Pseudospectral and Finite Volume Solutions of Burgers Equation

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Abstract *The Finite Volume method (FVM) and Fourier Pseudospectral method (FPM) are applied to solve non-linear evolution equation, named Burgers equation, with a viscosity ν= 0.2 [m^2/s] and advecting with the velocity c=4.0 [m/s]. The goal is to determine the behavior of the solutions periodic and of non-periodics boundary conditions with both methods. Towards the FVM, was used the upwind and central-difference interpolation scheme. Which was possible to show in periodic domain, with central-difference scheme, reach the better results than upwind scheme. In simulations obtained by FPM, the results confirm the high accuracy of Fourier Pseudospectral method and reach round-off errors only periodic domain. However non-periodic domain using immersed boundary method attaining second order convergence.*

Keywords: *Burgers equation, Fourier Pseudospectral method, Finite Volume method*

1. INTRODUCTION

The Burgers equation serves as a useful model for many interesting problems in applied mathematics. It models effectively several problems of a fluid flow nature, in which either shocks or viscous dissipation is a significant factor, (Smith, 1997). That equation has two specific regions: the high and low property variations, as shown at Fig. 5.

The Burgers equation consists by partial differential equation (PDE) in one spatial dimension. Which is similar to one-dimensional Navier-Stokes equation without the pressure term. Its name from the extensive research of Burgers (1939). It is a very important fluid dynamical model both the conceptual understanding of a class of physical flows and for testing numerical algorithms (Zhang *et al.*, 1997). In order to investigate the two differents numeric methodologies, finite volumes and Fourier pseudospectral, the Burgers equation was solved.

The finite volume method (FVM) is a discretization method which is well suited for the numerical simulation of various kinds of conservation laws. It is normaly used in several engineering fields: fluid mechanics, heat and mass transfer, among others. This methodogly is based on approach to physics of the problem represented by the PDE. The solution strategy of the FVM, is divide the domain into a number of control volume that corresponds to the mesh cells as Fig. 1 (Maliska, 1995; Patankar, 1980),

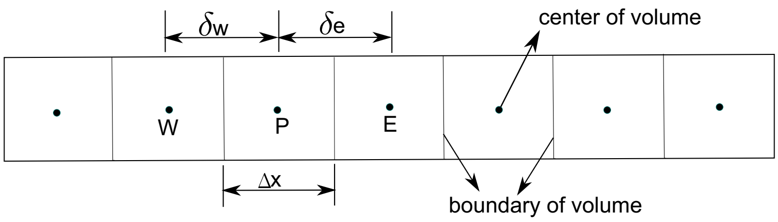

Figure 1. Scheme of the domain divisions performed with the finite volume method.

In Figure 1 the E is east, W west and P is "central-volume"; the Δx is the distance between the boundary volume and δx_e and δx_w are the distance between center of volumes. The approximate equations are obtained through a balance of property involved conservation, which are integrate at the control volume.

Fourier spectral method (FPM) is based at solution at integration the term of Fourier series (DFT) along all discrete domain. The DFT is used to evaluate spatial derivatives in place of conventional finite volume. In according of Roache (1978) the use of the DFT over M nodepoints corresponds to using M-th order trigonometric interpolation to evaluate the derivatives. This procedure is of "infinite order", in the sense that it may be shown to converge faster than any finite volume expression when all derivatives are continuous. Although it is, to require only periodic boundary condition.

FPM use physical and spectral domains, which are co-related by reciprocity relation. The spectral methods applied to problems with smooth solutions attain high order of spatial convergency rates, because it use all collocation points to calculate a derivative in one point, Fig. 2 (Basdevant, 1984; Canuto *et al.*, 2006).

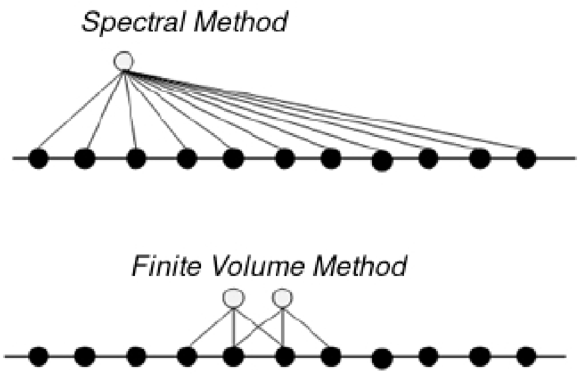

Figure 2. Scheme of spectral method and finite volume method used at derivative operations.

The aim of this paper is report, discuss and compare: the convergence order and accuracy, of numerical results of both methods. Thereunto in Burger equation was used the periodic boundary condition, in the domain (-π:π), and non-periodic boundary condition, in domain (-$\frac{\pi}{2}$:$\frac{\pi}{2}$). For this boundary condition was used the immersed boundary only at FPM. Due to it requeres solely the pereiodic boundary conditions.

2. THE BURGERS EQUATION ANALYTICAL SOLUTION

The one-dimensional Burgers equation is given by Eq.(1), where the u is the velocity profile [m/s], x is the position of domain [m] and ν is the viscosity coefficient [m^2/s]. The initial condition is given by Eq.(2), where the c is the advective velocity. The boundary conditions are two cases: the first is periodic and second one is non-periodic, imposed by immersed boundary method..

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \qquad in \quad \Omega, \qquad \forall t > 0. \tag{1}$$

$$u(x,0) = c + u_b(x,0) \tag{2}$$

The equation (3) is analytical solution, as sugested in Canuto *et al.* (2006);

$$u_b(x,t) = -2\nu\frac{\dfrac{\partial\phi}{\partial x}(x-ct,t+1)}{\phi(x-ct,t+1)}; \qquad \phi(x,t) = \sum_{n=-\infty}^{\infty} e^{\dfrac{-[x-(2n+1)\pi]^2}{4\nu t}} \tag{3}$$

3. NUMERICAL METHODS

3.1 Finite volume method

The finite volume method uses the integration around the control volume, Fig. 1. Accordingly with, Eq.(4) the Eq.(1) is integrated at space and time.

$$\frac{(u^{t+\Delta t} - u^t)}{\Delta t} = \int_w^e \frac{NLT}{\Delta x} - \nu\Big(\frac{u_E - u_P}{\delta x_e} - \frac{u_P - u_W}{\delta x_w}\Big)\frac{1}{\Delta x} \tag{4}$$

The non linear term at the Eq.(1) can be study by three differents scheme: advective $u(\nabla u)$, divergent $\frac{1}{2}(\nabla uu)$ and skew-simetric $\frac{1}{2}[u\nabla u + \frac{1}{2}(\nabla uu)]$, in agreements with Souza (2005) this work uses the skew-simetric form due to the stability.

Although the FVM used in this paper is second order spacial derivates. The time discretization scheme is explicit, and the time advance was is the classical fourth order Runge-Kutta as Canuto *et al.* (2006).

The non-linear term, discretizated by the skew-simetric scheme aforementioned is given by Eq.(5):

$$TNL = \frac{u_P(u_e - u_w)}{2\Delta x} + \frac{u_e u_e - u_w u_w}{2\Delta x} \tag{5}$$

Where the subscript e, w are the position of east and west face the principal volume ,Fig. (1). The u_P is the velocity at the main volume, and the u_e, u_w are the velocities at the faces. These velocities were obtained by upwind or central-difference scheme.

The upwind scheme assumes that value of the velocity u at an interface is equal to the value of the u upstream.

$$\begin{cases} if \quad u > 0 \quad u_e = u_P \\ if \quad u < 0 \quad u_e = u_E \end{cases} \tag{6}$$

Its the same to the west side. The central-difference is between the propriety at the central-behind volume and central-front volume. The velocity of the non-linear term is obttained by Eq.(7):

$$u_w = \frac{1}{2}(u_W + u_P) \qquad\qquad u_e = \frac{1}{2}(u_E + u_P) \tag{7}$$

3.2 Fourier Pseudospectral method

The solution of the Eq.(1) using Fourier pseudospectral method, is given by Eq.(8).

$$\frac{\widehat{u}^{t+\Delta t} - \widehat{u}^t}{\Delta t} = -ik\widehat{u * u} + (ik)^2\widehat{u} \tag{8}$$

Where $\widehat{u}$ is the velocity transformated to Fourier space using the DFT, k is the wave number, and i is the complex number $\sqrt{-1}$. It can be noted the convolution product at NLT, is use the Fourier pseudospectral method.

The advective term at Eq.(8) results at integral convolution, its requers high computational cost (memory storage and CPU time). Then to avoid this, we used the pseodospectral method. Therefore in the pseudospectral method was implemented at skew-simetric form, following sequencies:

First calculate de divergent form, $\frac{1}{2}(\nabla uu)$, likes :

1. calculate the multiplicate the velocity at physical space, as *uu*;

2. transform the product to spectral space, as $\widehat{\vec{u}\vec{u}}$;
3. calulate the derivative and multiplicate by $\frac{1}{2}$;

Second calculate the advective form, $u(\nabla u)$, likes:

1. transform the field $\vec{u}$ from physical space to spectral space;
2. calculate the velocity derivatives $ik\hat{\vec{u}}$;
3. calculate the inverse transformated the velocity derivatives and multiplicate with the velocity in the physical space;
4. transformate the derivates from the physical space to spectral space;

After obtained the advective and divergent form realize the arithmetic mean.

3.3 Immersed boundary method - IBM

The term immersed boundary method, is used in presente paper to reference a method developed by Peskin(1970), with the aim of simulated the cardiac mechanics valves and associated blood flow. Wherefore, is necessary use two domains, lagrangian and euleran domain. The first is independent of eulerian domain, wich get model any object.

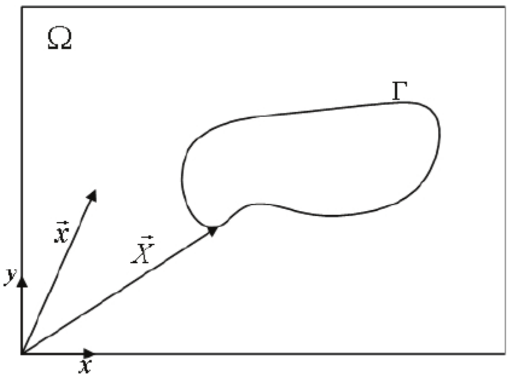

Figure 3. Scheme of domain used by the immersed boundary method.

The Fig.3 represents the domain used by calculate the immersed boundary, where $\overrightarrow{x}$ represents position any point at the field Eulerian (Ω) and $\overrightarrow{X}$ position any point at the field Lagrangian (Γ), (Mariano, 2011).

The spectral method works only periodic boundary conditions. Due to this, the Eq.(1) is modified:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + f_x = 0 \tag{9}$$

Where f_x is the source term, which represents the field force. In the present work it is the force from immersed boundary, which is written by:

$$f_x = \begin{cases} F_x(\overrightarrow{X}, t) & if \quad \overrightarrow{x} = \overrightarrow{X} \\ 0 & if \quad \overrightarrow{x} \neq \overrightarrow{X} \end{cases} \tag{10}$$

The Direct-Forcing method (DFM) is used to calculate the Lagrangian field. The DFM developed by Mohd-Yusof (1997) extracts the forcing directly from the numerical solution, which is determined by difference between the interpolated velocities in the boundary points and the desired at the physical boundary velocities. For the purpose of discussion of the general concepts, let us write the time-discretized Burgers equation, Eq.(9), in the following form:

$$\frac{u^{t+\Delta t} - u^* + u^* - u^t}{\Delta t} + rhs + f_x = 0 \tag{11}$$

Where *rhs* regroups the convective and viscous terms at some intermediate time level between t and t+Δt. The euleraian force term which yields the temporal parameter u* is then the Eq. (11) is solved in two steps, given by Eqs. (12) and (13) :

$$\frac{u^* - u^t}{\Delta t} + rhs = 0 \tag{12}$$

The lagrangian source term isgiven by Eq.(13) for X_l.

$$F_x = \frac{U^{t+\Delta t} - U^*}{\Delta t} \qquad\qquad \forall X_l \tag{13}$$

Where $U^{t+\Delta t}$ is the boundary condition and U^* is the temporal parameter interpolated. Lastly the update by the eulerian velocity is given by Eq.(14) where f_x is calculated by Eq.(10) .

$$u^{t+\Delta t} = u^* + f_x \Delta t \tag{14}$$

The major advantage of the discrete forcing concept is the absence of user specified parameters in the forcing and the elimination of associated stability constraints (Mittal and Iaccarino, 2005). Thus is possible to solve the Burgers equation with non-periodic boundary conditions using the pseudospectral method coupled with IBM.

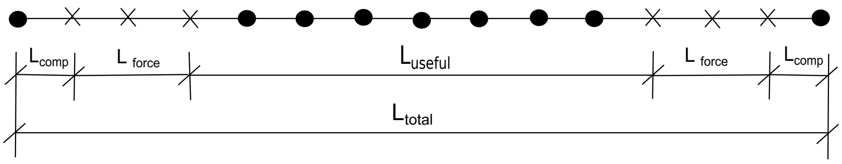

Figure 4. Scheme of Immersed boundary method, used in this work.

Figure 4 represents the one-dimensional domain, used in presente work, in order to study the Burgers equation for non-periodic boundary condition. The L_{total} is the all eulerian domain, which comprise by L_{force}, L_{useful}, and L_{comp}. L_{force} are forcing zone, where 'x' are lagrangian points, L_{comp} are the complementary domain, and L_{useful} is the useful domain.

4. RESULTS

Figure 5 shows the initial condition and the Burgers equation solution at time $\frac{\pi}{8}$[s], using grid with 64 points. At Fig. 5a is the solution for the periodic boundary condition and Fig. 5b is the non-periodic solution.

In the Figure 5a can be observed the numerical differences. As expected, the approach using FVM with upwind scheme, shows the results with more numerical diffusion and the results using central-difference approach is better than previous but had numerical oscilation at the discontinuity region. The result that close better with the analytical solution is the obtained by Fourier pseudospectral method .

Figure 6 the high accuracy of the spectral method for the discontinuos problem and periodic boundary conditions are presented. This figure shows that increase the points of the FPM, the maximum error tend to 10^{-15}, approaching the collocation method obtained by Canuto *et al.* (2006).

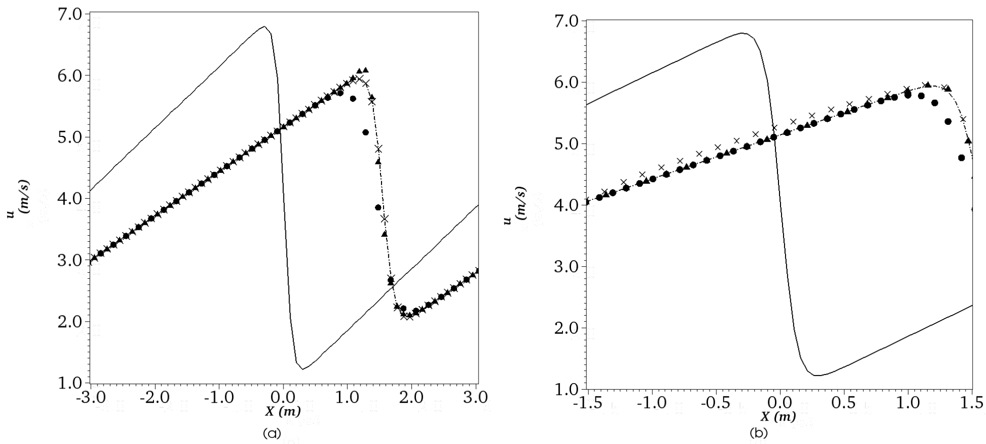

Figure 5. Reference solution of finite volume for 64 grid points. The solid line is the initial condition, the dash line is the analytical solution at $\frac{\pi}{8}$ [s] the triangule represents the solution of central-diference approach, the circle represents the solution of upwind apporach, and the cross symbol is the pseudospectral solution. (a) Periodic domain, (b) Non periodic domain.

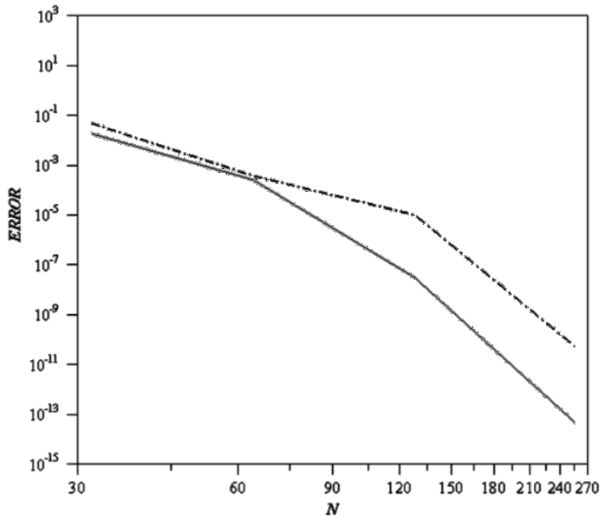

Figure 6. Maximum error of Fourier pseudospectral method (dash-point line) and Collocation spectral method by Canuto *et al.* (2006) (solid line)

The solution for the Burgers in a non periodic domain, is presented in Fig. 5b, with the 64 grid points and the c=4.0 [m/s]. To solve this problem using FPM was used the immersed boundary with directing forcing method, (Mariano, 2011). At Fig. 5b represents the non-periodic solution for both methodologies in study. For the pseudospectral method (dash-dot line) the Fig. 5b shows only useful domain Fig. 4.

Figure 7 is the spatial convergency of the both methodologies. At the Fig.(7a) represents the convergency order using the pseudospectral method, which was applied IBM at 1, 2 and 3 points on each side forcing zone domain (L_{force} in Fig. 4).

Figure 7a can be seen that the improve of the points of aplicantion the IBM, the order of convergence tending to second order and the accuracy is better. This is possible because when improve the number of points, ensures the derivated them. At Fig. 7b confirm the approach that upwind scheme to first order and the central-difference approach is the second order.

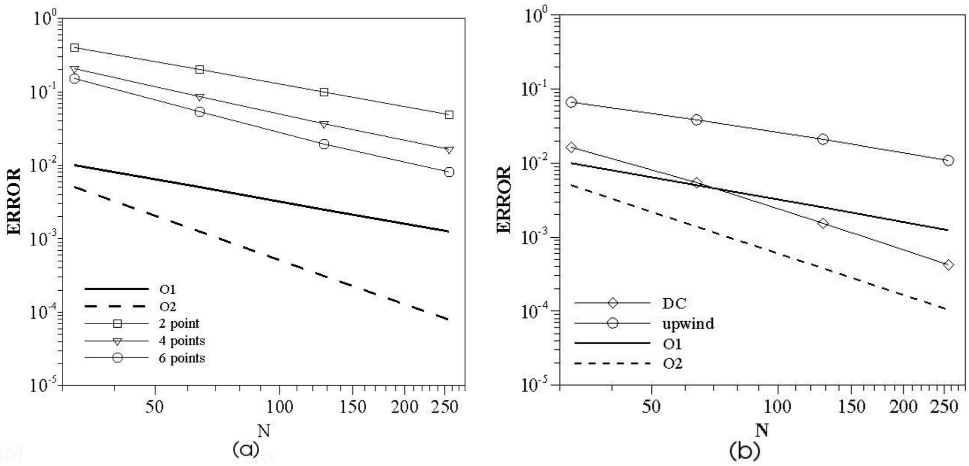

Figure 7. Reference the convergence order,the CFL= 0.001. (a) Pseudospectral method using IBM, (b) Finite Volume method

5. CONCLUSION

The central-difference interpolation scheme used at the finite volume method, showed better that the upwind results. It which shows high numerical dissipation Fig. 5. Therefore the central-difference interpolation scheme at the second order maintain the convergency order of finite volume.

As the computations reported in this paper, it was possible conclude that Fourier spectral methods are well suited to the calculation of problems with periodic boundary conditions. The spectral methods are better from this point of view and call for higher-order time marching schemes. The IBM used to include the non-periodic boundary conditions in this methodology, causes the decrease of accuracy, which is shows in Fig. 7a, second convergency order.

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