 

# AN A PRIORI UPPER BOUND ESTIMATE FOR STEADY-STATE CONDUCTION HEAT TRANSFER PROBLEMS WITH A LINEAR RELATIONSHIP BETWEEN TEMPERATURE AND CONDUCTIVITY

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***Abstract.*** *This paper presents an a priori upper bound estimate for the steady-state temperature distribution in a body with a temperature-dependent thermal conductivity. The discussion is carried out assuming linear boundary conditions (Newton law of cooling) and a thermal conductivity linearly dependent on the temperature. The estimates consist of a powerful result that, depending on the objectives, avoids the necessity of an expensive numerical simulation of a nonlinear heat transfer problem and may be more effective than usual approximations (in which heat sources and thermal conductivities are assumed constant).*

***Keywords***: *Nonlinear heat transfer, upper bound, Kirchhoff transform.*

## 1. INTRODUCTION

Conduction heat transfer problems are usually simulated assuming temperature independent thermal conductivity. Such an approximation is carried out in order to simplify the simulation of the considered problems. Nevertheless, even taking into account the dependence of the thermal conductivity on the temperature, it is possible to estimate “a priori” an upper bound for the temperature field, without requiring a complete simulation of the conduction heat transfer process. Sometimes the simulation of a complex nonlinear heat transfer problem is carried out only for verifying if the maximum temperature remains lower than a given bound. In such cases the simulation is no longer required if an upper bound estimate for the solution is already available.

Temperature dependent thermal conductivity is present in many problems with engineering relevance. For instance, concerning carbon nanotubes (known for their high thermal conductivities), the temperature-dependent thermal conductivity of crystalline ropes of single-walled carbon nanotubes decreases smoothly with decreasing temperature, and displays a linear temperature dependence below 30 K (Hone et al., 1999). Also, when thermal conductivity is considered as a function of temperature for a peaking behavior of the thermal conductivity is verified, before falling off at higher temperatures (Osman and Srivastava, 2001). Thermal properties play an important role in porous silicon, specially regarding its applications in optoelectronics. Geseley et al. (1997) verified that the thermal conductivity increases with temperature increase. Another important material presenting interest in optoelectronic and electronic applications is zinc oxide. The maximum thermal conductivities of the polycrystalline zinc oxide occur at about 60 K and their values are almost an order of magnitude lower than bulk ZnO (Alvarez-Quintana et al., 2010).

The main objective of this work is to provide an a priori upper bound estimate for the temperature distribution in homogeneous bodies with temperature-dependent thermal conductivity, subjected to a linear boundary condition (Newton’s law of cooling). This a priori estimate may be useful, for instance, when the main goal is to ensure that a (maximum admissible) temperature will not be reached. It is important to note that upper bounds for problems subjected to non-linear boundary conditions (conduction/radiation heat transfer) for constant thermal conductivity have already been proposed (Saldanha da Gama, 1997; Saldanha da Gama, 2000).

## 2. MECHANICAL MODELING

In this work the thermal conductivity is assumed as a linear function of the temperature as follows

$$k = \left(1+\beta T\right)k_0, \quad \text{with} \quad k_0 > 0, \quad k_0 \text{ and } \beta \text{ constants} \tag{1}$$

Equation (1) represents a first approximation for problems with temperature dependent thermal conductivity and makes sense if, and only if, $1+\beta T > 0$ everywhere (Incropera and Dewitt, 1996).

The classical steady-state conduction heat transfer process in a rigid and opaque body at rest, represented by the bounded open set $\Omega$ with boundary $\partial\Omega$, subjected to a linear boundary condition is mathematically described by (Slattery, 1999; Incropera and Dewitt, 1996)

$$\begin{aligned} &\operatorname{div}\left[k \operatorname{grad} T\right]+\dot{q}=0 && \text{in} \quad \Omega \\ &-k \operatorname{grad} T \cdot \mathbf{n}=h(T-T_\infty) && \text{on} \quad \partial\Omega \end{aligned} \tag{2}$$

in which **n** is the unit outward normal, $k$ is the thermal conductivity, $\dot{q}$ is the internal heat generation (per unit time and unit volume), $T_\infty$ is the environment temperature (constant and satisfying $1+\beta T_\infty > 0$), $h$ is the convection heat transfer coefficient (constant and positive-valued) and $T$ is the temperature (unknown). The conductivity $k$ is a linear function of the temperature $T$, given by equation (1).

The mathematical problem studied in this paper is obtained combining equations (1) and (2), in other words, the steady-state conduction heat transfer process mathematically represented by

$$\begin{aligned} &\operatorname{div}\left[\left(1+\beta T\right)k_0 \operatorname{grad} T\right]+\dot{q}=0 && \text{in} \quad \Omega \\ &-\left(k \operatorname{grad} T\right)\cdot \mathbf{n}=h(T-T_\infty) && \text{on} \quad \partial\Omega \end{aligned} \tag{3}$$

where the heat generation $\dot{q}$ is a given nonnegative-valued function. Since $k$ must be positive-valued everywhere, in order to preserve the physical meaning, Eq. (3) must be considered with the constraint $1+\beta T > 0$.

In order to work with an unconstrained description, equation (1) is now replaced by the following one

$$k=\bar{k}+\gamma\left|T-T_{REF}\right|, \quad \text{with} \quad \bar{k}>0 \quad \text{and} \quad \gamma>0 \tag{4}$$

in which $\bar{k}$ is a positive constant, $T_{REF}$ is a temperature of reference, $\gamma$ is a positive constant and "$|\ |$" denotes the "absolute value of". It is easy to see that when $k$ is given by equation (4) it is always positive-valued. Since in any actual heat transfer problem the temperature is considered within a given range, equation (4) may represent equation (1) for any real situation.

## 3. THE KIRCHHOFF TRANSFORM

The Kirchhoff transform may be defined in terms of the variable $\omega$ as follows (Arpaci, 1966)

$$\begin{aligned} &\omega=\hat{f}(T)=\int_{T_{REF}}^{T}\hat{k}\left(\xi\right)d\xi \\ &\text{So} \quad \omega=\hat{f}(T)=\int_{T_{REF}}^{T}\left[\bar{k}+\gamma\left|\xi-T_{REF}\right|\right]d\xi=\bar{k}\left(T-T_{REF}\right)+\frac{\gamma}{2}\left(T-T_{REF}\right)\left|T-T_{REF}\right| \\ &\text{and} \quad \operatorname{grad}\omega=k \operatorname{grad} T \quad \Rightarrow \quad \operatorname{div}\left(\operatorname{grad}\omega\right)+\dot{q}=0 \quad \text{in} \quad \Omega \end{aligned} \tag{5}$$

The inverse of the above Kirchhoff transform and the boundary condition can be written as

$$\begin{aligned} &T=\hat{f}^{-1}(\omega)=\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\omega+|\omega|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{|\omega|-\omega}{\gamma}}+T_{REF} \\ &-\left(\operatorname{grad}\omega\right)\cdot\mathbf{n}=h\left(\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\omega+|\omega|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{|\omega|-\omega}{\gamma}}+T_{REF}-T_\infty\right) \quad \text{on} \quad \partial\Omega \end{aligned} \tag{6}$$

## 4. ESTIMATING AN UPPER BOUND FOR THE TEMPERATURE $T$

Combining the last equation of Eq. (5) with the last one of Eq. (6) it comes that

$$\begin{aligned} &\operatorname{div}\left(\operatorname{grad}\omega\right)+\dot{q}=0 && \text{in} \quad \Omega \\ &-\left(\operatorname{grad}\omega\right)\cdot\mathbf{n}=h\left(\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\omega+|\omega|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{|\omega|-\omega}{\gamma}}+T_{REF}-T_\infty\right) && \text{on} \quad \partial\Omega \end{aligned} \tag{7}$$

The field $\Psi$ is defined below, where $\Omega^*$ is a conveniently chosen set given by

$$\operatorname{div}\left(\operatorname{grad}\Psi\right)+\sup_{\Omega}\left[\dot{q}\right]=0 \quad \text{in} \quad \Omega^*, \quad \text{with} \quad \Omega\subseteq\Omega^* \tag{8}$$

Thus,

$$\begin{aligned}&\operatorname{div}\left[\operatorname{grad}\left(\omega-\Psi\right)\right]\geq 0 && \text{in} \quad \Omega\\ &-\operatorname{grad}\left(\omega-\Psi\right)\cdot\mathbf{n}=h\left(\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\omega+|\omega|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{|\omega|-\omega}{\gamma}}+T_{REF}-T_\infty\right)+\operatorname{grad}\Psi\cdot\mathbf{n} && \text{on } \partial\Omega\end{aligned} \tag{9}$$

Since $\operatorname{div}\left[\operatorname{grad}\left(\omega-\Psi\right)\right]\geq 0 \quad \text{in} \quad \Gamma$, employing the divergence theorem, for any subset $\Gamma\subseteq\Omega$ with boundary $\partial\Gamma$, it comes that

$$\int_{\partial\Gamma}\operatorname{grad}\left(\omega-\Psi\right)\cdot\mathbf{n}\,dS\geq 0 \tag{10}$$

ensuring that the supremum of $\left(\omega-\Psi\right)$ in $\Gamma$ coincides with the supremum of $\left(\omega-\Psi\right)$ on $\partial\Gamma$. In addition, it may be concluded that $\sup_{\Omega}\left(\omega-\Psi\right)=\sup_{\partial\Omega}\left(\omega-\Psi\right)=\sup_{\partial\Omega^+}\left(\omega-\Psi\right)$, in which the nonempty subset $\partial\Omega^+\subseteq\partial\Omega$ is the following one: $\partial\Omega^+\equiv\left\{\mathbf{x}\in\partial\Omega \text{ such that } grad\left(\omega-\Psi\right)\cdot\mathbf{n}\geq 0\right\}$. The boundary condition gives rise to the following inequality (Taylor, 1958)

$$\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\sup_{\partial\Omega^+}\left[\omega\right]+\left|\sup_{\partial\Omega^+}\left[\omega\right]\right|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\left|\sup_{\partial\Omega^+}\left[\omega\right]\right|-\sup_{\partial\Omega^+}\left[\omega\right]}{\gamma}}\leq T_\infty-T_{REF}+\frac{1}{h}\sup_{\partial\Omega^*}\left\|\operatorname{grad}\Psi\right\| \tag{11}$$

Therefore,

$$\sup_{\Omega}\left[\omega\right]\leq\left\{\frac{\gamma}{2}\left|T_\infty-T_{REF}+\frac{1}{h}\sup_{\partial\Omega^*}\left\|\operatorname{grad}\Psi\right\|\right|+\bar{k}\right\}\left(T_\infty-T_{REF}+\frac{1}{h}\sup_{\partial\Omega^*}\left\|\operatorname{grad}\Psi\right\|\right)+\sup_{\Omega^*}\left[\Psi\right]-\inf_{\Omega^*}\left[\Psi\right] \tag{12}$$

In other words, once $\Psi$ is chosen, there is an a priori upper bound estimate for $\omega$ (since the supremum of $\omega$ is greater than or equal to $\omega$).

In addition, from the inverse of the Kirchhoff transform, it comes that

$$\sup_{\Omega}\left[T\right]=\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\sup_{\Omega}\left[\omega\right]+\left|\sup_{\Omega}\left[\omega\right]\right|}{\gamma}}-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{\left|\sup_{\Omega}\left[\omega\right]\right|-\sup_{\Omega}\left[\omega\right]}{\gamma}}+T_{REF} \tag{13}$$

So, taking Eq. (12) into account, it may be concluded that

$$\begin{aligned}\sup_{\Omega}\left[T\right]\leq&\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\frac{|\delta|\delta}{2}+\frac{\bar{k}}{\gamma}\delta+\frac{1}{\gamma}\left(\sup_{\Omega^*}\left[\Psi\right]-\inf_{\Omega^*}\left[\Psi\right]\right)+\left|\frac{|\delta|\delta}{2}+\frac{\bar{k}}{\gamma}\delta+\frac{1}{\gamma}\left(\sup_{\Omega^*}\left[\Psi\right]-\inf_{\Omega^*}\left[\Psi\right]\right)\right|}-\\ &-\sqrt{\left(\frac{\bar{k}}{\gamma}\right)^2+\left|\frac{|\delta|\delta}{2}+\frac{\bar{k}}{\gamma}\delta+\frac{1}{\gamma}\left(\sup_{\Omega^*}\left[\Psi\right]-\inf_{\Omega^*}\left[\Psi\right]\right)\right|-\frac{|\delta|\delta}{2}-\frac{\bar{k}}{\gamma}\delta-\frac{1}{\gamma}\left(\sup_{\Omega^*}\left[\Psi\right]-\inf_{\Omega^*}\left[\Psi\right]\right)}+T_{REF}\end{aligned} \tag{14}$$

where $\delta = T_\infty - T_{REF} + \frac{1}{h}\sup_{\partial\Omega^*}\|\operatorname{grad}\Psi\|$. Since, in the body $\Omega$, $T$ is less than (or equal to) its supremum in $\Omega$, inequality (14) represents an a priori upper bound estimate for $T$.

## 5. THE CHOICES FOR THE FIELD $\Psi$

Since $\Psi$ is any solution of Eq. (8), there are an infinite number of possible choices for $\Psi$. For instance, the four ones stated below (represented in terms of the usual rectangular Cartesian coordinates)

$$\Psi = -\frac{Cx^2}{2}, \quad \Psi = -\frac{Cz^2}{2}, \quad \Psi = -\frac{C\left(x^2+y^2\right)}{4} \quad \text{and} \quad \Psi = -\frac{C\left(x^2+y^2+z^2\right)}{6} \tag{15}$$

in which $C = \sup_{\Omega}\left[\dot{q}\right] = \text{constant}$.

Each of these choices may give rise to a different upper bound estimate, even for a same $\Omega^*$. Supposing that $\Omega^*$ is given by $\Omega_1^*$ or by $\Omega_2^*$ where

$$\begin{aligned} \Omega_1^* &\equiv \left\{(x,y,z) \quad \text{such that} \quad x^2+y^2<R^2 \text{ and } -L<z<L\right\} \\ \Omega_2^* &\equiv \left\{(x,y,z) \quad \text{such that} \quad -L<x<L, \ -L<y<L \ \text{ and } -L<z<L\right\} \end{aligned} \tag{16}$$

The results presented in table 1 illustrate the choices above.

Table 1. Some results for different choices for $\Psi$, with $\Omega^*$ given by Eq. (16).

| | $\sup_{\partial\Omega^*}\|\operatorname{grad}\Psi\| =$ | $\inf_{\Omega^*}\left[\Psi\right] =$ | $\sup_{\Omega^*}\left[\Psi\right] =$ |
|---|---|---|---|
| $\Psi = -\frac{C\left(x^2+y^2\right)}{4}$ with $\Omega^* \equiv \Omega_1^*$ | $\frac{CR}{2}$ | $-\frac{CR^2}{4}$ | 0 |
| $\Psi = -\frac{C\left(x^2+y^2+z^2\right)}{6}$ with $\Omega^* \equiv \Omega_1^*$ | $\frac{C}{3}\left(R^2+L^2\right)^{1/2}$ | $-\frac{C}{6}\left(L^2+R^2\right)$ | 0 |
| $\Psi = -\frac{C\left(x^2+y^2\right)}{4}$ with $\Omega^* \equiv \Omega_2^*$ | $CL\frac{\sqrt{2}}{4}$ | $-\frac{CL^2}{2}$ | 0 |
| $\Psi = -\frac{C\left(x^2+y^2+z^2\right)}{6}$ with $\Omega^* \equiv \Omega_2^*$ | $CL\frac{\sqrt{3}}{6}$ | $-\frac{CL^2}{2}$ | 0 |

## 6. FINAL REMARKS

This paper presented an a priori upper bound estimate for the steady-state temperature distribution for bodies in which there exits a linear relationship between the thermal conductivity and the temperature. In other words, for the class of problems considered here, it is possible to establish, without any simulation (exact or numerical) a value that is greater than (or equal to) the temperature in any point of the body.

Since the choice of the set $\Omega^*$ and of the function $\Psi$ influences the estimates, they must be conveniently chosen in order to minimize the difference $\sup_{\Omega^*}\left[\Psi\right] - \inf_{\Omega^*}\left[\Psi\right]$ as well as $\sup_{\partial\Omega^*}\|\operatorname{grad}\Psi\|$.

## 7. ACKNOWLEDGEMENTS

The authors R.M. Saldanha da Gama and M. L. Martins-Costa gratefully acknowledge the financial support provided by the Brazilian agency CNPq.

## 8. REFERENCES

Alvarez-Quintana, J. Martínez, E., Pérez-Tijerina, E., Pérez-García, S. A. and Rodríguez-Viejo, J, 2010, "Temperature dependent thermal conductivity of polycrystalline ZnO films", *J. Appl. Phys.*, vol. 107, pp. 063713.1-063713.4.

Arpaci, V.S., 1966, *Conduction Heat Transfer*, Addison-Wesley Publishing Company, Massachusetts.

Geseley, G., Linsmeieryx, J., Drachy, V. Frickey, J. and Arens-Fischerz, R., 1997, "Temperature-dependent thermal conductivity of porous silicon", *J. Phys. D: Appl. Phys.,* vol 30 pp. 2911–2916.

Hone, J., Whitney, M., Piskoti, C. and Zettl, A., 1999, Thermal conductivity of single-walled carbon nanotubes, Physical Review B, vol. 59/4, pp. 0163-1829/99/59**(**4**)**/2514**(**3).

Incropera, F. and Dewitt, P.D., 1996, *Introduction to Heat Transfer*, 3rd edition, John Wiley&Sons, New York.

Osman, M. A. and Srivastava, D., 2001, "Temperature dependence of the thermal conductivity of single-wall carbon nanotubes", *Nanotechnology*, vol. 12, pp. 21–24

Saldanha da Gama, R.M., 1997 "An easy procedure for solving steady-state conduction heat transfer problems with nonlinear boundary conditions", *Mech. Research Comm.*, vol. 24/2, pp. 197-202

Saldanha da Gama, R.M., 2000, "An upper bound estimate for a class of conduction heat transfer problems with nonlinear boundary conditions". *Int. Comm. Heat Mass Transfer*, Vol.27, No.7, pp.955-964.

Slattery, J.C., 1999, *Advanced Transport Phenomena*, Cambridge University Press, USA.

Taylor, A.E., 1958, Introduction to functional analysis, Wiley Toppan, Tokyo.

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