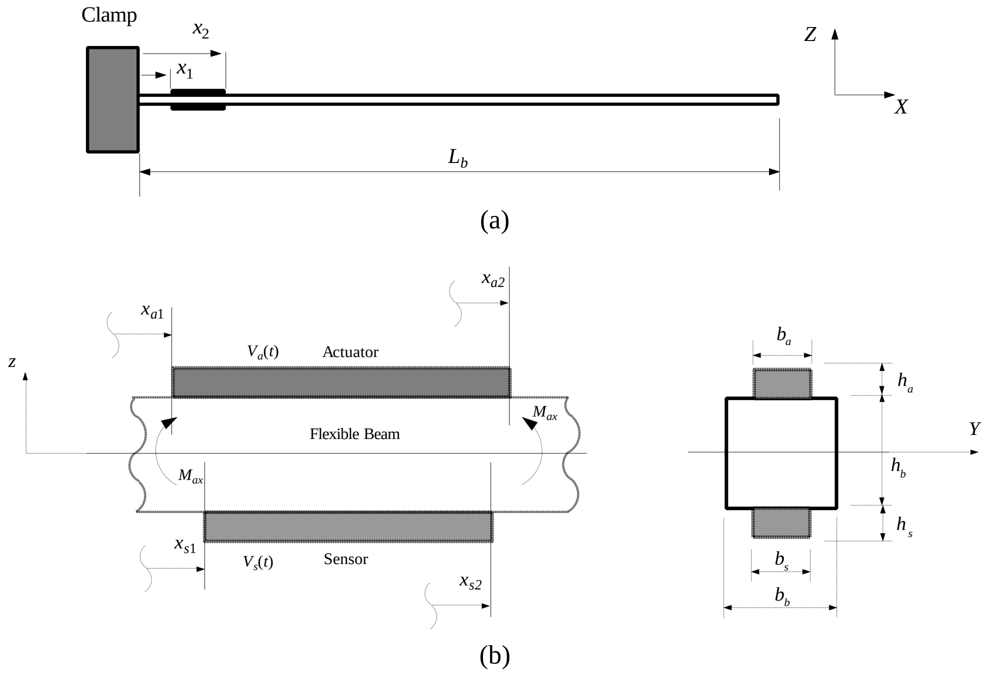

Figure 1. (a) Flexible cantilever beam containing piezo-sensor and actuator and (b) Geometrical properties of the piezoelectric elements.

A model of the structure is obtained through a modal analysis procedure. This approach finds a solution to the partial differential equation which describes the dynamics of the flexible structure:

$$r_b A_b \frac{\partial^2 w}{\partial t^2} + E_b I_b \frac{\partial^4 w}{\partial x^4} = q(x,t) \tag{1}$$

where, r_b, A_b, E_b and I_b represent density, cross-sectional area, Young's modulus of elasticity, and moment of inertia about the neutral axis of the beam, respectively.

2.1. Actuator Dynamics

The total distributed load $q(x,t)$ generated by the piezoelectric actuator when it is deformed is given by:

$$q(x,t) = M_{ax} \frac{\partial^2 R(x)}{\partial x^2} \tag{2}$$

where R is the generalized location function expressed as:

$$R(x) = H(x - x_1) - H(x - x_2) \tag{2.1}$$

and the H is the Heaviside function (H) given by (Soutas and Inman, 1999):

$$H(z - z_0) = \begin{cases} 1, \; se \;\; z \geq z_0 \\ 0, \; se \;\; z < z_0 \end{cases} \tag{2.2}$$

The bending moment acting on the beam (denoted by M_{ax}) is given by (Baz and Poh, 1988):

$$M_{ax} = \int_{h_b/2}^{h_b/2+h_a} s_{ax}(x,t) b_a x dx \tag{3}$$

where:

$$s_{ax} = E_a e_{ax} \tag{3.1}$$

and

$$e_{ax} = \frac{d_{31} V_a(t)}{h_a} \tag{3.2}$$

where d_{31} is the electric charge constant, E_a is the Young's modulus and h_a is the thickness of the PZT patch as shown in Fig. 1 (b).

The evaluation of Eq. (3) through Eqs. (3.1) and (3.2) gives the following expression:

$$M_{ax} = C_a V_a(t) \tag{4}$$

where C_a is a constant which depends on the geometry of the composite system:

$$C_a = \frac{1}{2} E_a d_{31} b_a (h_b + h_a) \tag{5}$$

The boundary conditions for the cantilever beam in Fig. 1 (a) are:

$$\begin{aligned} & w(0,t) = 0 \\ & E_b I_b \frac{\partial w(0,t)}{\partial x} = 0 \\ & E_b I_b \frac{\partial^2 w(L_b,t)}{\partial x^2} = 0 \\ & E_b I_b \frac{\partial^3 w(L_b,t)}{\partial x^3} = 0 \end{aligned} \tag{6}$$

To use the assumed mode technique, the function $w(x,t)$ is expanded as an infinite series as follows (Meirovitch, 1997):

$$w(x,t) = \sum_{i=1}^{\infty} f_i(x) h_i(t) \tag{7}$$

where $f_i(x)$ are the eigenfunctions satisfying the ordinary differential equations resulting from the substitution of Eq. (7) into Eq. (1) and (6), and $h_i(t)$ are the temporal functions.

The mode shapes for clamped-free boundary conditions, Eq. (6), are assumed to be expressed as (Meirovitch, 1997):

$$f_i(x) = C_i [sin(b_i x) - sinh(b_i x) + a_i (\cos(b_i x) - \cosh(b_i x))] \tag{8}$$

where the following equations define a_i and b_i, respectively:

$$a_i = \frac{\cos(b_i L_b) + \cosh(b_i L_b)}{sin(b_i L_b) - sinh(b_i L_b)} \tag{8.1}$$

$$1 + \cos(b_i L_b) \cos(b_i L_b) = 0 \tag{8.2}$$

and the constant C_i is determined by the condition expressed by the following expression:

$$\int_0^{L_b} f_i^2(x) dx = 1 \tag{9}$$

Substituting Eqs. (7) and (2) into (1) yields:

$$\sum_{i=1}^{\infty} \left[r_b A_b f_i(x) \ddot{h}_i(t) + E_b I_b f_i''''(x) h_i(t) \right] = M_{ax} \frac{\partial^2 R}{\partial x^2} \tag{10}$$

where (.) denotes time derivatives and (‘) represents space derivatives.

Pre-multiplying Eq. (10) by $\int_0^{L_b} f_i(x)dx$, the *i*-th equation, will be given by:

$$\left(r_b A_b \int_0^{L_b} f_i^2(x)dx \right)\ddot{h}_i(t)+\left(E_b I_b \int_0^{L_b} f_i(x)f_i^{''''}(x)dx \right)h_i(t)=M_{ax}\frac{\partial^2 R}{\partial x^2}\int_0^{L_b} f_i(x)dx \tag{11}$$

Substituting Eqs. (9) and (2.1) in Eq. (11), it is possible to obtain:

$$\ddot{h}_i(t)+w_{ni}^2 h_i(t)=k_a\left[f_i^{'}(x_{a2})-f_i^{'}(x_{a1})\right]V_a(t) \tag{12}$$

where:

$$w_{ni}^2=\frac{E_b I_b}{r_b A_b}\int_0^{L_b} f_i(x)f_i^{''''}(x)dx \tag{12.1}$$

$$k_a=\frac{C_a}{r_b A_b} \tag{12.2}$$

The natural frequencies w_{ni} are determined by using Eqs. (12.1) and (8):

$$w_{ni}^2=\frac{E_b I_b}{r_b A_b}b_i^4 \tag{12.3}$$

Modal damping (z_i) can be included in Eq. (12), as follows:

$$\ddot{h}_i(t)+2z_i w_{ni}\dot{h}(t)+w_{ni}^2 h_i(t)=k_a\left[f_i^{'}(x_{a2})-f_i^{'}(x_{a1})\right]V_a(t) \tag{13}$$

Now, by taking the Laplace transform of Eq. (13), the transfer function of this system is obtained:

$$\frac{w(x,s)}{V_a(s)}=\sum_{i=1}^{\infty}\frac{k_a f_i(x)\left[f_i^{'}(x_{a2})-f_i^{'}(x_{a1})\right]}{s^2+2z_i w_{ni}s+w_{ni}^2} \tag{14}$$

This equation describes the elastic deflection of the flexible beam due to a voltage applied to the actuating piezoelectric.

2.2. Sensor Dynamics

The net forcing of the beam is equivalent to two equal and opposite moments applied to the beam at the endpoints of the actuator, as shown in Fig. 2. Due to the piezoelectric effect, a strain-induced voltage appears in the sensors.

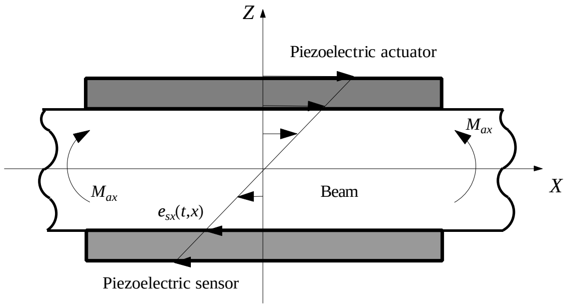

Figure 2. Piezoelectric distributed sensor-actuator-beam configuration and associated strain distribution.

The electric charge distribution $q_s(t)$, i.e., the charge per unit area, is given by (Pota and Alberts, 1995):

$$q_s(x,t)=\frac{k_{31}^2}{g_{31}}e_{sx}(x,t) \tag{15}$$

where k_{31} is the electromechanical coupling factor and g_{31} is the piezoelectric voltage constant in the *X* direction. Using Hooke's law for the beam deflection in the *X* direction (see Fig. 2), the expression for the strain in the sensor patch is obtained as (Fuller et al, 1996):

$$e_{sx}(x,t)=-\left(\frac{h_b}{2}+h_s\right)\frac{\partial^2 w}{\partial x^2} \tag{16}$$

The total charge accumulated on the sensing layer can be found by integrating $q_s(x,t)$ over the entire length of the piezoelectric element (Pota and Alberts, 1995):

$$Q_s(t)=\int_{x_{s1}}^{x_{s2}} b_s q_s(x,t)dx=-b_s\left(\frac{h_b}{2}+h_s\right)\frac{k_{31}^2}{g_{31}}\left.\frac{\partial w(x,t)}{\partial x}\right|_{x_{s1}}^{x_{s2}} \tag{17}$$

Substituting Eq. (7) into (17), yields:

$$Q_s(t)=k_s\sum_{i=1}^{\infty}h_i(t)\left[f_i^{'}(x_{s2})-f_i^{'}(x_{s1})\right] \tag{18}$$

where k_s is given by:

$$k_s=-b_s\left(\frac{h_b}{2}+h_s\right)\frac{k_{31}^2}{g_{31}} \tag{19}$$

The piezoelectric sensor is similar to an electric capacitor and the voltage across the two layers is given by the following formula:

$$V_s(t)=\frac{Q_s(t)}{C_s b_s(x_{s2}-x_{s1})} \tag{20}$$

where C_s is the capacitance per unit area of the piezoelectric sensor and $b_s(x_{s2}-x_{s1})$ is the surface area of the piezoelectric element.

Substituting Eq. (18) in Eq. (21), yields:

$$V_s(t)=k_s\sum_{i=1}^{\infty}h_i(t)\left[f_i^{'}(x_{s2})-f_i^{'}(x_{s1})\right] \tag{22}$$

Taking the Laplace transform of Eqs. (22) and (13), the expression for $V_s(s)$ in terms of the input voltage $V_a(s)$ is obtained:

$$\frac{V_s(s)}{V_a(s)}=\sum_{i=1}^{\infty}\frac{k_s k_a\left[f_i^{'}(x_{s2})-f_i^{'}(x_{s1})\right]\left[f_i^{'}(x_{a2})-f_i^{'}(x_{a1})\right]}{s^2+2z_i w_{ni}s+w_{ni}^2} \tag{23}$$

This equation is the transfer function of the circuit, i.e., the relation between the voltage applied to the actuator and the voltage induced in the piezoelectric sensor.

3. Taking into account out of bandwidth modes

As an infinite number of modes are taken into account in the above formulation, Eqs. (14) and (23) represent infinite-dimensional transfer functions.

In a typical control design scenario, the designer is often interested only in a particular bandwidth. Therefore, an approximate model of the system that best represents its dynamics in the prescribed frequency range is needed. A natural choice in this case is to simply ignore the modes which correspond to the frequencies that lie outside of the bandwidth of interest. Equation (23) is rewritten by limiting the number of modes considered to N.

$$\frac{V_s(s)}{V_a(s)}=G(s)=\sum_{i=1}^{N}\frac{F_i}{s^2+2z_i w_{ni}s+w_{ni}^2} \tag{24}$$

where F_i is given by:

$$F_i=k_s k_a\left[f_i^{'}(x_{s2})-f_i^{'}(x_{s1})\right]\left[f_i^{'}(x_{a2})-f_i^{'}(x_{a1})\right] \tag{24.1}$$

The drawback of this approach is that the truncated higher order modes may contribute to the low frequency dynamics in the form of distorting zero locations. Moheimani (1999) suggests a way of dealing with this problem. The idea is to allow for a constant feed through term in Eq. (24) to account for the compliance of omitted higher order modes of Eq. (23). That is, to approximate Eq. (24) or $G(s)$ by:

$$\hat{G}(s)=G(s)+K_c \tag{25}$$

where K_c is a constant that considers the effect of dynamical responses of higher order modes.

Moheimani (1999) found the optimal value of K_c so that the effect of higher order modes on the low frequency dynamics is minimized in some measure. The optimal value of K_c was found to be:

$$K_c = \frac{1}{2w_c} \sum_{i=N+1}^{N_t} \frac{F_i}{w_i} \ln\left(\frac{w_i + w_c}{w_i - w_c} \right) \tag{26}$$

where w_c represents the cut-off frequency chosen to lie within the interval: $w_c \in (w_N, w_{N+1})$.

In practice, a finite number of modes (N_t) is used to calculate the feed through term. So, K_c in Eq. (26) is calculated from $i = N+1$ to N_t with N_t is chosen so that the neglected dynamics in the model can be compensated sufficiently. However, choosing a large enough N_t is quite reasonable as its effect diminishes when $N_t \to \infty$ since the contribution of higher frequency modes is decreasing.

Similarly, Eq. (14) can be rewritten as:

$$\hat{G}(x,s) = \sum_{i=1}^{N} \frac{F_i f_i(x)}{s^2 + 2z_i w_{ni} s + w_{ni}^2} + K_x(x) \tag{27}$$

where:

$$K_x(x) = \frac{1}{2w_c} \sum_{i=N+1}^{N_t} \frac{F_i}{w_i} \ln\left(\frac{w_i + w_c}{w_i - w_c} \right) f_i(x) \text{ and } F_i = k_a \left[f_i'(x_{a2}) - f_i'(x_{a1}) \right]. \tag{27.1}$$

To this end, the analysis given here ignores the effect of modal dampings. However, it is a difficult exercise to determine these values during the modeling phase if the modal analysis is to be used.

4. Spatial H_∞ control of a flexible beam containing piezoelectric elements

This section is concerned with the problem of spatial H_∞ control for flexible structures. Consider a typical system of a flexible cantilever beam such as the one shown in Fig. 3. The system consists of only one piezoelectric actuator-sensor pair. Here, the purpose of the controller is to reduce the effect of disturbance $d(t)$ on the entire structure, using piezoelectric elements.

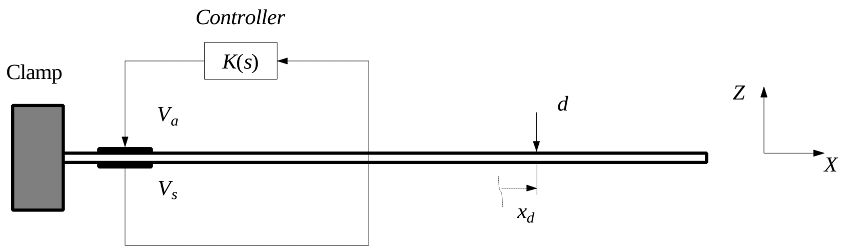

Figure 3. Typical feedback controller implementation for rejection of unwanted disturbance forces.

The problem of the elastic deflection (w) suppression in a flexible structure is the standard H_∞ control problem, as defined in (Doyle and Stein, 1981). The model of a flexible structure has two inputs (a disturbance d and a control V_a) and two outputs (a deformation vector w that should be minimized, and a measurement V_s). The state space equations, describing the structure with feedback controller can be obtained by using Eq. (13) and the corrected models for Eqs. (14) and (23) (Eqs. 25 and 27):

$$\dot{q}(t) = Aq(t) + B_1 d(t) + B_2 V_a(t) \tag{28}$$

$$w(x,t) = C_1(x) q(t) + D_{12}(x) V_a(t) \tag{29}$$

$$V_s(t) = C_2 q(t) + D_{21} d(t) \tag{30}$$

where:

$$q(t) = [h(t) \quad \dot{h}(t)]^T \tag{30.1}$$

$$A = \begin{bmatrix} 0_{N\times N} & I_{N\times N} \\ -w_{ni}^2 & -2z_i w_{ni} \end{bmatrix}, \; i = 1, 2, ..., N. \tag{30.2}$$

$$B_1 = \begin{bmatrix} 0_{N\times 1} \\ \overline{B}_1 \end{bmatrix} \tag{30.3}$$

$$\overline{B}_1 = \begin{bmatrix} f_1(x_d) \\ f_2(x_d) \\ \vdots \\ f_N(x_d) \end{bmatrix} \tag{30.4}$$

$$B_2 = k_a \begin{bmatrix} 0_{N\times 1} \\ \overline{B}_2 \end{bmatrix} \tag{30.5}$$

$$\overline{B}_2 = \begin{bmatrix} f_1^{'}(x_{a_2}) - f_1^{'}(x_{a_1}) \\ f_2^{'}(x_{a_2}) - f_2^{'}(x_{a_1}) \\ \vdots \\ f_N^{'}(x_{a_2}) - f_N^{'}(x_{a_1}) \end{bmatrix} \tag{30.6}$$

$$C_1(x) = \begin{bmatrix} \overline{C}_1 & 0_{1\times N} \end{bmatrix} \tag{30.7}$$

$$\overline{C}_1 = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_N(x) \end{bmatrix} \tag{30.8}$$

$$C_2 = k_s \begin{bmatrix} \overline{C}_2 & 0_{1\times N} \end{bmatrix} \tag{30.9}$$

$$\overline{C}_2 = \begin{bmatrix} f_1^{'}(x_{s_2}) - f_1^{'}(x_{s_1}) & f_2^{'}(x_{s_2}) - f_2^{'}(x_{s_1}) & \cdots & f_N^{'}(x_{s_2}) - f_N^{'}(x_{s_1}) \end{bmatrix} \tag{30.10}$$

$$D_{12}(x) = K_x(x), \text{ where } F_i = k_a\left[f_i^{'}(x_{a2}) - f_i^{'}(x_{a1})\right]. \tag{30.11}$$

$$D_{21} = K_c\text{, where } F_i = k_s\left[f_i^{'}(x_{s2}) - f_i^{'}(x_{s1})\right] f_i(x_d). \tag{30.12}$$

Then, the *Spatial Control* problem is to design a controller:

$$\dot{x}_k(t) = A_k x_k(t) + B_k V_s(t) \tag{31}$$

$$V_a(t) = C_k x_k(t) + D_k V_s(t) \tag{32}$$

such that the closed-loop system minimizes the spatial cost (Halim and Moheimani, 2002):

$$\inf_{K(.)\in U} \sup \frac{\int_0^\infty \int_\Re z(x,t)' Q(x) z(x,t)\,dx\,dt}{\int_0^\infty d(t)' d(t)\,dt} < g^2 \tag{33}$$

where g is called the disturbance attenuation factor (Doyle and Stein, 1981).

Here $Q(x)$ is a spatial weighting function that emphasizes the subset of $\Re$ over which disturbance rejection is of importance. It should be clear that the spatial H_∞ controller design to meet the performance index (33) a level of disturbance rejection over the entire $\Re$ in an average sense while emphasizing a subset of $\Re$ defined by $Q(x)$. In this particular application, $Q(x)$ = 1. In other words, the entire beam is weighted equally.

Condition (33) guarantees that a level of disturbance attenuation less than g will be achieved over the entire structure in an average sense.

It can be shown by the method in (Halim and Moheimani, 2002) that the above problem is equivalent to a standard H_∞ control problem for the following system:

$$\dot{q}(t) = Aq(t) + B_1 d(t) + B_2 V_a(t) \tag{34}$$

$$\tilde{w}(t) = \Pi q(t) + \Theta V_a(t) \tag{35}$$

$$V_s(t) = C_2 q(t) + D_{21} d(t) \tag{36}$$

where $\begin{bmatrix} \Pi & \Theta & \Theta \end{bmatrix} = \Gamma$. Here, Γ is any matrix that satisfies:

$$\Gamma^T \Gamma = \int_X \begin{bmatrix} C_1(x) \\ D_{12}(x) \end{bmatrix} Q(x) [C_1(x) \quad D_{12}(x)] dx \tag{37}$$

Based on (37) and using the orthogonality property in (9), the matrices Π and Θ can be obtained:

$$\Pi = \begin{bmatrix} I_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & 0_{N\times N} \\ 0_{1\times N} & 0_{1\times N} \end{bmatrix} \tag{38}$$

$$\Theta = \begin{bmatrix} 0_{2N\times 1} \\ \left(\sum_{i=N+1}^{N_t} (K_i)^2 \right)^{1/2} \end{bmatrix} \tag{39}$$

where $K_i = \dfrac{F_i}{2w_c w_i} \ln\left(\dfrac{w_i + w_c}{w_i - w_c} \right)$ and $F_i = k_a \left[f_i'(x_{a2}) - f_i'(x_{a1}) \right]$.

Hence, the system in (34)-(36) can be solved using a standard H_∞ control technique (Doyle and Stein, 1981). The spatial H_∞ controller can be regarded as a controller that reduces structural vibration in a spatially averaged sense.

It can be observed that the H_∞ control problem associated with the system described in (34)-(36) is nonsingular. This is due to the existence of feed through terms from the disturbance to the measured output and from the control signal to the performance output.

Designing a H_∞ controller for the system (34)-(36) may result in a very high gain controller (Halim and Moheimani, 2002). This could be attributed to the fact that the term Θ in (35) does not represent a physical weight on the control signal. Rather, it represents the effect of truncated modes on the in-bandwidth dynamics of the system, which is important in ensuring the robustness of the close-loop system. This problem can be fixed by introducing a weight on the control signal. This can be achieved by rewritten (34)-(36) as:

$$\dot{q}(t) = Aq(t) + B_1 d(t) + B_2 V_a(t) \tag{40}$$

$$\tilde{w}(t) = \begin{bmatrix} \Pi \\ 0 \end{bmatrix} q(t) + \begin{bmatrix} \Theta \\ R \end{bmatrix} V_a(t) \tag{41}$$

$$V_s(t) = C_2 q(t) + D_{21} d(t) \tag{42}$$

where R is a weighting matrix with compatible dimensions.

The matrix R serves as a weighting factor to balance the controller effort with respect to the degree of vibration reduction that can be achieved. Setting R with smaller value might lead to higher vibration reduction but at the expense of a higher controller gain. In practice, one has to make a compromise between the level of vibration reduction an controller gain by choosing a suitable R. The scalar R can then be determined to find a controller with sufficient damping properties and robustness. *Matlab m -Analysis* and *Synthesis Toolbox* was used to calculated the spatial H_∞ controller based on the system in (40)-(42).

5. Numerical Simulations

In order to evaluate the proposed control methodology, a flexible cantilever aluminium beam type structure containing one set of sensor/actuator PVDF/PZT ceramic piezoelectric elements symmetrically bonded to both sides of the beam is considered (see Fig. 3). The characteristics of the numerical model[1] (dimensions and material properties) are listed in Table 1.

Table 1. Characteristics of the piezo-structure.

		Piezoelectric		
Properties	**Units**	**Sensor**	**Actuator**	**Beam**
E (Young's modulus)	*Gpa*	2	69	65
r (density)	Kg/m^3	1780	7700	2711
d_{31} (Charge constant)	m/V	23×10^{-12}	-179×10^{-12}	----
g_{31} (Voltage constant)	mV/N	216×10^{-3}	10×10^{-3}	----
k_{31} (coupling coef.)	----	0.12	0.34	----
b (width)	$m \times 10^{-3}$	10	20	30
h (thickness)	$m \times 10^{-3}$	0.205	0.254	3
L (Length)	$m \times 10^{-3}$	30	50	700

[1] The damping ratio (z) of 0.3% was assumed for all the modes.

The *PZT* actuator patch and *PVDF* sensor film were attached to the beam surface at optimal location points (x_{s1} = 0.0 *m* ; x_{a1} = 0.0 *m*), as presented in (Abreu et al, 2003).

In this paper, a *SISO* controller is designed for the purpose of controlling only the first five vibration modes of the beam (Eqs. 28-30). Hence, the model is truncated to include only the first five bending modes. The effect of out of bandwidth modes has to be taken into consideration to correct the locations of the in bandwidth zeros (Moheimani, 1999) of the truncated model as discussed in Section 3. Using Eq. (26), the feed through term is then calculated:

$$D_{21} = K_c = -6.16\times10^{-2} \tag{43}$$

where N_t = 200.

Assuming x_d = 0.10 *m* (see Fig. 3), a robust controller (Eqs. 31 and 32) is designed to minimize the H_∞ norm of the transfer function (see Eq. 33) from the disturbance input to a *particular point along the beam* (x = 0.3). Using Eq. (27.1), where $K_x = 1.32\times10^{-8}$, Eqs. (28)-(30) are then evaluated to illustrate the performance of the proposed controller to reduce the transversal vibrations in this specific point (where $g = 7.18\times10^{-7}$). The Fig. 4 shows the H_∞ norm of the close-loop system as a function of the beam length.

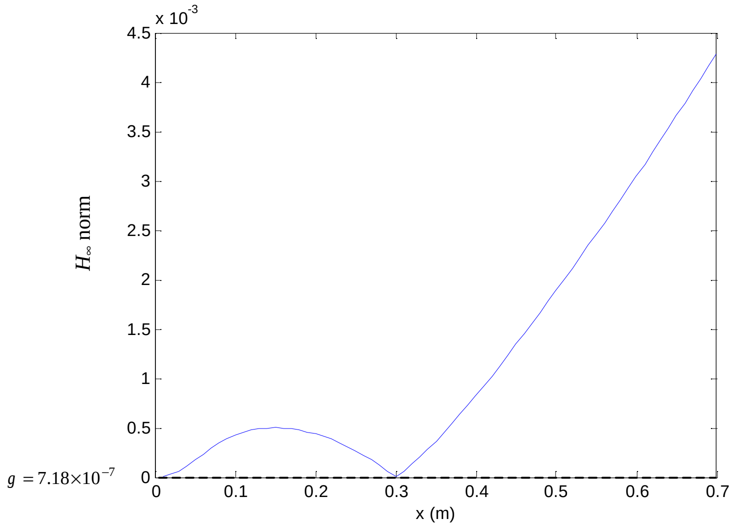

Figure 4. The H_∞ norm of the disturbance to various locations along the beam.

This controller does not guarantee disturbance reduction at other locations along the beam. One approach to attenuating the disturbance more evenly along the beam is to increase the number of error outputs in Eq. (29). Therefore, a new controller is designed for a single input (V_a), multiple output system (w). To illustrate this approach, we assume that:

$$x = \{0.3 \quad 0.6\} \tag{44}$$

which results C_1 matrix having two rows. As before, the Fig. 5 shows the corresponding H_∞ norm as a function of the beam position (where $g = 1.86\times10^{-6}$).

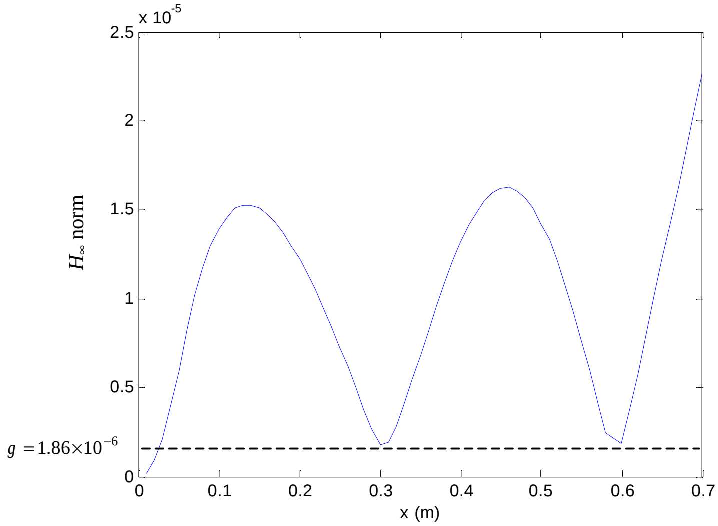

Figure 5. The H_∞ norm of the disturbance to various locations along the beam.

The special interest in this moment is the problem of guaranteeing a level of disturbance attenuation over the entire beam. As explained in section 4, the *spatial controller* can be obtained by using Eqs. (34)-(36). Figure 6 shows the numerical result of the H_∞ norm as a function of the beam position (where $g = 1.18\times10^{-6}$).

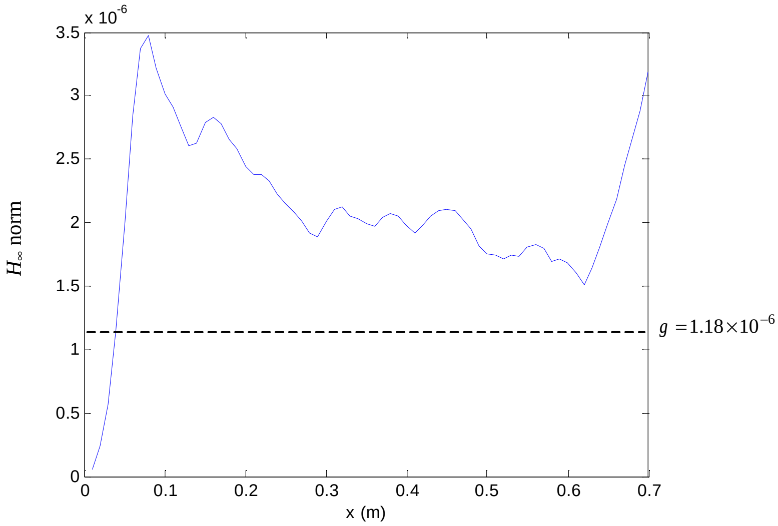

Figure 6. The H_∞ norm of the disturbance to various locations along the beam.

Figure 6 clearly demonstrates the effect of the proposed spatial H_∞ controller in reducing the vibration of the beam. It is obvious that the H_∞ norm of the entire beam has been reduced by the action of the controller in a more uniform manner (see Figs. 4 and 5). The Figs. 4 and 5 shows the effectiveness of the controller in local reduction of the H_∞ norm, especially at and around x = 0.3 and x = 0.6. This is expected since the purpose of this controller is to minimize vibration at these points. Comparing Figs. 4 and 5 and Fig. 6, it can be concluded that the spatial H_∞ norm as it minimizes the vibration throughout the entire structure, i.e., the spatial H_∞ controller reduces structural vibration in an average sense.

6. Conclusions

A spatial H_∞ controller was design and numerically implemented on a flexible beam type structure containing piezoelectric sensors and actuators. It was observed that such a controller resulted in suppression of transverse vibrations of the entire structure by minimizing the spatial H_∞ norm of the closed-loop system. The controller was obtained by solving a standard H_∞ control problem for a finite-dimensional system. Feed through terms were added to correct the locations of in bandwidth zeros of the system. It was shown that the spatial H_∞ control has an advantage over the local reduction in minimizing structural vibration of the entire structure. The spatial H_∞ control minimizes the H_∞ norm of the entire structure more uniformly, while the other H_∞ control minimizes the H_∞ norm more locally. It is important to say that the methodology presented in this paper can be extended to more sophisticated structures, such as thin plates. The methodology presented here can then be extended to explicitly allow for the residual uncertainty description in dealing with robustness issue. This work and the experimental tests however, are relegated to the future.

7. References

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