SOURCE RECONSTRUCTION FROM BOUNDARY DATA: THE THERMAL WAVE MODEL

Nilson Costa Roberty, nilson@nuclear.ufrj.br
Marcelo S. L. Rainha, rainha@nuclear.ufrj.br
Denis M. de Sousa, denis@con.ufrj.br
Programa de Engenharia Nuclear - COPPE/UFRJ

Abstract. *The purpose of this work is to presents some preliminary results in the investigation of the source function reconstruction for a model based on the thermal wave equation. The thermal wave equation is a high propagation velocity with damping wave model based on the telegraph equation. A time discretization based on the Backward Differentiation formulae and Houbolt scheme is combined with a finite element implementation of the stationary modified Helmholtz equations to formulates a sequential model for reconstruction of characteristics star shape sources inside an arbitrary regular domain . Using Green's formula we establish a reciprocity gap functional relating directly the boundary integral of Cauchy data with the domain integral of test functions in the unknown characteristic source support. The test functions are solutions of a homogeneous modified Helmholtz equation. The source and solution is then sequentially reconstructed at each time step using the Cauchy boundary data and the previous step reconstructed sources and solutions. Two based problems are solved: the source centroid determination and the source shape identification. Numerical investigation of the problem in arbitrary domain are presented. Degradation of the methodology for smalls propagations velocities are also investigated.*

Keywords: *thermal wave heat equation, star shape sources, reciprocity gap functional, centroid and shape capture*

1. INTRODUCTION

The direct transient source initial second order boundary value problem consists in finding $u(x,t)$ with $(x,t) \in \Omega \times [0,T), T > 0$ given a boundary input $g(x,t)$ with $(x,t) \in \Gamma \times [0,T)$, an initial input $u_0(x)$ and $u_0'(x)$ with $x \in \Omega$ and a source distribution $f(x,t)$ with $(x,t) \in \Omega \times [0,T)$ that verifies the problem :

$$(P_{g,f}) \begin{cases} \frac{1}{c}\frac{\partial^2 u}{\partial^2 t} + \alpha \frac{\partial u}{\partial t} - \Delta u = f, & \text{in } \Omega \times [0,T); \\ u = u_0, & \text{in } \Omega \times \{0\}; \\ \frac{\partial u}{\partial t} = u_0', & \text{in } \Omega \times \{0\}; \\ u = g, & \text{on } \Gamma \times [0,T) \end{cases} \tag{1}$$

Heat, wave and telegraph equations will results from particular cases of the parameters c and α. It is well known that this direct problem is well posed with unique solution for regular data.

The inverse source problem that we address consists in the recovery of the source $f(x,t) \in \Omega \times [0,T)$, knowing the initial data in Ω and the Cauchy data in the boundary Γ for $t \in [0,T)$. We consider that the unique information available is given by only one measurement, say, the Neumann boundary measurements

$$\partial_\nu u = g^\nu, \text{ on } \Gamma \times [0,T). \tag{2}$$

corresponding to $g = 0$, on $\Gamma \times [0,T)$, where ν is the boundary domain exterior normal.

2. FINITE DIFFERENCE SCHEMES

Time finite differences schemes applied the transient problems like Eqs. (1) results in a sequence of equations based in the modified Helmholtz model. Based on this fact, we present an algorithm for reconstruction of a moving transient star shape source

$$f(x,t) = \chi_{\omega(t)}(x), \text{ in } \Omega \times [0,T) \tag{3}$$

where $\omega(t) \in \Omega, t \in [0,T)$ is a representation of the source boundary. For one -dimensional problems it is a set with two points. For two or tree-dimensional problems it is a moving Lipschitz parametric curve or surface in which the parameter has been omitted.

Let us consider a partition of the time interval $[0,T]$ into N subintervals of length $\tau > 0$. Let $\{t_0, t_1, t_2, ..., t_n, t_{n+1}, ...t_N\}$ be the knots of this partition, with $t_0 = 0$ and $t_N = T$. For $t_n < t < t_{n+1}, n = 0, 1, N-1$ we use finite differences scheme approach for the discretization of Eqs.1.

We define the θ-scheme for a transient equation $\frac{\partial u}{\partial t}(x,t) = u'(x,t)$, a linear $0 < \theta \leq 1$ weighted approximation first order forward difference based on the time knots spatial field by

$$\frac{\partial u}{\partial t}(x,t) \cong \frac{u(x,t_{n+1}) - u(x,t_n)}{t_{n+1} - t_n} \cong \delta_\theta(u')(x) := \theta u'(x,t_{n+1}) + (1-\theta)u'(x,t_n), \ x \in \Omega \tag{4}$$

for $n = 0, ..., N$.

The Houbolt-scheme is an fully implicit unconditionally stable time integration scheme based on a cubic-Lagrange interpolation. It makes regression with the last four estimation of the field $u(x,t)$ to approximates its first and second time derivatives,

$$\frac{\partial u_{n+1}}{\partial t} \cong \frac{1}{6\tau}(11u(x,t_{n+1} - 18u(x,t_n) + 9u(x,t_{n-1}) - 2u(x,t_{n-2})) \tag{5}$$

$$\frac{\partial^2 u_{n+1}}{\partial t^2} \cong \frac{1}{\tau^2}(2u(x,t_{n+1} - 5u(x,t_n) + 4u(x,t_{n-1}) - u(x,t_{n-2})) \tag{6}$$

By denoting $u_n(x)$ and $\chi_n(x)$, with $x \in \Omega$, the approximate solution and the characteristic source at the time t_n, the finite difference-scheme for Eq. 1 and Eq. 2 may be written as

$$(H_{g_{n+1},f_n+\chi_{n+1}})\begin{cases} -\Delta u_{n+1} + \lambda u_{n+1} = f_n + \chi_{n+1}, & \text{in } \Omega; \\ u_{n+1} = g_{n+1}, & \text{on } \Gamma. \end{cases} \tag{7}$$

$$(N_{g^\nu_{n+1}})\left\{ \partial_\nu u_{n+1} = g^\nu_{n+1}, \quad \text{on } \Gamma. \right. \tag{8}$$

for $n = 0, ..., N$. Here

$$\lambda = \begin{cases} \frac{1}{\theta^2 c^2 \tau^2} + \frac{\alpha}{\theta\tau}, & \text{for the } \theta - \text{scheme ;} \\ \frac{2}{c^2\tau^2} + \frac{11\alpha}{6\tau}, & \text{Houbolt} - \text{scheme .} \end{cases} \tag{9}$$

is the modified Helmholtz parameter for the finite differences-schemes and

$$f_n = \begin{cases} -\frac{1-\theta}{\theta}(-\Delta u_n + \lambda u_n - \chi_n) + \frac{\lambda}{\theta} u_n + \frac{1}{\theta^2 c^2 \tau}\frac{\partial u}{\partial t}|_n & \text{for the } \theta - \text{scheme ;} \\ \frac{(30+18\alpha c^2\tau)}{6c^2\tau^2} u_n + \frac{(24+9\alpha c^2\tau)}{6c^2\tau^2} u_{n-1} + \frac{(6+2c^2\tau)}{6c^2\tau^2} u_{n-2}, & \text{Houbolt} - \text{scheme .} \end{cases} \tag{10}$$

is a source term based on previous step calculations. Convergence theorems for this kind of transient algorithm may be see in [[**?**]]. Note that in the θ-scheme case,

$$f_n = -\frac{1-\theta}{\theta} f_{n-1} + \frac{\lambda}{\theta} u_n + \frac{1}{\theta^2 c^2 \tau}\frac{\partial u}{\partial t}|_n \tag{11}$$

$$\frac{\partial u}{\partial t}|_n = -\frac{1-\theta}{\theta}\frac{\partial u}{\partial t}|_{n-1} + \frac{u_n - u_{n-1}}{\theta\tau} \tag{12}$$

may be defined in a recurrent way and the set up for the initial framework may be given through the initial fields $u_0(x), u_0'(x), \chi_0(x)$. We also may determined $\chi_0(x), u_0(x)$ for arbitrary $u_0'(x)$ by solving a initial stationary inverse source problem

$$(H_{g_0,\chi_0})\begin{cases} -\Delta u_0 = \chi_0, & \text{in } \Omega; \\ u_0 = g_0, & \text{on } \Gamma. \end{cases} \tag{13}$$

$$(N_{g^\nu_0})\left\{ \partial_\nu u_0 = g^\nu_0, \quad \text{on } \Gamma. \right. \tag{14}$$

In the Houbolt-scheme case, the set up of initial framework is done by a Taylor expansion near the initial values

$$\begin{cases} u_{n-1} = u_0 - \tau u_0' & \text{for } n = 0; \\ u_{n-2} = u_0 - 2\tau u_0' & \text{for } n = 0, 1. \end{cases} \tag{15}$$

We may further improves the finite difference scheme sequence by splitting the solution of Eq. 8 in a solution $u^{f_n,g_{n+1}}_{n+1}$ for an auxiliary problem $H_{f_n,g_{n+1}}$ that depends only on information knower at knot time t_n and so may be solved before the reconstruction of the characteristic source χ_{n+1} and an complementary auxiliary problem $H_{0,\chi_{n+1}}$ with solution $u^{\chi_{n+1},0}_{n+1}$ depending on the unknown source at time t_{n+1}.

$$u_{n+1} = u^{f_n,g_{n+1}}_{n+1} + u^{\chi_{n+1},0}_{n+1} \tag{16}$$

The θ-scheme using all prior information existing at the beginning of the time step $t_n \leq t \leq t_{n+1}$ is

$$(H_{0,\chi_{n+1}})\begin{cases} -\Delta u^{\chi_{n+1},0}_{n+1} + \lambda u^{\chi_{n+1},0}_{n+1} = \chi_{n+1}, & \text{in } \Omega; \\ u^{\chi_{n+1},0}_{n+1} = 0, & \text{on } \Gamma. \end{cases} \tag{17}$$

$$(N_{g^\nu_{n+1} - \partial_\nu u^{f_n,g_{n+1}}_{n+1}})\left\{ \partial_\nu u^{\chi_{n+1},0}_{n+1} = g^\nu_{n+1} - \partial_\nu u^{f_n,g_{n+1}}_{n+1}, \quad \text{on } \Gamma. \right. \tag{18}$$

for $n = 0, ..., N$. The sequence of modified Helmholtz source inverse problems given by Eq. 18 and Eq. 17 may be solved starting with $n = 0$ reconstructing the star shape source $\chi_n(x)$ for the time knots sequence $n = 1, ..., N$ showing its movement inside the domain Ω.

3. AUXILIARY FOURIER SERIES SOLUTIONS

3.1 Direct thermal wave problem

When the external domain Ω is a box $(0,1)^d \in R^d$, where $d = 1, 2, 3$ are the physical domain, and the Dirichlet boundary condition is homogeneous, $g = 0$, the transient problem has an explicit Fourier sine solution

$$u(x_1, ..., x_d, t) = \sum_{i=1}^{d} \sum_{n_i=1}^{N_i} c_{(n_1...n_d)}(t) \prod_{i=1}^{d} \sin(n_i \pi x_i) \tag{19}$$

Let

$$u_{0,(n1,...,n_d)} = \left(\frac{2}{\pi}\right)^d \int_0^1 ... \int_0^1 u_0(x_1, ..., x_d) \prod_{i=1}^{d} \sin(n_i \pi x_i) dx_1 ... dx_d \tag{20}$$

$$u'_{0,(n1,...,n_d)} = \left(\frac{2}{\pi}\right)^d \int_0^1 ... \int_0^1 u'_0(x_1, ..., x_d) \prod_{i=1}^{d} \sin(n_i \pi x_i) dx_1 ... dx_d \tag{21}$$

$$\chi_{\omega(t),(n1,...,n_d)} = \left(\frac{2}{\pi}\right)^d \int_0^1 ... \int_0^1 \chi_{\omega(t)}(x_1, ..., x_d) \prod_{i=1}^{d} \sin(n_i \pi x_i) dx_1 ... dx_d \tag{22}$$

the respective Fourier coefficients of initial data and characteristic source. The series coefficients in Eq. 19 are given by

$$c_{(n_1...n_d)}(t) = u_{0,(n_1,...,n_d)} \cosh\left(ct\sqrt{\frac{\alpha^2 c^2}{4} - \pi^2 \sum_{l=1}^{d} n_l^2}\right) \exp\left(-\frac{\alpha c^2}{4} t\right) +$$

$$\left(u_{0,(n_1,...,n_d)} + \frac{\alpha c^2}{4} u'_{0,(n_1,...,n_d)}\right) \frac{\sinh\left(ct\sqrt{\frac{\alpha^2 c^2}{4} - \pi^2 \sum_{l=1}^{d} n_l^2}\right)}{c\sqrt{\frac{\alpha^2 c^2}{4} - \pi^2 \sum_{l=1}^{d} n_l^2}} \exp\left(-\frac{\alpha c^2}{4} t\right) +$$

$$\int_0^t \frac{c \sinh\left(c(t-\tau)\sqrt{\frac{\alpha^2 c^2}{4} - \pi^2 \sum_{l=1}^{d} n_l^2}\right)}{c\sqrt{\frac{\alpha^2 c^2}{4} - \pi^2 \sum_{l=1}^{d} n_l^2}} \exp\left(-\frac{\alpha c^2}{4}(t-\tau)\right) \chi_{\omega(\tau),(n1,...,n_d)} d\tau \tag{23}$$

In the singular case in which $\lim_{c \to \infty}$and the propagation velocity parameter c is infinity we obtain the solution to the heat equation are given by

$$c_{(n_1...n_d)}(t) = \exp(-t\pi^2 \sum_{i=1}^{d} n_i^2) u_{0,(n_1,...,n_d)} + \int_0^t \exp(-(t-\tau)\pi^2 \sum_{i=1}^{d} n_i^2) \chi_{\omega(\tau),(n1,...,n_d)} d\tau. \tag{24}$$

For an specific source and an appropriated dimension, the synthetic Neumann data (2) to be used in the reconstruction inverse problem, may be calculated as the normal boundary derivative of solution (19) .

3.2 Direct Helmholtz problem

We consider the problem given by Eq. 7 without the characteristic part of the source again in a box $(0,1)^d \in R^d$, where $d = 1, 2, 3$. Without loss of generality for the inverse source problem we may consider the zero Dirichlet data case. The explicit Fourier sine solution is

$$u_{n+1}^{f_n,0}(x_1, ..., x_d) = \sum_{i=1}^{d} \sum_{n_i=1}^{N_i} c_{(n_1...n_d)}^{f_n,0} \prod_{i=1}^{d} \sin(n_i \pi x_i) \tag{25}$$

Let

$$f_{n,(n1,...,n_d)} = \left(\frac{2}{\pi}\right)^d \int_0^1 ... \int_0^1 f_n(x_1, ..., x_d) \prod_{i=1}^{d} \sin(n_i \pi x_i) dx_1 ... dx_d \tag{26}$$

then

$$c_{(n_1...n_d)}^{f_n,0} = \frac{f_n(x_1, ..., x_d)}{\pi^2 \sum_{i=1}^{d} n_i^2 + \lambda} \tag{27}$$

We must mention here that for general domains Ω with arbitrary geometry, these explicit solutions may not exists.

4. RECIPROCITY GAP FUNCTIONAL

The reciprocity functional for the Helmholtz Problem depends only on boundary values of the solution and its properties are derived from elementary properties of the Green's theorem. Let v in be the space of Helmholtz functions $H_\lambda(\Omega) = \{v : -\Delta v + \lambda v = 0\}$. The reciprocity gap functional for the Cauchy data in the sequence of modified Helmholtz problems given by Eq.7 with additional Neumann data Eq. 8 is

$$R^\lambda_{\chi_{n+1}}(v_\lambda) = \int_{\partial\Omega} (g^\nu_{n+1} - \partial_\nu u^{f_n,g_{n+1}}_{n+1}) v_\lambda d\sigma, (\text{ for } v_\lambda \in H_\lambda(\Omega)). \tag{28}$$

It is a direct consequence of Green's theorem that

$$R^\lambda_{\chi_{n+1}}(v_\lambda) = \int_{\Omega} \chi_{\omega(t_{n+1})}\, v_\lambda dx, (\text{ for } v_\lambda \in H_\lambda(\Omega)). \tag{29}$$

5. CENTROID DETERMINATION

The approximated centroid $\bar{x} = (\bar{x_1}, ..., \bar{x_d})$ position of a sub domain $\omega_{n+1} \subset \Omega \in R^d$ (meta centroid) as the solution of Eq. 29 at the functions $\{v_{\lambda,i} = \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa}$, i=1,...,d$\}$

$$R^\lambda_{\chi_{n+1}}(v_{\lambda,i}) = \int_{\partial\Omega} (g^\nu_{n+1} - \partial_\nu u^{f_n,g_{n+1}}_{n+1}) \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa} d\sigma = 0, \text{ for } i = 1, ..., d. \tag{30}$$

Eq. 30 is obviously equivalent to

$$R^\lambda_{\chi_{n+1}}(v_{\lambda,i}) = \int_{\omega_{n+1}} \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa} dx = 0, \text{ for } i = 1, ..., d. \tag{31}$$

Note that Eq. 31 may be rewrite with more resemblance with the classical centroid definition

$$\bar{x_i} = \frac{\int_{\omega_{n+1}} x_i \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa(x_i - \bar{x_i})} dx}{\int_{\omega_{n+1}} \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa(x_i - \bar{x_i})} dx}, \text{ for } i = 1, ..., d. \tag{32}$$

Is easy to see from the behavior of the hyperbolic sinc function

$$\text{sinch}(\kappa(x_i - \bar{x_i})) = \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa(x_i - \bar{x_i})}$$

that in the $\lim_{\kappa \to 0}$ its coincides with the harmonic centroid definition. We also notes a small variability of the meta centroid for the κ-range used in the present model and that if there exists symmetry of the star shape boundary curve or surface with all coordinates directions, then the meta centroid is independent of the parameter κ and coincides with the usual harmonic centroid that occurs when $\kappa = 0$. If there is symmetry for a generic direction $l = (l_1, ..., l_d) \in S^{d-1}$, then the meta centroid variability with respect with the classical centroid will be zero. Fortunately, the observed variability of asymmetrical domains for values of κ used to be used in the finite differences-scheme reconstruction procedure is small and the source domain remains star shape with respect to the meta centroid if its is star shaped with respect to the classical centroid. Based on this, we will use the meta centroid as the center of coordinates in the shape reconstruction procedure.

5.1 Non linear least squares solution to meta centroid equations

Since the Neumann data in Eq. (30) frequently are noised, the least square non linear method may be used to formulates an unconstrained minimizing problem for the determination of coordinates $\bar{x_i}$ of the centroid. If necessary, classical regularizations methods such as the Tikhonov method may be adapted for the stabilization and improvement of the algorithm. Without any regularization than truncation the problem of centroid determination in the modified Helmholtz equation with boundary Dirichlet data zero and g^ν_{n+1} Neumann data on the boundary is

$$\bar{x}^\kappa_i = \arg\min\{\| \int_\Gamma \frac{\sinh(\kappa(x_i - y_i))}{\kappa} g^\nu_{n+1} d\sigma(x) \|^2 | y \in \Omega\} \text{ for } i = 1, ..., d. \tag{33}$$

6. SHAPE DETERMINATION

An star shape domain with respect to an internal point is by definition a region for which each ray emerging from this interior point cross the boundary once one time. If we combine two antipodal rays, we see that it may seen as a kind of generalization of the concept of interval. Is this fact that suggest the adoption of polar or spherical coordinates in modeling the shape reconstruction problem. In this work we will fix the model for the two dimensional problem. It is not difficult to see that generalization for three dimensions is straightforward, even if it involves a computational procedure much more expensive. Let $r_{n+1}(\varphi)$ the parametric radial representation of the star shape boundary in a coordinates system centered in the already determined meta centroid and let us choose modified Bessel test functions and substitutes it in the polar form of Eq. 29

$$R^{\lambda}_{\chi_{n+1}}\left(\frac{2^m m! I_m(\kappa\rho)\exp(im\theta)}{\kappa^m}\right) = \int_0^{2\pi}\left[\int_0^{r_{n+1}(\theta)} \frac{2^m m! I_m(\kappa\rho)\exp(im\theta)}{\kappa^m}\right]\rho d\rho d\theta. \tag{34}$$

for $m = 0, 1, ..., \infty$. The reciprocity gap functional Eq. 28 calculated at the domain boundary with zero Dirichlet data for the same set of test functions is

$$I_{\partial\Omega}(m,\kappa,\bar{x}_i,g^{\nu}_{n+1}) := \int_{\partial\Omega} \frac{2^m m! I_m(\kappa\sqrt{\sum_{i=1}^{d}(x_i-\bar{x}_i)^2})\exp(im\theta)}{\kappa^m} g^{\nu}_{n+1}(x)d\sigma(x). \tag{35}$$

Equating, for each m, this two equation will form a system for the shape reconstruction problem in the two dimensional problem. We will denote

$$R^{\kappa}_m = R^{\lambda}_{\chi_{n+1}}\left(\frac{2^m m! I_m(\kappa\rho)\exp(im\theta)}{\kappa^m}\right) \tag{36}$$

and omit the time index n for a while.

6.1 Reciprocity gap functional representation with Fourier series in two dimensional problems

Let us consider the Fourier series associated with the source boundary

$$r(\varphi) = c_0 + \sum_{\substack{j=-\infty,\\ j\neq 0}}^{+\infty} c_j e^{ij\varphi}. \tag{37}$$

It is a positive real valued function for which $c_{-j} = \bar{c}_j, j = 1, 2, ...,$ c_0 is positive, and since

$$r_{\min} = \min\{r(\varphi), \varphi \in [0, 2\pi)\} > 0$$

we have $\left|\sum_{j=-\infty, j\neq 0}^{+\infty} c_j e^{ij\varphi}\right| < c_0$.

Based on this we will consider the Taylor series expansion of $\frac{2^m m! I_m(\kappa\rho)}{\kappa^m}$ in a neighborhoods of the value $\rho = c_0$. This may be done through derivatives of the modified Bessel functions, but we have found to be more practical to make a combination of the Taylor series near zero with the binomial expansion. By defining for $\kappa > 0$

$$b^{\kappa}_{|n|,l,m} = \begin{cases} 0, & \text{if } |n| + 2l + 2 - m < 0; \\ \frac{(\kappa^2/4)^l |n|!(|n|+2l+1)}{l!(|n|+l)!m!(|n|+2l+2-m)!}, & \text{else.} \end{cases} \tag{38}$$

with $l = 0, \infty$; $m = 0, \infty$; $n = -\infty, +\infty$. In the case $\kappa = 0$ we define

$$b^{\kappa}_{|n|,l,m} = \begin{cases} \frac{(|n|+1)}{l!(|n|+l)!m!(|n|+2-m)!}, & \text{if } l = 0; \\ 0, & \text{else.} \end{cases} \tag{39}$$

We obtain the double series that heritages convergence properties from the modified Bessel first kind cylinder functions

$$R^{\kappa}_n[r] = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b^{\kappa}_{|n|,l,m} c_0^{(|n|+2l+2-m)} \int_0^{2\pi}\left(\sum_{\substack{j=-\infty;\\ j\neq 0}}^{\infty} c_j e^{ij\theta}\right)^m e^{in\theta}d\theta.$$

By evaluating analytically the integral we obtain the algebraic system Fourier series based reciprocity functional by noting that (36) has expansion

$$R^{\kappa}_n[r] = 2\pi \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} b^{\kappa}_{|n|,l,m} c_0^{(|n|+2l+2-m)} V^m_n(c) \tag{40}$$

where we have defined the grade m polynomial with star-shaped boundary parametric curve Fourier coefficients $\{c_n; n = -\infty, +\infty\}$

$$V_n^m(c) = \begin{cases} \delta_{n,0}, & \text{if } m = 0; \\ (1 - \delta_{n,0})c_{-n}, & \text{if } m = 1 \\ \sum_{\substack{j_1=-\infty; \\ j_1 \neq 0}}^{\infty} \dots \sum_{\substack{j_{m-1}=-\infty; \\ j_{m-1} \neq 0}}^{\infty} (\prod_{t=1}^{m-1} c_{j_t}) c_{-n-\sum_{t=1}^{m-1} j_t}, & \text{if } m \geq 2 \end{cases} \quad (41)$$

Note that the main Fourier coefficient c_0 is out of the sum and that the smoothness of the star-shaped domain boundary parametric curve assures a high decay of the Fourier coefficients, and it can be approximated by $3th$ or $5th$ grade polynomials.

7. RESULTS AND CONCLUSIONS

Preliminary results shows that the model is only appropriated for high velocities of propagation, for with the transient heat equation is appropriated.

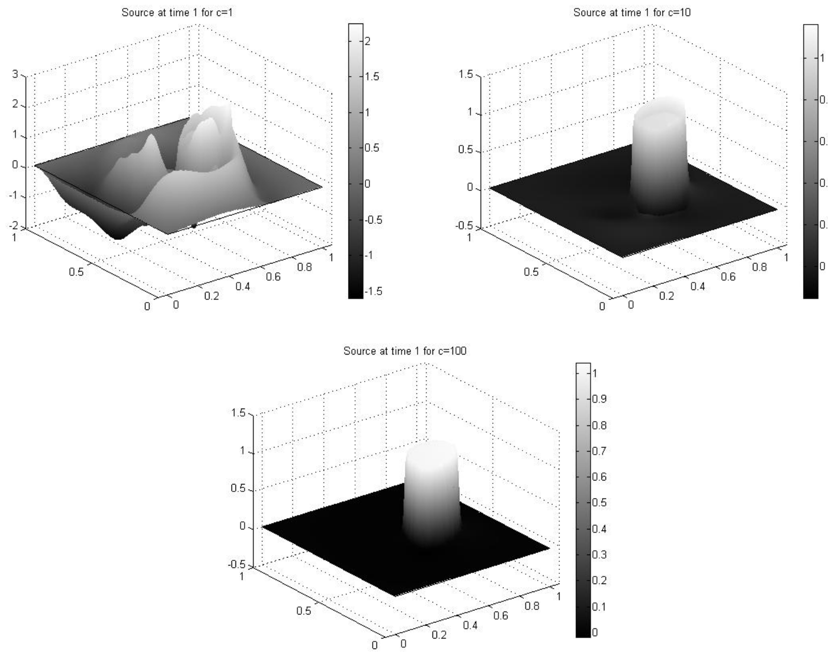

Figure 1. Transient source reconstruction behavior for the thermal wave model with velocity $c = 1, 10$ and 100

8. ACKNOWLEDGEMENTS

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