ANALYSIS OF FREE CONVECTION UTILIZING A SECOND-ORDER TIME ACCURATE FINITE ELEMENT FORMULATION STABILIZED BY LOCAL TIME-STEPS

Curi, M.F., marcosfilardy@hotmail.com
Laboratory of Transmission and Technology of Heat, Federal University of Rio de Janeiro, P.O. Box 68503, Rio de Janeiro, RJ 21945-970, Brazil

De Sampaio, P.A.B., sampaio@ien.gov.br
Nuclear Engineering Institute/CNEN, P.O. Box 68550, Rio de Janeiro, RJ 21945-970, Brazil

Gonçalves Jr, M.A., milton.a.g.j@gmail.com
Center for Parallel Computations, Federal University of Rio de Janeiro, P.O. Box 68516, Rio de Janeiro, RJ 21945-970, Brazil

Abstract. *A stabilized finite element method for the solution of viscous flow and heat transfer is presented. An equation for pressure is derived from a second-order time accurate Taylor-Galerkin procedure that combines the mass and the momentum conservation laws. At each time-step, once the pressure has been determined, the velocity field and the temperature field are computed solving discretized equations obtained from another second-order time accurate scheme and a least-squares minimization of momentum and energy residual. Thus, the procedure leads to a stabilized finite element method suitable for the simulation of heat transfer problems in free convection. The terms that stabilize the finite element method arise naturally from the process, rather than being introduced a priori in the variational formulation. Local time-steps, chosen according to the time-scales of convection-difusion of momentum and energy, play the role of stabilization parameters. Numerical solutions of some representative examples demonstrate the application of the proposed stabilized formulation, where good agreement with previously published experimental and computational results have been obtained.*

Keywords: *Finite element method, computational fluid dynamic, stabilized finite element method, free convection, second-order time accurate methods*

1. INTRODUCTION

In this work a finite element method for quasi-incompressible viscous flows and heat transfer is presented. In a recent work (Gonçalves Jr. and De Sampaio, 2010) the authors presented a stabilized finite element formulation for the solution of incompressible flows, where the time discretization precedes the spatial discretization. In this paper we extend those ideas to derive a stabilized finite element method suitable for dealing with free convection flows. Here a finite element method for quasi-incompressible viscous flows and heat transfer is presented. However, here the time discretizations employed are improved to second order accuracy. In the present method an equation for pressure is derived from a second-order time accurate Taylor-Galerkin procedure that combines the mass and the momentum conservation laws.

At each time-step, once the pressure has been determined, the velocity and temperature fields are computed solving discretized equations obtained from another second-order time accurate scheme and a least-squares minimization of spatial momentum and energy residuals. In order to introduce the correct amount of stabilization everywhere on the domain of analysis, the time-step must be defined locally, leading to spatially varying time-step distributions. The procedure proposed in De Sampaio (2006) is followed. In this case the use of local time-steps and the required synchronization scheme are embedded in the method. The result is a method that resembles well known stabilized formulations that employ a single time-step for the whole domain and a local definition of stabilization parameters, but whose origins are based on the use of local time-steps combined with a synchronization scheme.

The terms that stabilize the finite element method, controlling wiggles and circumventing the Babuška-Brezzi condition (Brezzi and Fortin, 1991), arise naturally from the process, rather than being introduced a priori in the variational formulation. Generating the desired stabilization effect, without compromising the consistency of the approximation. For stabilization formulations details see De Sampaio (1991, 1993, 2005, 2006).

The method demonstrated good agreement with previously published experimental (Churchill and Chu, 1975 and Hyman et al., 1953) and numerical results (De Vahl Davis, 1983 and Barakos et al., 1994).

2. PHYSICAL MODEL

We consider a continuum model for quasi-incompressible viscous flows including buoyancy forces and heat transfer. The problem is defined on the open bounded domain Ω, with boundary Γ, contained in nsd-dimensional Euclidean space. The flow is governed by the quasi-incompressible Navier-Stokes equations and an energy convection-diffusion equation. These are written using the summation convention for *a=1,2,...,nsd e b=1,2,...,nsd,* in Cartesian coordinates:

$$\rho\left[\frac{\partial u_a}{\partial t}+u_b\frac{\partial u_a}{\partial x_b}\right]-\frac{\partial \tau_{ab}}{\partial x_b}+\frac{\partial p}{\partial x_a}+\rho\beta g_a\left(T-T_0\right)=0 \tag{1}$$

$$\frac{1}{\rho c_s{}^2}\frac{\partial p}{\partial t}+\frac{\partial u_a}{\partial x_a}=0 \tag{2}$$

$$\rho c\left[\frac{\partial T}{\partial t}+u_b\frac{\partial T}{\partial x_b}\right]+\frac{\partial q_b}{\partial x_b}=0 \tag{3}$$

The dependent variables are the velocity, pressure and temperature fields represented by u_a, p and T, respectively. The sound speed is denoted by c_s. The fluid specific heat is represented by c. Note that the viscous stress is given by $\tau_{ab}=\mu\left(\partial u_a/\partial x_b+\partial u_b/\partial x_a\right)$, where μ is the fluid viscosity. The heat flux is given by $q_b=-\kappa\,\partial T/\partial x_b$, where κ is the fluid thermal conductivity. The fluid density (at the reference temperature T_0) is denoted by ρ. The volumetric expansion coefficient of the fluid is $\beta=-\rho^{-1}\,\partial\rho/\partial T$

Velocity and traction boundary conditions are prescribed by given data on non-overlapping boundary partitions Γ_{ua} and Γ_{ta}, such that $\Gamma_{ua}\bigcup\Gamma_{ta}=\Gamma$, according to:

$$u_a=\overline{u_a}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_{ua}, \tag{4}$$

$$\left(-p\delta_{ab}+\tau_{ab}\right)n_b=\overline{t_a}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_{ta}, \tag{5}$$

where δ_{ab} is the Kronecker delta and n_b denotes Cartesian components of the outward normal vector at the boundary.

Temperature and heat flux boundary conditions are prescribed by given data on non-overlapping boundary partitions Γ_T and Γ_q, such that $\Gamma_T\bigcup\Gamma_q=\Gamma$, according to:

$$T=\overline{T}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_T, \tag{6}$$

$$q_b n_b=\overline{q}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_q, \tag{7}$$

Pressure and normal velocity conditions are prescribed by given data on non-overlapping boundary partitions Γ_p and Γ_G, such that $\Gamma_p\bigcup\Gamma_G=\Gamma$, according to:

$$p=\overline{p}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_p, \tag{8}$$

$$u_b n_b=\overline{G}(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_G. \tag{9}$$

2.1. Governing Equations In Non-Dimensional Form

Here the variables are non-dimensionalized with respect to reference scales conveniently chosen from the problem data. The non-dimensional velocity, pressure and temperature are represented by $u'_a=u_a/u_0$, $p'=p/\rho u_0^2$ and $T'=\left(T-T_0\right)/\left(T_{max}-T_{min}\right)$, respectively. Note that u_0 is the velocity reference scale and $T_{\max}$ and $T_{\min}$ are the maximum and minimum temperatures in the problem. The spatial co-ordinates are non-dimensionalized with respect to the reference length L, i.e., $x'_a=x_a/L$. The non-dimensional time is represented by $t'=t\,u_0/L$. The gravity field is non-dimensionalized with respect to its modulus $g'_a=g_a/\|\mathbf{g}\|$.

In terms of the non-dimensional variables the governing equations become:

$$\frac{\partial u'_a}{\partial t'}+u'_b\frac{\partial u'_a}{\partial x'_b}-\frac{1}{Re}\frac{\partial}{\partial x'_b}\left(\frac{\partial u'_a}{\partial x'_b}+\frac{\partial u'_b}{\partial x'_a}\right)+\frac{\partial p'}{\partial x'_a}+Rig'_aT'=0 \tag{10}$$

$$M^2\frac{\partial p'}{\partial t'}+\frac{\partial u'_a}{\partial x'_a}=0 \tag{11}$$

$$\frac{\partial T'}{\partial t'}+u'_b\frac{\partial T'}{\partial x'_b}-\frac{1}{RePr}\frac{\partial}{\partial x'_b}\left(\frac{\partial T'}{\partial x'_b}\right)=0 \tag{12}$$

where $M=u_0/c_s$ is the Mach number, $Re=\rho\|\mathbf{u}\|L/\mu$ is the Reynolds number, $Ri=\beta(T_{\max}-T_{\min})\|\mathbf{g}\|L/u_0^2$ is the Richardson number, $Pr=c\mu/\kappa$ is the Prandtl number.

For free convection, we have to obtain the velocity time scale indirectly, defining it as $u_0=\sqrt{\beta(T_{\max}-T_{\min})\|\mathbf{g}\|L}$. Thus, the Reynolds and Richardson numbers that appear in the non-dimensional equations become $Ri=1$ and $Re=\sqrt{Ra/Pr}$, respectively, where $Ra=\rho^2c\|\mathbf{g}\|\beta(T_{\max}-T_{\min})L^3/\mu\kappa$ is the Rayleigh number. The non-dimensional boundary conditions remain the same forms presented previously.

3. STABLE FINITE ELEMENT FORMULATION

3.1. Pressure Equation

To obtain an equation for pressure update, we use a Taylor series for pressure in time. The spatial discretization is performed with Lagrangian linear triangular elements in 2D. For the problem variables, we have: $\hat{u}_a^n=N_ju_{aj}^n$, $\hat{T}^n=N_jT_j^n$, $\hat{p}^n=N_jp_j^n$, and $\Delta\hat{p}^n=N_j\Delta p_j$. The pressure change during the time step Δt is represented by $\Delta p=p^{n+1}-p^n$, where the superscript n e $n+1$ indicates the time level. Note that N_j represents the shape functions of the finite element and the variables with the subscript j are nodal values.

Employing the classical Galerkin method, using Green's identity and introducing the boundary conditions we obtain:

$$\int_\Omega M^2N_i\frac{\Delta\hat{p}}{\Delta t}d\Omega+\int_\Omega\frac{\Delta t}{2}\frac{\partial N_i}{\partial x_a}\frac{\partial\Delta\hat{p}}{\partial x_a}d\Omega=-\int_\Omega N_i\frac{\partial\hat{u}_a^n}{\partial x_a}d\Omega-\int_\Omega\frac{\Delta t}{2}\frac{\partial N_i}{\partial x_a}\left[\hat{u}_b^n\frac{\partial\hat{u}_a^n}{\partial x_b}+\frac{\partial\hat{p}^n}{\partial x_a}+Rig_a\hat{T}^n\right]-\int_{\Gamma_G}\frac{N_i}{2}\left(\overline{G}^{n+1}-\overline{G}^n\right)d\Gamma \tag{13}$$

Mathematical details are explored in Gonçalves Jr. and De Sampaio (2010). Substituting $\Delta\hat{p}=N_j\Delta p_j$ in Eq. (13), obtain a symmetric equation system for calculating the nodal values of the pressure update:

$$\lfloor\mathbf{A}_{\mathbf{pp_{ij}}}\rfloor\{\mathbf{\Delta p_j}\}=\{\mathbf{F_{p_i}}\} \tag{14}$$

3.2. Velocity and Temperature Equations

Once the pressure field has been determined, we use a second order accurate time discretization for the momentum and energy balance to obtain equations for the velocity and temperature update.

Discretizations with respect to time are given by:

$$\frac{\Delta u_a}{\Delta t_M}+\frac{1}{2}\left(u_b^n\frac{\partial\Delta u_a}{\partial x_b}+\Delta u_b\frac{\partial u_a^n}{\partial x_b}\right)-\frac{1}{2Re}\frac{\partial}{\partial x_b}\left(\frac{\partial\Delta u_a}{\partial x_b}+\frac{\partial\Delta u_b}{\partial x_a}\right)+\frac{Rig_a\Delta T}{2}=-\left[u_b^n\frac{\partial u_a^n}{\partial x_b}+\frac{\partial p^{n+1/2}}{\partial x_a}-\frac{\partial\tau_{ab}^n}{\partial x_b}+Rig_aT^n\right]+0\left(\Delta t^2\right) \tag{15}$$

$$\frac{\Delta T}{\Delta t_E}+\frac{1}{2}\left(u_b^n\frac{\partial\Delta T}{\partial x_b}+\Delta u_b\frac{\partial T^n}{\partial x_b}\right)-\frac{1}{2RePr}\frac{\partial}{\partial x_b}\left(\frac{\partial\Delta T}{\partial x_b}\right)=-\left[u_b^n\frac{\partial T^n}{\partial x_b}+\frac{\partial q_b^n}{\partial x_b}\right]+0\left(\Delta t^2\right) \tag{16}$$

The velocity change during the time step Δt_M is represented by $\Delta u_a = u_a^{n+1} - u_a^n$. The temperature change during the time step Δt_E is $\Delta T = T^{n+1} - T^n$. The pressure field at time level $n+1/2$ is written as $p^{n+1/2} = p^n + \Delta p/2$.

Consider the following spatial discretization of the problem variables: $\hat{u}_a^n = N_j u_{aj}^n$, $\Delta\hat{u}_a = N_j \Delta u_{aj}$, $\hat{T}^n = N_j T_j^n$ and $\Delta\hat{T} = N_j \Delta T_j$. Using the discretized field of variables we can write, after minimizing the squared discretization residuals of Equation. (15) and (16) and combining traction and heat flux boundary conditions (Curi et al., 2011), the following expressions:

$$\int_\Omega \left[N_i + \frac{\Delta t_M}{2} \hat{u}_b^n \frac{\partial N_i}{\partial x_b} \right] \hat{R}_c d\Omega + \int_\Omega \frac{\Delta t_M}{2} \frac{\partial \hat{u}_a^n}{\partial x_c} N_i \hat{R}_a d\Omega + \int_\Omega \frac{\Delta t_E}{2} \frac{\partial \hat{T}^n}{\partial x_c} N_i \hat{E} d\Omega + \int_{\Gamma_{t_c}} N_i \left[(-p\delta_{cb} + \tau_{cb}) n_b - \bar{t}_c(\mathbf{x},t) \right]^{n+1/2} d\Gamma = 0 \quad \forall \text{ free } \Delta u_{ci} \tag{17}$$

$$\int_\Omega \left[N_i + \frac{\Delta t_E}{2} \hat{u}_b^n \frac{\partial N_i}{\partial x_b} \right] \hat{E} d\Omega + \int_\Omega \frac{\Delta t_M}{2} R_i g_a N_i \hat{R}_a d\Omega + \int_{\Gamma_q} N_i \left[\bar{q}(\mathbf{x},t) - q_b n_b \right]^{n+1/2} d\Gamma = 0 \quad \forall \text{ free } \Delta T_i \tag{18}$$

Note that the weight functions present in the first terms of these Eqs. (17) and (18) have the same structure as the SUPG weight function method of Brooks and Hughes (1982). The remaining weight functions, affecting the second and third term in Eq. (17) and affecting the second term in Eq. (18), are specific of the method presented here.

Using Green's identity, in Eqs. (17) and (18) we obtain a symmetric equation system to solve for the nodal values of velocity and temperature update. For 2D problems, we have the following system:

$$\begin{bmatrix} \mathbf{A_{uu\,ij}} & \mathbf{A_{uv\,ij}} & \mathbf{A_{uT\,ij}} \\ \mathbf{A^T_{uv\,ij}} & \mathbf{A_{vv\,ij}} & \mathbf{A_{vT\,ij}} \\ \mathbf{A^T_{uT\,ij}} & \mathbf{A^T_{vT\,ij}} & \mathbf{A_{TT\,ij}} \end{bmatrix} \begin{Bmatrix} \mathbf{\Delta u_j} \\ \mathbf{\Delta v_j} \\ \mathbf{\Delta T_j} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_{u\,i}} \\ \mathbf{F_{v\,i}} \\ \mathbf{F_{T\,i}} \end{Bmatrix} \tag{19}$$

It is important to note that the time-discretization is performed before the spatial discretization, which is performed using standard finite elements C_0. The terms multiplied by Δt_M and Δt_E in Eqs. (17) and (18) are responsible for controlling the spatial oscillations (wiggles) in convection-dominated flows, and stabilize the computation, regardless of the restrictions Babuška-Brezzi on the choice of interpolation spaces for velocity and temperature. In particular, the use of the equal order of interpolation for all variables adopted here become possible through a proper choice of Δt_M and Δt_E. It is important to remark that rather than being proposed a priori, the stabilization terms appear naturally in this method from least squares minimization of the time-discretized momentum and energy square residuals with respect to the temperature and velocity degrees of freedom (with free nodal values).

3.3 Local Time-Steps and Synchronization

In this paper we propose an alternative way to choose the time step. Instead of using the method proposed by De Sampaio (1991, 1993, 2005) we chose the time step according to the minimum values of the characteristic time scales of convection and diffusion , i.e., $\Delta t_M = \min(t_c, t_{dM})$ and $\Delta t_E = \min(t_c, t_{dE})$, where $td_M = \rho h_e^2/6\mu$ and $td_E = \rho h_e^2/6\kappa$, are the momentum and energy diffusion time-scales respectively, and $tc = h_e/\|\mathbf{u}^n\|$ is the convection time scale. Here h_e is the mesh with local size (De Sampaio, 1991).

Because we have optimal time-steps that vary with position and according to the quantity transported (momentum or energy), we have to resort to a special scheme to synchronize the time advance of the computation. In this paper, we adopt the procedure introduced in De Sampaio (2006). It is based on selecting a synchronization time-step Δt^*, which will be the same for all flow variables and for all domain (in fact, the usual concept for a time-step).

The synchronization time-step is chosen to be quite close to the minimum problem time-step and is calculated as $\Delta t^* = 0.999 \min(\Delta t_M, \Delta t_E)$. The time-step Δt^* is the time step used for synchronizing the advancement of the numerical simulation. Let $\Delta\hat{u}_a$, $\Delta\hat{p}$ and $\Delta\hat{T}$ be the variable changes obtained when using the appropriate local time-steps to solve Eqs. (14) and (19). On the other hand, let us denote the variable changes from time t^n to the time $t^n + \Delta t^*$ (the synchronization time) as $\Delta\hat{u}_a^*$, $\Delta\hat{p}^*$ e $\Delta\hat{T}^*$. Thus, keeping the same rate of change, we have the following relations:

$$\frac{\Delta\hat{u}_a^*}{\Delta t^*} = \frac{\Delta\hat{u}_a}{\Delta t_M} \tag{20}$$

$$\frac{\Delta\hat{p}^*}{\Delta t^*} = \frac{\Delta\hat{p}}{\Delta t_M} \tag{21}$$

$$\frac{\Delta\hat{T}^*}{\Delta t^*} = \frac{\Delta\hat{T}}{\Delta t_E} \tag{22}$$

where $\Delta\hat{u}_a^* = \hat{u}_a(\mathbf{x}, t^n + \Delta t^*) - \hat{u}_a(\mathbf{x}, t^n)$, $\Delta\hat{u}_a = \hat{u}_a(\mathbf{x}, t^n + \Delta t_M) - \hat{u}_a(\mathbf{x}, t^n)$, $\Delta\hat{p}^* = \hat{p}(\mathbf{x}, t^n + \Delta t^*) - \hat{p}(\mathbf{x}, t^n)$, $\Delta\hat{p} = \hat{p}(\mathbf{x}, t^n + \Delta t_M) - \hat{p}(\mathbf{x}, t^n)$, $\Delta\hat{T}^* = \hat{T}(\mathbf{x}, t^n + \Delta t^*) - \hat{T}(\mathbf{x}, t^n)$, and $\Delta\hat{T} = \hat{T}(\mathbf{x}, t^n + \Delta t_E) - \hat{T}(\mathbf{x}, t^n)$.

In practice, the computation based on local time-steps and the synchronization phase does not have to be performed separately. The the synchronization phase, represented by Eqs. (20), (21) and (22), can be embedded in Equations (14) and (19). Thus, the synchronized solution at $t^n + \Delta t^*$ can be obtained directly solving the following symmetric equations:

$$\left[\mathbf{A}^*_{\mathbf{pp_{ij}}}\right]\left\{\Delta\mathbf{p}^*_{\mathbf{j}}\right\} = \left\{\mathbf{F_{p_i}}\right\} \tag{23}$$

$$\begin{bmatrix} \mathbf{A}^*_{\mathbf{uu\,ij}} & \mathbf{A}^*_{\mathbf{uv\,ij}} & \mathbf{A}^*_{\mathbf{uT\,ij}} \\ \mathbf{A}^{*\mathbf{T}}_{\mathbf{uv\,ij}} & \mathbf{A}^*_{\mathbf{vv\,ij}} & \mathbf{A}^*_{\mathbf{vT\,ij}} \\ \mathbf{A}^{*\mathbf{T}}_{\mathbf{uT\,ij}} & \mathbf{A}^{*\mathbf{T}}_{\mathbf{vT\,ij}} & \mathbf{A}^*_{\mathbf{TT\,ij}} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{u}^*_{\mathbf{j}} \\ \Delta\mathbf{v}^*_{\mathbf{j}} \\ \Delta\mathbf{T}^*_{\mathbf{j}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_{u\,i}} \\ \mathbf{F_{v\,i}} \\ \mathbf{F_{T\,i}} \end{Bmatrix} \tag{24}$$

The solution procedure is semi-segregated., in the sense that pressure is computed first and independently at each step but the solution of the velocity components and temperature are coupled, as shown in Equation (24). After each step the problem variables are updated and the computation proceeds until the specified final analysis time is reached. All equation systems in the present formulation involve symmetric positive-definite matrices. The equation system is solved using the conjugate gradient method with Jacobi pre-conditioner, in an Element-by-Element (EBE) implementation. The computer code used here was developed for the study of 2D problems, where linear triangles were used to interpolate all dependent variables. The code also uses dynamic adaptive finite element meshes generated using Bowyer's algorithm (Bowyer, 1981), guided by the error estimation of Zienkiewicz and Zhu (1987). Was also employed parallel programming optimized for high performance on distributed parallel computing systems.

4. NUMERICAL EXAMPLES

4.1. Free Convection around a Hot Cylinder

This example shows the external free convection flow that develops around a hot horizontal cylinder. The fluid in contact with the cylinder is initially at rest at temperature $T_{min} = T_0$. The temperature at the cylinder surface is $T_{max} = T_0 + \Delta T$. Non-slip velocity boundary conditions are applied at the cylinder surface. The cylinder diameter d is chosen as the reference scale for length. The numerical results are parameterized with respect to the Prandtl and Rayleigh numbers.

The transients ran from $t = 0$ to $t = 70d/u_0$, where $u_0 = \sqrt{\beta(T_{max} - T_{min})\|g\|d}$. We have performed simulations at $\mathrm{Pr} = 0.71$ and $\mathrm{Ra} = 10^4, 5\mathrm{x}10^4, 10^5, 1.5\mathrm{x}10^5, 2\mathrm{x}10^5, 5\mathrm{x}10^5, 10^6, 10^7$ and 10^8, using minimum element size from $0.02d$ until $0.001d$. At the end of all numerical simulations we obtained in the more refined meshes involved up to 300000 elements. Figure 1 presents the temperature iso-lines and final adaptive mesh at $t = 70d/u_0$ for $\mathrm{Ra} = 5\mathrm{x}10^4$ and $\mathrm{Pr} = 0.71$ with minimum element size as $h_{min} = 0.02d$.

We also compared the mean Nusselt number, on the cylinder surface with experimental correlations available in literature. According to Churchill and Chu (1975) and Hyman et al. (1953), experimental data in the laminar range of the mean Nusselt number on the cylinder surface is well correlated by the following correlations respectively:

$$\langle Nu \rangle = 0.36 + \frac{0.518 Ra^{1/4}}{\left[1 + \left(0.559/Pr\right)^{9/16}\right]^{4/9}} \tag{25}$$

$$\langle Nu \rangle = 0.53\left[\left(\frac{\Pr}{\Pr + 0.952}\right)\mathrm{Ra}\right]^{1/4} \tag{26}$$

Figure 2 presents a comparison of our predictions for the mean Nusselt number, given by $\langle Nu \rangle = \langle q_w \rangle d / \kappa \langle T_{max} - T_{min} \rangle$, where $\langle q_w \rangle$ is the mean heat flux from the cylinder, using all Rayleigh numbers with the experimental data provided from Eqs. (25) and (26). Note that we have obtained a fairly good agreement with the experimental correlations.

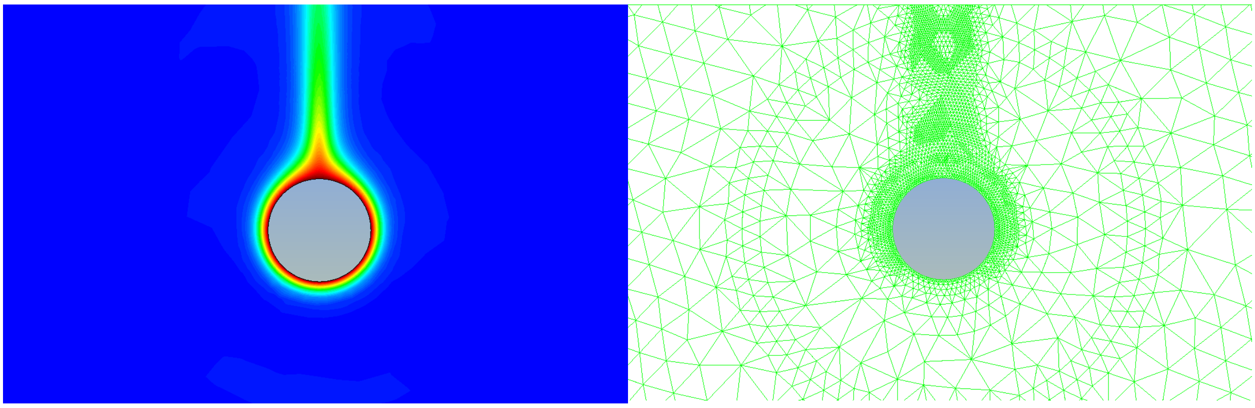

Figure1: Temperature field and final adaptive mesh at $t = 70d/u_0$ for $\mathrm{Ra} = 5x10^4$ and $\mathrm{Pr} = 0.71$

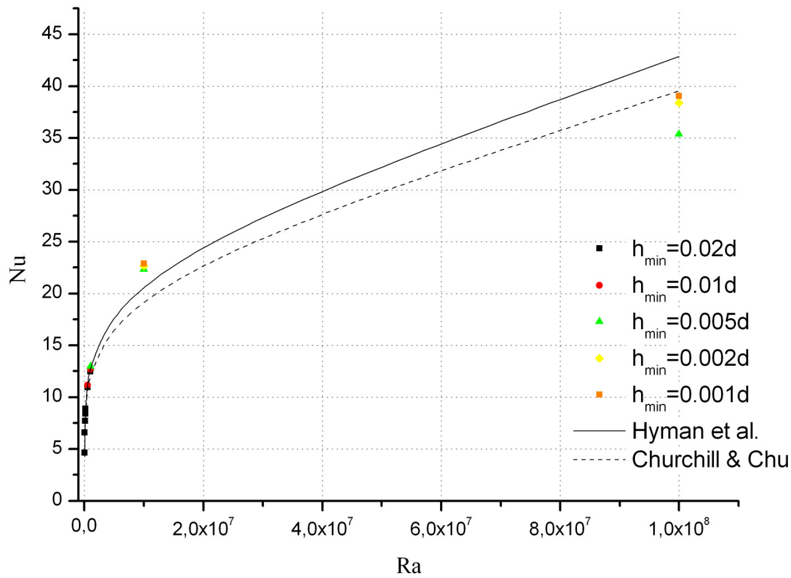

Figure 2: Comparison between the present results with those of Churchill and Chu ande Hyman et al..

4.2. Free Convection in a Square Cavity

We consider a square cavity centered on *x=0* and *y=0* with length L (reference spatial scale), thermally insulated at the top and bottom walls. A reference pressure $p = 0$ is imposed at the center of the cavity. Non-slip velocity boundary conditions are imposed on all walls. The cavity contains a fluid that is initially at rest at temperature T_0. The left wall temperature is $T = T_0 + \Delta T/2$ and the right is at temperature $T = T_0 - \Delta T/2$. These boundary conditions at opposite parallel walls generate a free convection flow inside the cavity and. Once again, the numerical results are parameterized with respect to the Prandtl and Rayleigh numbers.

The transients ran from $t = 0$ to $t = 60L/u_0$, where $u_0 = \sqrt{\beta(T_{max} - T_{min})\|g\|L}$, with minimum element size of $0.005L$.. In all cases this has been long enough to obtain convergence to steady-state. Figure 4 and Fig. 5 presents the final adaptive mesh, temperature field, pressure field and velocity field for $Ra = 10^5$ and $Ra = 10^6$ respectively. We could note that for sufficiently high Rayleigh numbers, thermal stratification occurs.

Table 1 compares our results for the mean, maximum and minimum Nusselt numbers with Barakos et al. (1994) and the benchmark provided by De Vahl Davis, (1983). The use of adaptive meshes in our computations allowed obtaining results that agree within less than 2.2% with the benchmark data using much finer meshes. We could note that the local Nusselt number increases with the Rayleigh number, as expected by the theory.

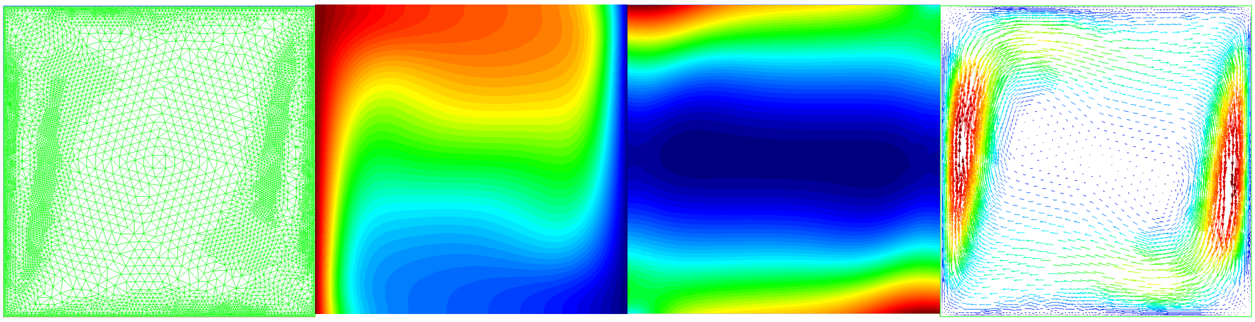

Figure 3: Adaptive mesh, temperature field, pressure field and velocity field for $Ra = 10^5$ and $Pr = 0.71$

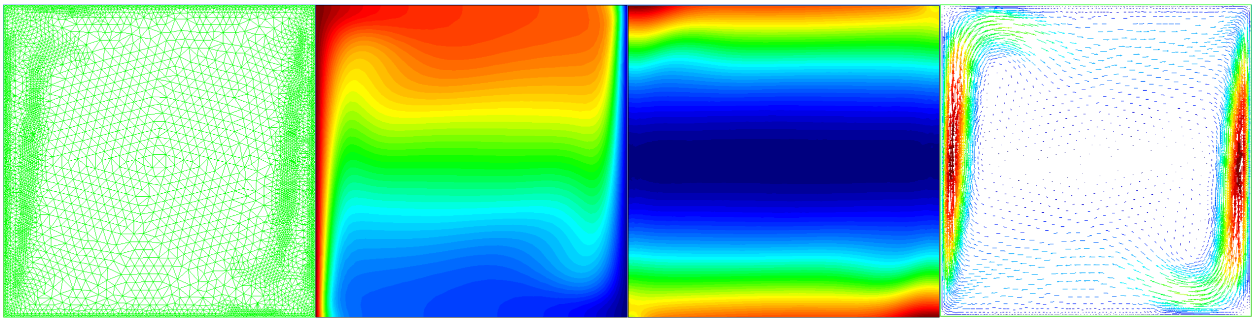

Figure 4: Adaptive mesh, temperature field, pressure field and velocity field for $Ra = 10^6$ and $Pr = 0.71$

Table 1: Mean and maximum Nusselt numbers: comparison between the present results with those of De Vahl Davis, (1983).

	$Ra = 10^4$			$Ra = 10^5$			$Ra = 10^6$		
	Nu max.	*Nu* min.	*Nu* mean	*Nu* max.	*Nu* min.	*Nu* mean	*Nu* max.	*Nu* min.	*Nu* mean
De Vahl Davis, (1983)	3.528	0.586	2.243	7.717	0.729	4.519	17.923	0.989	8.799
Barakos et al. (1994)	3.539	0.583	2.245	7.636	0.773	4.510	17.442	1.001	8.806
Present method	**3.529**	**0.584**	**2.244**	**7.723**	**0.721**	**4.519**	**17.528**	**0.971**	**8.807**

5. CONCLUDING REMARKS

A second-order time accurate finite element formulation has been presented. The mass and momentum balances have been combined in a Taylor series for pressure. This is discretized in space with the Galerkin method and results in an equation suitable for computing the pressure update. Momentum balance and energy balance time-discretization are carried out with finite differences. A least square minimization of spatial residuals is performed to obtain equations for the velocity and temperature update. The proposed method introduces automatically the stabilization terms required to control wiggles in convection dominated problems and for circumventing Babuška-Brezzi restrictions on the choice of interpolating spaces for velocity and pressure. The approach leads to a partially coupled system, where pressure degrees of freedom are solved first and then the velocity and temperature degrees of freedom are computed simultaneously. The update of velocity components and temperature are obtained solving the coupled equation system shown in Equation (24). Numerical examples have been presented, covering free convection flow and heat transfer. Comparison of our results with the benchmark numerical solutions and with experimental heat transfer data shows the good performance of the stabilized formulation proposed here.

6. ACKNOWLEDGEMENTS

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8. RESPONSIBILITY NOTICE

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