A GENERAL VIEW OF CONSTRUCTING FINITE VOLUME METHODOLOGIES FOR FLUID FLOW SIMULATIONS

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1. INTRODUCTION

The Computational Fluid Dynamics area experienced in the last three decades noteworthy numerical developments that influenced considerable the course of the research in the field. The pioneering work realized at the Los Alamos Scientific Laboratory by Harlow and Welch (1965) and Amsden and Harlow (1970), the work of Chorin (1967,1971) and Patankar and Spalding (1972), among other, move the solution of the partial differential equations from the stream-function/vorticity class of methods to the primitive variable methods used up today. This move allowed the development of methods for three-dimensional problems with convergence rates far better than the methods using stream function and vorticity, because of the direct application of velocity conditions at the boundaries. The pressure-velocity coupling methods using the segregated approach developed in the 70s and 80s (Patankar, 1981, van Doormal and Raithby, 1985) are still in practice today.

In the 70s, applied mathematicians and engineers were clearly divided in those solving fluid flow problems, using finite-difference methods, and those solving elasticity problems using finite element methods. Much of the methods in the so-called finite-difference group were already finite-volume methods, but not recognized as this. A serious drawback was present in the 70s in the finite-difference suite of methods for fluid flow problems. Only geometries fitting to the orthogonal coordinate systems could be easily solved. The need for solving fluid flow problems in complex geometries motivated another spectacular move in the research activities around the world. It was the development of boundary-fitted methods for complex geometries, launched by the work of Thompson et al (1974,1976,1977) based on the pioneering works of Winslow (1967), Chu (1971), Barfield (1971), Amsden and Hirt (1973), Gudonov and Prokopov (1968), amon others. Nowadays, practically all methods available in the commercial packages embodied the boundary-fitted option.

In this period it took place also another important step in the CFD area with the beginning of massive developments of finite-element techniques for fluid flow problems, trying to overcome the difficulties of using the Galerkin formulation for strongly advective flows, giving rise to the Petrov-Galerkin and streamline upwinding Petrov-Galerkin family of methods (Brooks and Hughes, 1982, Hughes and Mallet, 1986).

To reach the scenario of the present days, the development of methods for unstructured grids was another important move, which began in the 80s and seems to be the route to obtain generality and flexibility in fluid flow calculations. In this field, important contribution were made by Baliga and Patankar (1980), Schneider and Raw (1986)], among others, for convective heat transfer applications, and Jameson and Mavriplis (1986)

working on methods using triangular grids for aerodynamics applications. An overall view of the CFD field shows that the major effort in the research activities nowadays are dedicated to the development of unstructured grids, mixing numerical technologies encountered in finite-volume and finite-element methods. The tendency, it seems, is to have the conservation principles obeyed at discrete level in all methodologies.

In this paper the conservation equations are integrated in an arbitrary control volume laid out in an unstructured grid framework. The cell vertex and cell centered approaches for constructing the control volume, as well as the role played by the integration points in the different constructions is discussed. The specialized equations for control volumes created from elements using the method of the medians are derived. Brief comments are made about to obtain the specialized equations for boundary-fitted and Voronoi grids. The use of the element concept in finite volume techniques, perhaps the most promising route for developing general schemes, is presented in details and forms the main part of the paper.

2. WHY FINITE-VOLUME METHODS?

The task of a numerical technique is to transform a partial differential equation in a system of *n* linear algebraic equation with *n* unknowns, one for each node. The process of obtaining algebraic equations requires the integration of the partial differential equations in consideration. The several numerical methods available for solving partial differential equations differ on the form of realizing this integration. In fact, all methods can be viewed as derived from the weighted residual approach, being the weighting function the parameter which defines the type of the method. The weighted residual method states that

$$\int_V \mathcal{L}(\phi) W dV = 0 \qquad (1)$$

where W is the weighting function and ϕ the approximation for the unknown variables. If $W=1$ for the control volume in consideration and $W=0$ elsewhere, and the equation is in its divergence form, the method is recognized as the finite volume method, since is represents the balance of the property over the control volume. Since this principle is applied for all control volumes, it results in a system of equation which conserves the property at a control volume level. Recalling that solving the partial differential equation exactly, signifies to obey the conservation principles at point-level, solving it using a finite volume means to obey the conservation principles at discrete levels. Fig 1, tries to represent this idea of reaching the point level conservation by successive conserved solutions at discrete (finite) levels. Therefore, using a finite-volume method, the discrete solution is on the way to reach the point-level solution, which is, of course, the goal of any numerical method. The finite-volume equations are, therefore, the natural discrete equations for the conservation principles of physics.

The satisfaction of the conservation principles at discrete level assures that there are no sources or sinks of the property at the interfaces. This condition, plus the enforcement of the boundary conditions at the boundary, means that the solution will be driven only by the boundary conditions and the governing differential equation, and not by spurious source or sink terms generated inside the domain by the application of a non-conservative numerical scheme.

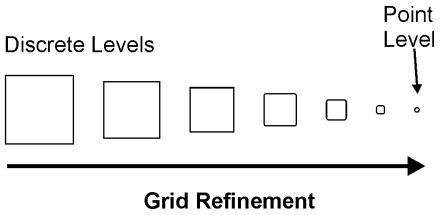

Figure 1 – Conservation at discrete levels

Another strong argument for seeking conservative schemes at discrete level is the physics of the phenomena being solved. The advective transport of momentum, energy and any other scalar is made by the mass flow. Therefore, to ensure conservation of any other scalar, it is required that mass conservation be strictly enforced in each time interval. In addition, this renders to the method robustness during the solution procedure. The balance of the property ϕ at the control volume shown in Fig. 2, given by

$$\dot{m}\phi\big|_{entra} - \dot{m}\phi\big|_{sai} + \dot{g}^{\phi}\Delta V = \frac{\Delta}{\Delta t}(m\phi)_{VC} \qquad (2)$$

where g^{ϕ} is the generation term for the property ϕ in consideration (Maliska, 2003), demonstrates the connection between mass flow and the transport of any scalar ϕ.

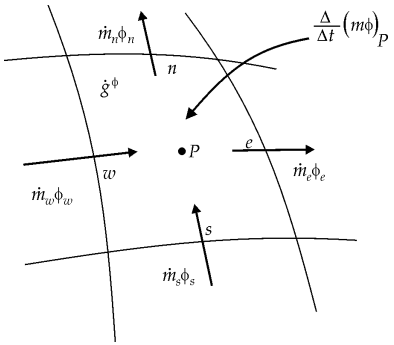

Figure 2 – Conservation of ϕ in a irregular control volume

Another very helpful characteristic of finite volume techniques is the possibility of checking the conservation principles in coarse grids. Doing this, the painful task of finding implementation errors is extremely simplified.

The next step of our development is this work is the creation of the control volumes where the conservation principles are applied.

3. ELEMENTS AND CONTROL VOLUMES

In a finite volume methodology the domain is covered by non-overlaping control volumes where the balances are made. Different finite volume methodologies have different ways of creating the control volumes. However, the starting point is always the grid obtained from a grid generator. Consider now Figs. 3 and 4, where two grids are shown. For simplicity only two-dimensional situations will be considered, but the analysis also holds for three dimensional cases. Based on these grids one can now define the ***element***. The definition of the element precedes the creation of the control volume in any discrete methodology. The element is a geometrical entity which covers the computational domain with no overlapping and no void spaces after assembling.

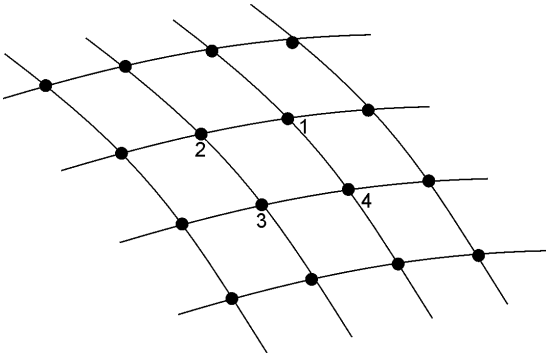

Figure 3 – Structured curvilienar grid

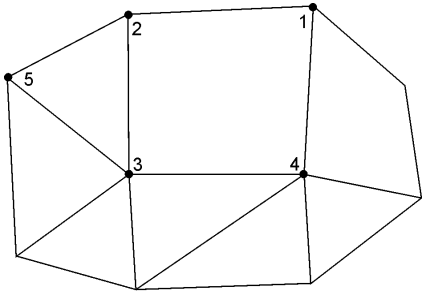

Figure 4 – Unstructured grid with triangles and quadrilaterals

Fig. 3 shows a structured curvilinear grid where the element 1234 is outlined, while Fig. 4 depicts an unstructured grid formed by triangular (235) and quadrilateral

(1234) elements. For the grid shown in Fig. 4 a local coordinate system is required. The definition of the integration point and the construction of the control volume referred to the element is now considered.

3.1 Integration Points

The integration points are always on the surface integral of the control volumes where all fluxes are calculated for performing the conservation balances. It is clear that the accuracy of the method will increase by increasing the number of integration points at the surface. Several integration rules can be used to better represent the fluxes at the surface integral using the available integration points and other interpolation points. The calculation of any property or geometrical information at one integration point should depend only on values of the variable and geometrical data stored at the nodes which defines the element. In Fig. 7 the integration points *ip1* and *ip2* belongs to the element 1234.

3.2 The Cell Centered Construction

The classical finite volume method using structured grids, which used grids ranging from Cartesian to curvilinear non-orthogonal shapes, uses the element as control volume for the integration of the partial differential equation, as can be seen in Fig. 5. In this case the unknown is on the center of the element and the integration points are in each mid-face of the element. Since the integrated conservation equations over the control volume require the specification of the fluxes at the control volume boundaries, the fluxes are calculated at the integration points.

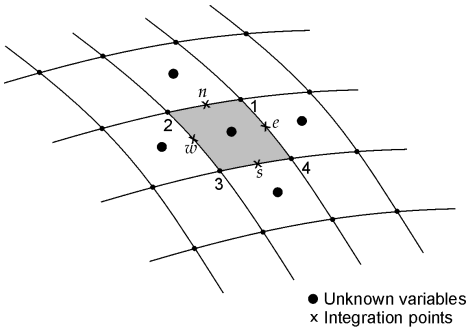

Figure 5 – Control volume coincident with the element in the classical finite volume method

In this construction, if the ϕ variable is to be calculated in the integration point *s*, for example, ϕ values from, at least two different elements will be needed. This poses difficulties in having a clean computer implementation of the algorithm, since a element-by-element assembling will be not possible. In addition, the fluxes at the interfaces are calculated using only one integration point, reducing the accuracy of the method for a given grid. There are several methodologies using structured, as well as unstructured grids, which use the element as control volume (Maliska, 2003).

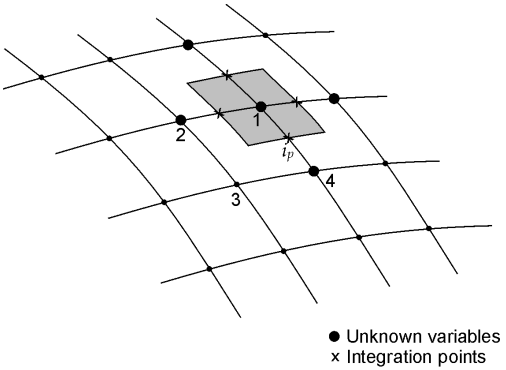

Figure 6 – Cell-vertex construction without using the element

3.3 The Cell Vertex Construction

In the cell vertex construction the control volumes are created joining the center of the elements to its medians. The resulting control volume is formed by portions (sub-control volumes) of the neighboring elements, as can be seen in Fig. 6. In the case shown, the integration points can be defined as before, that is, only one for each interface, as shown in Fig. 6. This approach, besides the fact of reducing accuracy by using only one integration point for each face, its location is ambiguous, since it lies in the interface between two elements. The other alternative, and the preferable one, is to locate two integration points in each element, like *ip1* and *ip2*, belonging to the element 1234, as shown in Fig.7.

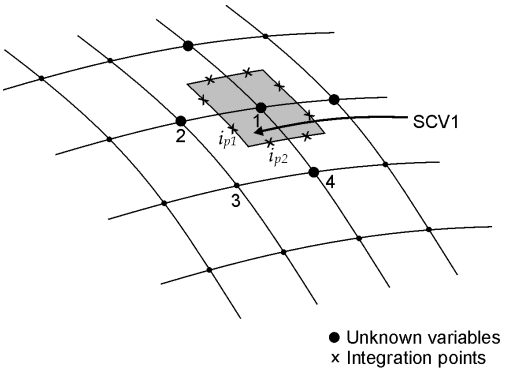

Figure 7- Cell-vertex construction using the element

In this case, all fluxes at one specified integration point can be calculated using data from the element where the integration point lies. This allows the creation of the approximate equation for the unknown located at the center of the control volume by assembling the neighboring elements. The computer implementation can now take

advantage of the element-by-element assembling technique. Oriented object programming, for example, becomes easy with this formulation.

The cell vertex method can be generalized to deal with triangular and quadrilateral elements, laid-out in an unstructured fashion, as shown in Fig. 8, where a nodal point can have variable number of neighbors. Again, the construction of the control volumes is made by joining the center of the elements to its medians.

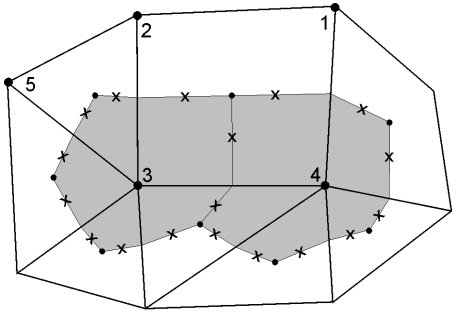

Figure 8 - Integration points in an unstructured mesh with triangular and quadrilateral elements

The above discussion demonstrated that the construction of the control volumes for any method can always be related to the elements defined in Figs. 3 and 4. It was also seen that using the element as control volume does not render to the method the desired characteristic of having all required parameters for fluxes calculation referred to a single element. This means that in the cell centered method the advantages offered by the element are not explored. These advantages are encountered in the cell vertex method choosing the integration point lying inside the element and not at its boundary. In other words, one can say that the integration points are properly located when they are in the same position as the Gauss points are in a finite element formulation.

The finite volume methodologies which create the control volume by assembling sub-control volumes belonging to different elements are called in the literature CVFEM-Control Volume Finite element Methods. This denomination is misleading and conveys the reader to view the methodology as being a finite element technique which uses control volumes for the integration of the equations. In fact, it is a finite volume methodology, whose only similarity with the finite element method is the use of the elements and the shape functions for the geometrical definition and for the variables interpolation. A better denomination would be Element-based Finite Volume Method (EbFVM), since it is a finite volume methodology which borrows from the finite element technique the concept of elements. Finite element formulations that obey the conservation principles at discrete levels are called mixed finite element methods. See, for instance, Raviart and Thomas (1977), Arbogast et al (1992), among others.

In the following section the conservation equation for a variable ϕ is integrated over a control volume formed by sub-control volumes of quadrilateral elements. Triangular elements will not be considered here, since the procedure is similar. The reader is referred to the works of Baliga and Patankar, (1980) for details.

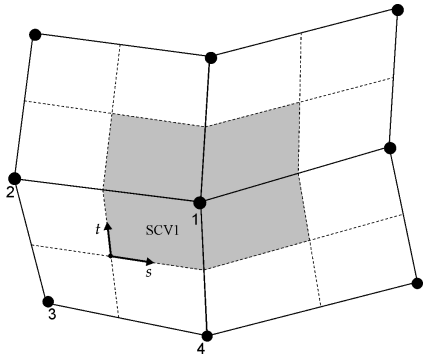

Figure 9 – Control volume 1, elements and the local coordinate (*s,t)*

3.4. Geometrical relations of the element/Local coordinate transformation

Consider Fig. 9 where a control volume for node 1, the element 1234 and its SCV1 (sub-control volume1) which contributes in forming the control volume are shown. The rectangular element in the transformed plane in the local coordinates ,*(s,t)* ranges from $-1 \leq s \leq 1$ and $-1 \leq t \leq 1$., as shown in Fig. 10, where it is shown the transformed domain and the element physical domain with the four sub-control volumes. Defining *x* and *y* as the global coordinate system one can write the following coordinate transformation

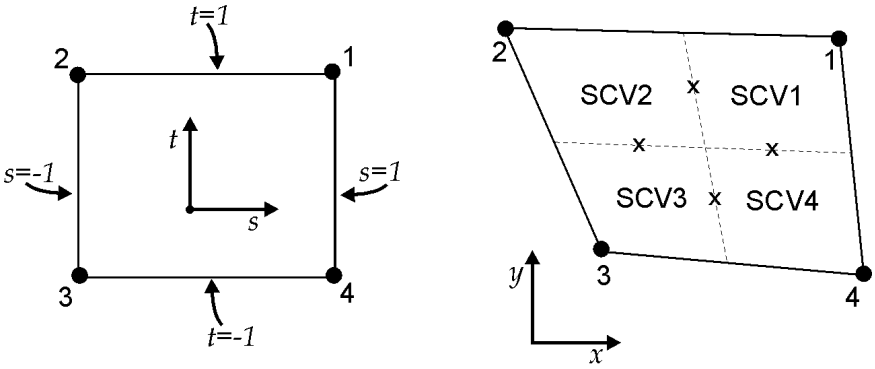

Figure 10 – Transformed (s,t) and physical (x,y) domains.

$$s = s(x, y) \tag{3}$$

$$t = t(x, y) \tag{4}$$

The *ds* and *dt* differentials can be written as

$$ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy \tag{5}$$

$$dt = \frac{\partial t}{\partial x} dx + \frac{\partial t}{\partial y} dy \tag{6}$$

or in matrix form by

$$\begin{bmatrix} ds \\ dt \end{bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{7}$$

or

$$\left[d^T\right] = [A]\left[d^F\right] \tag{8}$$

where d^T and d^F are the lengths in the transformed and physical domains. Conversely, one can write

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix} \tag{9}$$

or

$$\left[d^T\right] = [A]\left[d^F\right] \tag{10}$$

Using Eqs.(8) and (10) one obtains

$$[A] = [B]^{-1} = J \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial x}{\partial s} \end{bmatrix} \tag{11}$$

from where one obtains the metrics of the transformation

$$\frac{\partial s}{\partial x} = J\frac{\partial y}{\partial t} \tag{12}$$

$$\frac{\partial s}{\partial y} = -J\frac{\partial x}{\partial t} \tag{13}$$

$$\frac{\partial t}{\partial x} = -J\frac{\partial y}{\partial s} \tag{14}$$

$$\frac{\partial t}{\partial y} = J\frac{\partial x}{\partial s} \tag{15}$$

and the Jacobian is given by

$$J = \det[A] = \frac{1}{\det[B]} \tag{16}$$

or

$$J = \left[\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right]^{-1} \tag{17}$$

Defining x_i and y_i as the coordinates of a general node i one can write

$$x(s,t) = \sum_{i=1}^{4} N_i(s,t)x_i \tag{18}$$

$$y(s,t) = \sum_{i=1}^{4} N_i(s,t)y_i \tag{19}$$

where the shape functions are defined by

$$N_1(s,t) = \frac{1}{4}(1+s)(1+t) \tag{20}$$

$$N_2(s,t) = \frac{1}{4}(1-s)(1+t) \tag{21}$$

$$N_3(s,t) = \frac{1}{4}(1-s)(1-t) \tag{22}$$

$$N_4(s,t)=\frac{1}{4}(1+s)(1-t) \tag{23}$$

The interpolation of ϕ and its derivatives inside the element are given by

$$\phi(s,t)=\sum_{i=1}^{4} N_i(s,t)\phi_i \tag{24}$$

$$\frac{\partial\phi}{\partial x}=\sum_{i=1}^{4}\left.\frac{\partial N_i}{\partial x}\right|_{s,t}\phi_i \tag{25}$$

$$\frac{\partial\phi}{\partial y}=\sum_{i=1}^{4}\left.\frac{\partial N_i}{\partial y}\right|_{s,t}\phi_i \tag{26}$$

The derivatives of the shape functions related to *s* and *t* are

$$\frac{\partial N_i}{\partial s}=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s} \tag{27}$$

$$\frac{\partial N_i}{\partial t}=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial t} \tag{28}$$

Solving for the derivatives of N_i with respect to *x* and *y*, one gets

$$\frac{\partial N_i}{\partial x}=J\left[\frac{\partial N_i}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial N_i}{\partial t}\frac{\partial y}{\partial s}\right] \tag{29}$$

$$\frac{\partial N_i}{\partial y}=J\left[\frac{\partial N_i}{\partial t}\frac{\partial x}{\partial s}-\frac{\partial N_i}{\partial s}\frac{\partial x}{\partial t}\right] \tag{30}$$

where the Jacobian was already given in Eq.(17). The metrics of the transformation required in Eqs. (29) and (30) are given by Eqs. (12)-(15). We have now all the geometrical relations required for integrating the conservation equations.

4. GOVERNING EQUATIONS

The partial differential equation system of interest is the mass, momentum and energy conservation, given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0 \tag{31}$$

$$\frac{\partial \left(\rho u_i\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j u_i\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + S^{u_i} \tag{32}$$

$$\frac{\partial \left(\rho T\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j T\right) = \frac{\partial}{\partial x_j}\left(\frac{k}{c_p}\frac{\partial T}{\partial x_j}\right) + S^T \tag{33}$$

The above equations and any other scalar like turbulent kinetic energy and its dissipation, species concentration, for example, can be represented by the following general equation

$$\frac{\partial \left(\rho \phi\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j \phi\right) = \frac{\partial}{\partial x_j}\left(\Gamma^{\phi}\frac{\partial \phi}{\partial x_j}\right) + S^{\phi} \tag{34}$$

where the meaning of Γ^{ϕ} and ϕ can be found in any publication in the computational fluid dynamics area.

5. INTEGRATION OF THE CONSERVATION EQUATIONS

The integration of the conservation equation can now be realized over any of the control volume previously discussed, giving rise to different known methodologies. The integration of the conservation equation for an arbitrary control volume results in

$$\frac{M_P \phi_P - M_P^0 \phi_P^0}{\Delta t} = \sum_{ip}\left(\Gamma^{\phi}\frac{\partial \phi}{\partial \vec{n}}\right)_{ip}\Delta s_{ip} - \sum_{ip}\rho\left(\vec{V}\cdot\vec{n}\right)\phi_{ip}\cdot\Delta s_{ip} + S_P\phi_P\Delta V + S_c\Delta V \tag{35}$$

where the sub-index P means the node P (node 1 in this example), ip the integration points, M_p is the mass inside the sub-control volume and the superscript o means the previous time level. The first two terms in the right hand side of the equation represents the diffusive and advective fluxes and through the control volume surfaces. The above integration applies to any kind of control volume, giving rise to different methods. In this work the Element-based Finite Volume Method (EbFVM) will be described.

5.1 Element-Based Finite Volume Method (EbFVM)

Only the basic ideas and the main steps of the EbFVM will be described here. Details can be found in Schneider and Raw, (1986) and Raw, (1985). Eq.(35), substituting

the normal derivative by their Cartesian components, and substituting the dot product of the velocity and the normal vector by the normal velocity to the surface integral, reads

$$\frac{M_P\phi_P - M_P^0\phi_P^0}{\Delta t} + \sum_{ip}\dot{m}_{ip}\phi_{ip} = \sum_{ip}\left(\Gamma_{eff}^{\phi}\frac{\partial\phi}{\partial x_j}\Delta n_j\right)_{ip} + S_P\phi_P\Delta V + S_c\Delta V \qquad (36)$$

In this equation the volume involved is the volume of the SCV1, and Δn_j is equal do Δy when $j=1$ and $-\Delta x$ when $j=2$. The mass flow rate at the integration point is calculated by

$$\dot{m}_{ip} = \left(\rho u_j \Delta n_j\right)_{ip}^{0} \qquad (37)$$

where u_j are the Cartesian components of the velocity vector. Recall that the sub-index j implies summation in the derivative term of Eq.(35) as well as in Eq.(36).

The equation for each control volume is obtained assembling the sub-control volumes related to the specific node. For the control volume centered in node 1, for example, the element 1234 contributes with the sub-control volume 1 (SCV1) with fluxes calculated at integration points ip_1 and ip_2.

5.1.1. Interpolation functions

The next step in the formulation is the determination of the ϕ values and its derivatives at the integration points. The interpolation function is the main decision of the numerical method. If an exact interpolation function is employed, the resulting approximate equations are truncation error free. Of key importance is the interpolation function for the advective terms, since truncation errors in the form of numerical diffusion or numerical oscillation may appear. As reported in Maliska, (1995), interpolation with a diffusive character, like the well known upwind schemes, will introduce numerical diffusion, while high order interpolation schemes will introduce numerical oscillations. The interpolation function also plays an important rule in coupling the equations. For example, if all variables appear in the interpolation function, the resulting system of equations will involve all variables, becoming a system prone to be solve simultaneously.

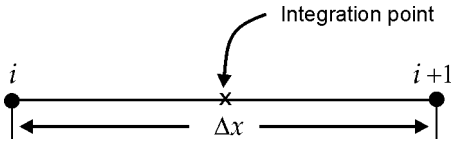

Figure 11 – 1D example for the construction of the interpolation function (Raw, 1985)

To exemplify the determination of the interpolation function consider Fig. 11 and $\phi=u$ in a 1D example, where the velocity u at the nodes are denoted by U and at the integration point by u. One wants to find u at the integration point based on the velocities at

the nodes. To create the interpolation function, an adequate form of the momentum equation is used, as

$$\rho u \frac{du}{dx} = -\frac{dP}{dx} + \mu \frac{d^2 u}{dx^2} \tag{38}$$

Approximating the advection term by a one-sided differencing and the diffusion and pressure term by a central scheme, one gets

$$\rho u^0 \frac{(u - U_i)}{\frac{\Delta x}{2}} = -\frac{P_{i+1} - P_i}{\Delta x} + \mu \frac{(U_i - 2u + U_{i+1})}{\left(\frac{\Delta x}{2}\right)^2} \tag{39}$$

Isolating u one obtains

$$u = U_i \left(\frac{\mathrm{Re} + 2}{\mathrm{Re} + 4}\right) + U_{i+1} \left(\frac{2}{\mathrm{Re} + 4}\right) + \frac{\Delta x}{2\rho u^0 \left(1 + \frac{4}{\mathrm{Re}}\right)} \left(\frac{P_i - P_{i+1}}{\Delta x}\right) \tag{40}$$

where $Re = \rho u^o \Delta x / \mu$, is the cell Reynolds number. It can be seen that the pressure is present in the interpolation function and, therefore, will contribute for promoting a better coupling between pressure and velocity. If Re tends to zero, the equation becomes

$$u = \frac{1}{2} U_i + \frac{1}{2} U_{i+1} + \frac{\Delta x}{8\mu} \left(\frac{P_i - P_{i+1}}{\Delta x}\right) \tag{41}$$

Pressure, of course, is still present, and if it is removed one recovers the pure CDS scheme only involving the velocities at the nodes. If Re tends to infinity, one gets

$$u = U_i + \frac{\Delta x}{2\rho u^0} \left(\frac{P_i - P_{i+1}}{\Delta x}\right) \tag{42}$$

and if the pressure is removed, one recovers the well known upwind scheme for the velocity.

It is then clear that a proper interpolation function may be extremely helpful for creating tight coupling between the equations. The same can be done using 2D and 3D forms of the momentum equation to obtain the interpolation function. Following it is presented a general form for developing the interpolation function in two and three dimensions.

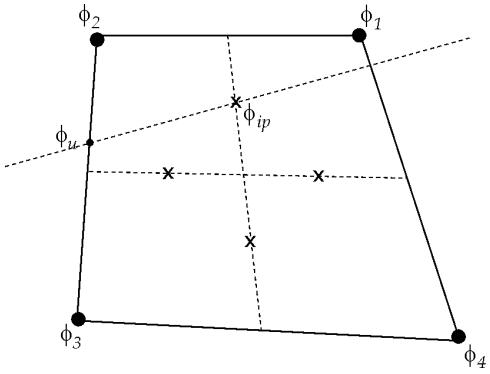

Figure 12 –Determination of ϕ ate the integration point

Consider Fig. 12, where an element is shown with a integration point and its upwind counterpart along the streamline. Applying a Fourier series along *s*, one obtains

$$\phi_{ip} = \phi_u + \frac{\partial \phi}{\partial s}\Delta s + \mathbf{O}\left(\Delta s^2\right) \tag{43}$$

where s is along the streamline and ϕ_u can be found by many different schemes. It can involve the variable at the four nodes of the element, only two of them, and even using a scheme that may involve the surrounding integration points, requiring them the solution a 4x4 matrix in 2D situations for each element. The reader is referred to the works of Raw (1985), Ávila Souza (2000), for details. The key question is the determination of $\Delta\phi/\Delta s$ in Eq. (43), which can be found from the approximation of a proper version of the momentum equation. For example, writing the momentum equation along *s* in the form

$$\rho V \frac{\Delta \phi}{\Delta s} = \mu \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] + S \tag{44}$$

one can find the value for $\Delta\phi/\Delta s$ as function of the relevant terms in the equation, by

$$\frac{\Delta \phi}{\Delta s} = \frac{\mu}{\rho V} \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] + S \tag{45}$$

Recall that, if ϕ is *u*, the pressure term is in the term S. Eq.(43) is an open road for developing several interpolation functions with second order accuracy. Back to Eq.(36), finding now the derivatives of ϕ at the integration points, and assembling the contribution of all SCVs, one obtains, after finding the equations for the boundary control volumes

$$A^{uu} u + A^{uv} v + A^{up} p = B^u \tag{46}$$

$$A^{uv}u + A^{vv}v + A^{vp}p = B^{v} \tag{47}$$

$$A^{up}u + A^{vp}v + A^{pp}p = B^{p} \tag{48}$$

The above system of equation is now suitable for a simultaneous solution. The fully coupled solution is the alternative for overcoming the difficulties with the segregated solutions using the SIMPLE-like methods. In these classes of methods, due to the segregated nature of the solution, the coupling between pressure and velocity requires very small time steps in order to achieve convergence. Another already well-known route to gain robustness and computer time is the use of conservative multigrid methods, such that the CPU time increase linearly with the increasing of grid number.

5.1.2 – Boundary conditions

Fig. 13 depicts a boundary control volume for P where the conservation balance should be made in order to find the approximate equation. Note that this control volume is made of only two sub-control volumes. Boundary conditions are applied following the same procedure applied for internal control volumes, that is, the fluxes through the boundaries are evaluated with the available boundary condition at that wall.

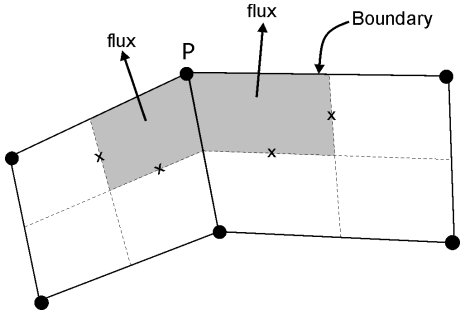

Figure 13 – Boundary control volume

5.2 Boundary-Fitted and Voronoi Discretizations

First of all, it should be put clear that what one is calling a boundary-fitted method is the classical finite volume method which uses a structured curvilinear grid as shown in Fig. 5 an 6 and with only one integration points in the middle of the interface. Therefore, starting from Eq.(35) and with the control volume shown in Fig. 5 or 6, it is easy to recover the formulation for structured non-orthogonal curvilinear grids. The classical method (Maliska, 1995), previous to the integration, transforms the conservation equation from the (x,y) coordinate system to the (ξ,η) global curvilinear coordinate system. Then the integration is performed in the new coordinates, called computational domain.

Recalling that *s* and *t* will be the well known ξ and η coordinates, replacing the normal derivative as a function of ξ and η, using the transformation relations and

recognizing that ***V.n*** is the contravariant velocity component, one recovers exactly the transformed equations, that is the conservation equation for ϕ written in the ξ,η global coordinate system. Details can be seen in Maliska (2003).

The methodology using Voronoi diagrams can also be recovered from Eq.(35). In this methodology the control volumes are created as in Fig 8, using a Delaunay triangulation with the property of being locally orthogonal. For example, in Fig. 14, the line joining the nodes 1 and 3 crosses the surface area AB orthogonally. Therefore, two alternatives for creating the equations exist.

One of them is to use the two integration points, as shown in Fig.14 for control volume 2, using a linear interpolation for the triangular element, as in Baliga and Patankar, (1980), or to place only one integration point as shown in control volume 1, as done in Maliska and Vasconcellos (2000), Maliska and Maliska Jr. (1994), Marcondes and Maliska (1995). In this case, the normal derivative can be found using variables stored at the nodes 1 and 3, not requiring the linear interpolation involving nodes 1, 2 and 3, required when two integrations points per element is used. Details can be found in Maliska (1995).

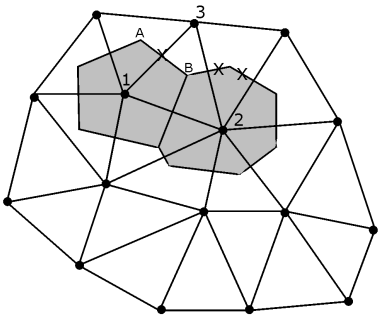

Figure 14 – Delaunay triangulation and Voronoi diagrams

6. CONCLUSIONS

The possibility of solving large systems of algebraic equations using the now available high speed computers has powered the development of numerical techniques in the last three decades. The CFD area, a research discipline in the beginning of the 70s, when supercomputers were needed to simulate simple aerodynamic problems, became a powerful tool for engineering design in modern days. Simulation of the turbulent Navier-Stokes equations with few million nodes is now possible with small computers.

This enormous growth in the development of numerical techniques, in the other hand, facilitated the appearance of new methods and misleading nomenclature and somewhat confusion among the methods. The well defined characteristics of the finite differences and finite element methods also disappeared, due to the merge of facilities of these methods in each other. As an example of the misleading nomenclature is the CVFEM-Control Volume Finite Element Method, the name given to the finite volume method which uses the element, a geometrical entity used in finite element methods to deal with all geometrical information of the computational domain. This nomination conveys the reader

to wrongly interpret the method as being a conservative finite element. A better name would be EbFVM-Element-based Finite Volume Method.

This paper discussed some of the fundamental issues when defining the finite volume method characteristics according to the construction of the control volumes and its relation with the element. It become clear that it is always possible to related the control volume method to the element. However, a genuine element-based method is the one which defines all the required parameters based on the element and creates the conservation equation by assembling the contributions of all surrounding elements.

It was also seen that all developments using structured and unstructured grids can depart from the same integrated conservation equation. The role of the integration points was also discussed, putting clear its importance and the major role played by the interpolation function.

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