THE UPWIND GENERALIZED INTEGRAL TRANSFORM TECHNIQUE APPLIED TO REGENERATIVE HEAT EXCHANGER ANALYSIS

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Abstract. *This work proposes a novel methodology for solving convective heat transfer problems using integral transforms: The Upwind Generalized Integral Transform Technique (UDS-GITT). The method is illustrated by solving the heat transfer problem in a counter flow regenerator. The advective terms are approximated using an upwind discretization scheme (UDS) and the transformation of the problem is carried-out using the Generalized Integral Transform Technique (GITT). A validation table displays the convergence of effectiveness, showing good agreement with the literature. The results show that UDS-GITT has significant reduction of the oscillations compared to traditional GITT if a proper discretization step-size (δ) is chosen.*
Keywords: *Upwind Generalized Integral Transform Technique, Mixed Formulation, Convective Heat Transfer Problem, Regenerative Heat Exchangers*

1. NOMENCLATURE

x	spatial coordinate
t	time
τ	dimensionless dwell time
NTU	number of transfer unit
C_r^*	matrix to fluid capacity ratio
T	fluid temperature
T_S	solid temperature
T_H	homogeneous fluid temperature
T_{SH}	homogeneous solid temperature
$\overline{T}_H$	transformed homogeneous fluid temperature
$\overline{T}_{SH}$	transformed homogeneous solid temperature
T_{in}	inlet temperature
T_o	initial fluid temperature distribution
T_{so}	initial solid temperature distribution
N	norm

Greek Symbols

μ	eigenvalues
ψ	eigenfunctions
ε	effectiveness
δ	discrete step-size
Λ	discrete derivative

Subscripts

n, m	GITT indexes

2. INTRODUCTION

The optimization of computational solution codes has played a significant role in the implementation of numerical methods, since maintaining CPU costs at a minimum level has always been a major concern. Moreover, numerical simulations of periodic heat transfer problems involve applications of great significance. Within the realm of heat regenerators (Shah and Sekulic, 2002), computational simulations have been demonstrated to have an advantageous effect on the construction and operation of these types of exchangers.

Discrete methodologies appear, in general, as main option to solve a variety of heat and mass transfer problems. Although most studies involve different discrete solution methods, only a few investigations are based on formulations that use the Generalized Integral Transform Technique (GITT) (Cotta, 1993).

Some of the recent applications of the Generalized Integral Transform Technique include heat transfer in combined pressure-electroosmotic driven flows (Sphaier, 2012), conjugate convection-condution heat transfer using a single domain formulation (Knupp *et al.*, 2012), convective heat transfer in flows within wavy walls (Silva *et al.*, 2011; Castellões *et al.*, 2010), wind-induced vibrations on overhead conductors (Matt, 2009), hyperbolic heat conduction problems (Monteiro *et al.*, 2009), transient diffusion in heterogeneous media (Naveira-Cotta *et al.*, 2009), heat and mass transfer in adsorbed gas storage (Hirata *et al.*, 2009), atmospheric pollutant dispersion (Almeida *et al.*, 2008), dispersion in rivers and channels (de Barros and Cotta, 2007), heat transfer in magnetohydrodynamics (Lima *et al.*, 2007), solution of the Navier-Stokes equations (de Lima *et al.*, 2007) and the boundary layer equations (Paz *et al.*, 2007), stability analysis in natural convection (de B. Alves *et al.*, 2002), applications to irregular geometries (Sphaier and Cotta, 2002), among others. Recently (Sphaier *et al.*, 2011), a unified integral-transform algorithm was proposed, facilitating the application of the technique and thereby enabling a wider number of users to adopt this eigenfunction-based solution methodology. In the realm of heat exchangers it should be mentioned the works (Scofano-Neto and Cotta, 1993) and (Scofano-Neto and Cotta, 1992) which employed lumped-differential schemes and GITT for double-pipe heat exchangers.

This work proposes a novel methodology based on integral transform technique and upwind discretization of the advective term (GITT-UDS) to solve the heat transfer problem of a counter flow rotary regenerator. The main goal of the advective term upwind discretization is the introduction of a numerical diffusion (more dissipation), smoothing the results and reducing the oscillations which are generally seen in GITT solutions.

3. PROBLEM FORMULATION

The studied problem is that of heat transfer in a rotary regenerator. The traditional formulation, using the $\varepsilon - \mathrm{NTU}_0$ method developed by Coppage and London (1953), employed in this kind of problem is shown bellow.

$$\tau \frac{\partial T(x,t)}{\partial t} + \frac{\partial T(x,t)}{\partial x} = \mathrm{NTU}[T_S(x,t) - T(x,t)] \qquad \text{for} \qquad 0 \le x \le 1 \qquad \text{and} \qquad 0 \le t \le 1 \tag{1a}$$

$$C_r^* \frac{\partial T_S(x,t)}{\partial t} = \mathrm{NTU}[T(x,t) - T_S(x,t)] \qquad \text{for} \qquad 0 \le x \le 1 \qquad \text{and} \qquad 0 \le t \le 1 \tag{1b}$$

$$T(0,t) = T_{\text{in}}, \tag{1c}$$

$$T(x,0) = T_o(x), \tag{1d}$$

$$T_S(x,0) = T_{so}(x) \tag{1e}$$

The previously described methodology provides a transient solution to a single process, or simply a single-blow solution. The first process is started from the initial conditions already mentioned, and from there on the final temperature distribution of every process are used as initial conditions to the next one. However, for a counterflow arrangement, the initial conditions are fed into the next process with the change of variable:

$$x_{\text{next}} = 1 - x_{\text{current}} \tag{2}$$

The periodicity of the problem appears in the inlet conditions of each process:

$$T_{\text{cold}}(x,0) = T_{\text{hot}}(1-x,1) \qquad \text{for the cold period} \tag{3}$$

$$T_{\text{hot}}(x,0) = T_{\text{cold}}(1-x,1) \qquad \text{for the hot period} \tag{4}$$

the same concept is applied to the solid temperature:

$$T_{S,\text{cold}}(x,0) = T_{S,\text{hot}}(1-x,1) \qquad \text{for the cold period} \tag{5}$$

$$T_{S,\text{hot}}(x,0) = T_{S,\text{cold}}(1-x,1) \qquad \text{for the hot period} \tag{6}$$

A periodic, quasi-steady-state solution of the transport problem considered is obtained by repeatedly solving the presented formulation, alternating the inlet conditions between the cold and hot cyles. This iterative solution proceeds until an acceptable quasi-steady-state solution is reached. The selected stopping criterion is when the both effectiveness, hot and cold, reach the same value ($\varepsilon = \varepsilon_{\text{hot}} = \varepsilon_{\text{cold}}$). The effectiveness of each process are given by:

$$\varepsilon_{\text{hot}} = 1 - \int_0^1 T_{\text{hot}}(1,t)\,\mathrm{d}t, \tag{7}$$

$$\varepsilon_{\text{cold}} = \int_0^1 T_{\text{cold}}(1,t)\,\mathrm{d}t \tag{8}$$

4. SOLUTION APPROACH

As usual in integral transform formulations, two variable changes are proposed in order to homogenize the boundary condition.

$$T(x,t) = T_H(x,t) + T_{\text{in}}, \tag{9}$$

$$T_S(x,t) = T_{SH}(x,t) + T_{\text{in}} \tag{10}$$

Resulting in the following homogeneous system.

$$\tau \frac{\partial T_H(x,t)}{\partial t} + \frac{\partial T_H(x,t)}{\partial x} = \mathrm{NTU}[T_{SH}(x,t) - T_H(x,t)] \tag{11a}$$

$$C_r^* \frac{\partial T_{SH}(x,t)}{\partial t} = \mathrm{NTU}[T_H(x,t) - T_{SH}(x,t)] \tag{11b}$$

$$T_H(0,t) = 0, \tag{11c}$$

$$T_H(x,0) = T_o(x) + T_{\text{in}}, \tag{11d}$$

$$T_{SH}(x,0) = T_{so}(x) + T_{\text{in}} \tag{11e}$$

The idea behind this proposed mixed methodology is to introduce an upwind numerical approach replacing the original advective derivative and then apply the transformation of the problem. This procedure is the only step that differs from the traditional GITT. In other words, the advection term is replaced by:

$$\frac{\partial T_H(x,t)}{\partial x} = \Lambda(T_H, x, \delta) \tag{12}$$

where $\Lambda(T_H, x, \delta)$ is the approximation rule of T_H derivative in x direction which naturally depend on the step-size δ. Applying the proposed approach (12) to the system (11), the partially discretized equation is obtained:

$$\tau\frac{\partial T_H(x,t)}{\partial t} + \Lambda(T_H, x, \delta) = \mathrm{NTU}[T_{SH}(x,t) - T_H(x,t)] \tag{13a}$$

$$C_r^*\frac{\partial T_{SH}(x,t)}{\partial t} = \mathrm{NTU}[T_H(x,t) - T_{SH}(x,t)] \tag{13b}$$

$$T_H(0,t) = 0, \tag{13c}$$

$$T_H(x,0) = T_o(x) + T_{\text{in}}, \tag{13d}$$

$$T_{SH}(x,0) = T_{so}(x) + T_{\text{in}} \tag{13e}$$

As customary done in integral transform schemes, the solution approach consists on seeking an eigenfunction expansion solution based on a integral transform pair. For this specific problem, two transform pairs are needed, one for the fluid temperature:

$$\text{Inversion} \quad \rightarrow \quad T_H(x,t) = \sum_{n=1}^{\infty} \frac{\overline{T}_{Hn}(t)\,\psi_n(x)}{N_n} \tag{14a}$$

$$\text{Transform} \quad \rightarrow \quad \overline{T}_{Hn}(t) = \int_0^1 T_H(x,t)\,\psi_n(x)\,\mathrm{d}x \tag{14b}$$

and another for the solid temperature:

$$\text{Inversion} \quad \rightarrow \quad T_{SH}(x,t) = \sum_{n=1}^{\infty} \frac{\overline{T}_{SHn}(t)\,\psi_n(x)}{N_n} \tag{14c}$$

$$\text{Transform} \quad \rightarrow \quad \overline{T}_{SHn}(t) = \int_0^1 T_{SH}(x,t)\,\psi_n(x)\,\mathrm{d}x \tag{14d}$$

in which ψ are orthogonal solutions of a Sturm-Liouville eigenvalue problem associated with the x direction. An usual choice within the considered framework is the simple one-dimensional Helmholtz problem:

$$\psi_n''(x) + \mu^2\psi_n(x) = 0, \tag{15}$$

$$\psi_n(0) = 0, \qquad \psi_n'(1) = 0, \tag{16}$$

$$N_n = \int_0^1 \psi_n^2\,\mathrm{d}x \tag{17}$$

where the last equation represents the norm of the eigenfunction. This eigenproblem has the simple solution:

$$\psi_n(x) = \sin(\mu_n\,x), \qquad \text{with} \qquad \mu_n = (n - \pi/2)\pi \qquad \text{for} \qquad n = 1, 2, 3\ldots \tag{18}$$

Applying the integral operator $\int_0^1 \bullet\,\psi_n\,\mathrm{d}x$ to the transformed system (13) and using the transforms (equations (14b) and (14d)) to the transformable terms and the inversions (equations (14a) and (14c)) to non-transformable terms, the transformed system is obtained:

$$\tau\frac{\partial \overline{T}_{Hn}(t)}{\partial t} + \sum_{n=1}^{\infty} A_{m,n}\,\overline{T}_{Hm}(t) = \mathrm{NTU}(\overline{T}_{SHn}(t) - \overline{T}_{Hn}(t)) \tag{19}$$

$$C_r^*\,\frac{\partial \overline{T}_{SHn}(t)}{\partial t} = \mathrm{NTU}(\overline{T}_{Hn}(t) - \overline{T}_{SHn}(t)) \tag{20}$$

where

$$A_{m,n} = \frac{1}{N_m}\int_0^1 \psi_m'(x)\,\psi_n(x)\,\mathrm{d}x \tag{21}$$

The transformed initial condition for the first process is given by:

$$\overline{T}_{Hn}(0) = \int_0^1 (T_o(x) - T_{\text{in}})\psi_n \, \mathrm{d}x, \tag{22}$$

$$\overline{T}_{SHn}(0) = \int_0^1 (T_o(x) - T_{\text{in}})\psi_n \, \mathrm{d}x \tag{23}$$

and for the following processes are:

$$\overline{T}_{H,cold_n}(0) = \sum_{m=1}^{\infty} \frac{\overline{T}_{H,hot_m}(1) \int_0^1 \psi_m(1-x)\,\psi_n(x)\,\mathrm{d}x}{N_m} + \int_0^1 \psi_n(x)\,\mathrm{d}x \tag{24}$$

$$\overline{T}_{H,hot_n}(0) = \sum_{m=1}^{\infty} \frac{\overline{T}_{H,cold_m}(1) \int_0^1 \psi_m(1-x)\,\psi_n(x)\,\mathrm{d}x}{N_m} - \int_0^1 \psi_n(x)\,\mathrm{d}x \tag{25}$$

$$\overline{T}_{SH,cold_n}(0) = \sum_{m=1}^{\infty} \frac{\overline{T}_{SH,hot_m}(1) \int_0^1 \psi_m(1-x)\,\psi_n(x)\,\mathrm{d}x}{N_m} + \int_0^1 \psi_n(x)\,\mathrm{d}x \tag{26}$$

$$\overline{T}_{SH,hot_n}(0) = \sum_{m=1}^{\infty} \frac{\overline{T}_{SH,cold_m}(1) \int_0^1 \psi_m(1-x)\,\psi_n(x)\,\mathrm{d}x}{N_m} - \int_0^1 \psi_n(x)\,\mathrm{d}x \tag{27}$$

For this work, the discrete operator Λ is chosen to be:

$$\Lambda(f, x, \delta) = \frac{f(x) - f(x-\delta)}{\delta} \tag{28}$$

The solution of the transformed potentials is obtained by truncating the infinite system representation (19) to a finite order and employing a commercially or publicly available dedicated ODE solver. In this work, the NDSolve routine, available in the Mathematica software was employed.

5. RESULTS AND DISCUSSION

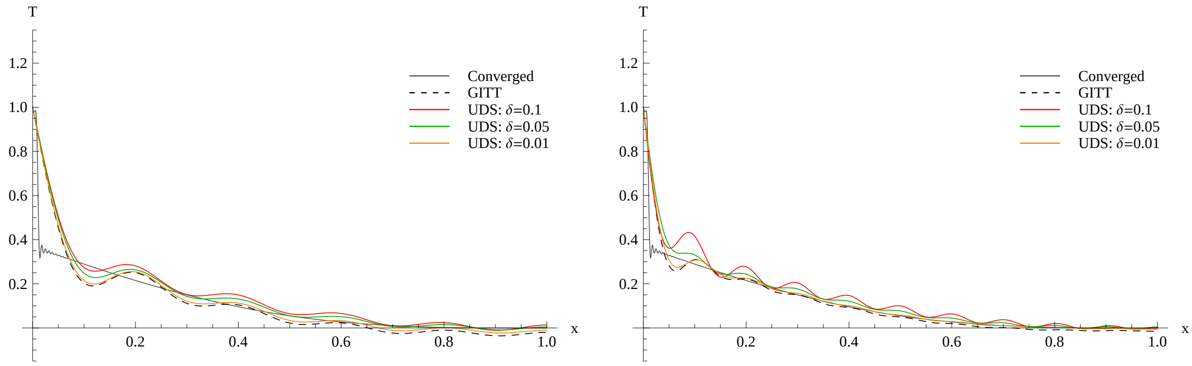
This section presents the results of the heat transfer problem of a counter flow regenerator solved by GITT and the proposed mixed formulation. All the results shown in this section were calculated for quasi-steady-state and with: $\tau = 10^{-3}$, $\text{NTU} = 10$ and $C_r^* = 1$. Although not showed here, in the final paper, a validation table will display the effectiveness convergence with the truncation order ($n_{\max}$) for GITT and a comparison of the final result with the literature (Shah and Sekulic, 2002). For code validation purpose, the table 1 shows the effectiveness convergence with the truncation order ($n_{\max}$) for GITT[†] and a comparison of the final result with the reference work (Shah and Sekulic, 2002)[‡].

Table 1. Effectiveness convergence for traditional GITT.

$n_{\max}$	$\varepsilon^{\dagger}$
50	0.7404
100	0.7392
150	0.7388
200	0.7386
250	0.7385
300	0.7384
Reference[‡]	0.7375

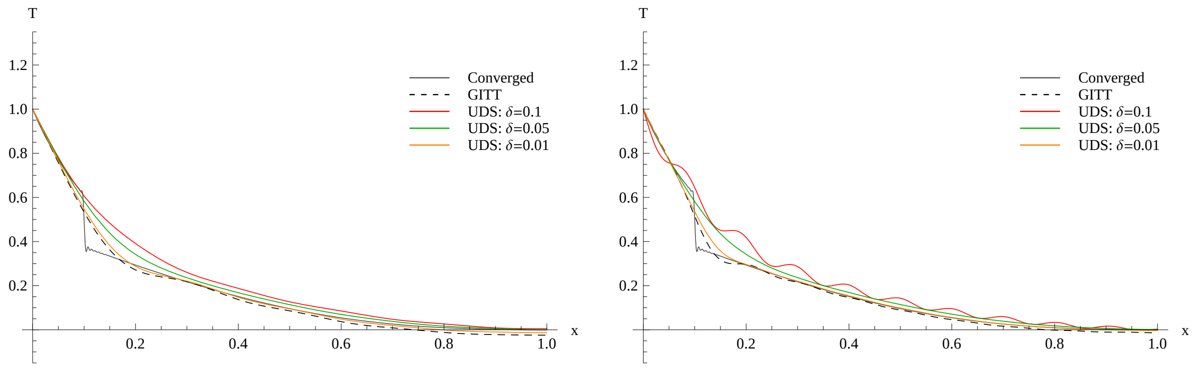
The following figures display the results for the temperature profiles for different values of the discretization parameter (δ), different times (t), different truncation order ($n_{\max}$). In all figures a fully-converged GITT solution is plotted in gray and traditional GITT in dashed black for comparisons.

Observing figure 1, it can be seen that in both subfigures, (a) and (b), that the orange curve, representing $\delta = 0.01$, is the one that most approaches the converged solution in gray. As can also be seen from these figures, as the δ parameter is increased the oscillations are reduced. However, a higher spreading (slope reduction) of the temperature profile at the wave front is seen due to the additional numerical diffusion introduced by the UDS approximation. In spite of these observations, as the size of the discretization parameter is reduced, the GITT-UDS curves converge to the GITT curve, such that the actual amount of numerical diffusion can be user-controlled. When comparing subfigures (a) and (b), one



(a) $n_{\max} = 10$ (b) $n_{\max} = 20$

Figure 1. Hot period fluid temperature profile for several step-sizes and $t = 0.00001$.



(a) $n_{\max} = 10$ (b) $n_{\max} = 20$

Figure 2. Hot period fluid temperature profile for several step-sizes and $t = 0.0001$.

clearly notices that the higher truncation order reduces the amplitude of oscilations and increases its frequency, due to the higher-order eigenvalues included in the series representation.

Figure 2 display similar results as the previous one. The first thing that can be notice is that for higher times the oscillation is lower. The observation of the curves leads to a conclusion that an optimum value of δ should depend on the truncation order, naturally decreasing with the value of $n_{\max}$. This can be specially seen on the red curves which with $n_{\max} = 10$ does not oscillate as much as it does when $n_{\max} = 20$. In other words, $\delta = 0.1$ is very large number for $n_{\max} = 20$.

6. CONCLUSION

In this work, a mixed formulation for solving convective heat transfer problems using integral transforms and upwind approximation for the advective term, called GITT-UDS, is proposed. In order to test and verify this new approach, the heat transfer problem of a counter flow regenerator is solved.The idea behind the methodology is to employ a discrete upwind approximation to the advective term to introduce numerical diffusion and hence causing a reduction of the oscillation in the solution.

The transformation of the whole problem with the approximated term was carried-out and a first-order upwind approximation (UDS) was introduced. The main objective of the proposed formulation was trying to improve the GITT convergence in regions where traditional GITT solution lead to a highly oscillatory behavior. This is achieved by introducing numerical diffusion to the advective terms, as the GITT has better results for diffusion problems.

It was observed that the solutions with discrete advective term, UDS produced a significant reduction in the oscillation of the solutions. Despite the reduction of oscillations provided by the proposed scheme, it became clear that an appropriate value of the discretization parameter must be selected to avoid an exceedingly spreading of the solution due to numerical diffusion. The results indicated that this value must decrease with an increasing truncation order; however, it can also depends on other parameters such as the dimensionless time. As a result, an important suggestion for further research involves the determination of the optimum value of δ. One can even propose a variable δ, which is function of x, in order to adapt the approach locally inside the domain. This proposition is similar to the one used with the flux limiters introduced by TVD (Total Variation Diminishing) schemes.

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8. REFERENCES

Almeida, G.L., Pimentel, L.C.G. and Cotta, R.M., 2008. "Integral transform solutions for atmospheric pollutant dispersion". *Environmental Modeling and Assessment*, Vol. 13, No. 1, pp. 53–65.

Castellões, F.V., Quaresma, J.N.N. and Cotta, R.M., 2010. "Convective heat transfer enhancement in low Reynolds number flows with wavy walls". *International Journal of Heat and Mass Transfer*, Vol. 53, No. 9-10, pp. 2022–2034.

Coppage, J.E. and London, A.L., 1953. "The periodic-flow regenerator-a summary of design theory". *Transactions of the ASME*, Vol. 75, pp. 779–787.

Cotta, R.M., 1993. *Integral Transforms in Computational Heat and Fluid Flow*. CRC Press, Boca Raton, FL.

de B. Alves, L.S., Cotta, R.M. and Pontes, J., 2002. "Stability analysis of natural convection in porous cavities through integral transforms". *International Journal of Heat and Mass Transfer*, Vol. 45, No. 6, pp. 1185–1195.

de Barros, F.P.J. and Cotta, R.M., 2007. "Integral transforms for three-dimensional steady turbulent dispersion in rivers and channels". *Applied Mathematical Modelling*, Vol. 31, pp. 2719–2732.

de Lima, G.G.C., Santos, C.A.C., Haag, A. and Cotta, R.M., 2007. "Integral transform solution of internal flow problems based on navier-stokes equations and primitive variables formulation". *International Journal for Numerical Methods in Engineering*, Vol. 69, No. 3, pp. 544–561.

Hirata, S.C., Couto, P., Lara, L.G. and Cotta, R.M., 2009. "Modeling and hybrid simulation of slow discharge process of adsorbed methane tanks". *International Journal of Thermal Sciences*, Vol. 48, No. 6, pp. 1176–1183.

Knupp, D., Naveira-Cotta, C. and Cotta, R., 2012. "Theoretical analysis of conjugated heat transfer with a single domain formulation and integral transforms". *International Communications in Heat and Mass Transfer*.

Lima, J.A., Quaresma, J.N.N. and Macêdo, E.N., 2007. "Integral transform analysis of MHD flow and heat transfer in parallel-plates channels". *International Communications in Heat and Mass Transfer*, Vol. 34, pp. 420–431.

Matt, C., 2009. "On the application of generalized integral transform technique to wind-induced vibrations on overhead conductors". *International Journal for Numerical Methods in Engineering*, Vol. 78, No. 8, pp. 901–930. ISSN 1097-0207.

Monteiro, E.R., Macedo, E.N., Quaresma, J.N.N. and Cotta, R.M., 2009. "Integral transform solution for hyperbolic heat conduction in a finite slab". *International Communications in Heat and Mass Transfer*, Vol. 36, No. 4, pp. 297–303.

Naveira-Cotta, C.P., Cotta, R.M., Orlande, H.R.B. and Fudym, O., 2009. "Eigenfunction expansions for transient diffusion in heterogeneous media". *International Journal of Heat and Mass Transfer*, Vol. 52, pp. 5029–5039.

Paz, S.P.A., Macedo, E.N., Quaresma, J.N.N. and Cotta, R.M., 2007. "Eigenfunction expansion solution for boundary-layer equations in cylindrical coordinates: simultaneously developing flow in circular tubes". *Numerical Heat Transfer, Part A: Applications*, Vol. 52, No. 12, pp. 1123–1149.

Scofano-Neto, F. and Cotta, R.M., 1992. "Lumped-differential analysis of concurrent flow double-pipe heat exchanger". *The Canadian Journal of Chemical Engineering*, Vol. 70, No. 3, pp. 592–595.

Scofano-Neto, F. and Cotta, R.M., 1993. "Improved hybrid lumped-differential formulation for double-pipe heat-exchanger analysis". *Journal of Heat Transfer (ASME)*, Vol. 115, No. 4, pp. 921–927.

Shah, R.K. and Sekulic, D.P., 2002. *Fundamentals of Heat Exchanger Design*. John Wiley & Sons, New York, NY.

Silva, R.L., Quaresma, J.N.N., Santos, C.A.C. and Cotta, R.M., 2011. "Integral transforms solution for flow development in wavy wall ducts". *International Journal of Numerical Methods for Heat & Fluid Flow*, Vol. 21, No. 2, pp. 219–243. ISSN 0961-5539.

Sphaier, L.A., 2012. "Integral transform solution for heat transfer in parallel-plates micro-channels: Combined electroosmotic and pressure driven flows with isothermal walls". *International Communications in Heat and Mass Transfer*, Vol. DOI: 10.1016/j.icheatmasstransfer.2012.05.010. Accepted for Publication in in May 2012.

Sphaier, L.A. and Cotta, R.M., 2002. "Analytical and hybrid solutions of diffusion problems within arbitrarily shaped regions via integral transforms". *Computational Mechanics*, Vol. 29, No. 3, pp. 265–276.

Sphaier, L.A., Cotta, R.M., Naveira-Cotta, C.P. and Quaresma, J.N.N., 2011. "The UNIT algorithm for solving one-dimensional convection-diffusion problems via integral transforms". *International Communications in Heat and Mass Transfer*, Vol. 38, pp. 565–571.

9. RESPONSIBILITY NOTICE