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THE GRAETZ PROBLEM FOR A FENE-P FLUID IN A PIPE

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Abstract. *The development of the thermal boundary layer in a pipe for a FENE-P fluid is investigated using the method of separation of variables. The ensuing Sturm-Liouville problem is then solved for the eignevalues by means of an adequate solver, while the ordinary differential equations for the eigenfunctions and their derivatives are calculated with a fourth order Runge-Kutta method. Solutions are presented for two different boundary conditions and viscous dissipation effects are included: imposed wall temperature and imposed wall heat flux. The physical properties are considered to be independent of temperature, the fluid dynamics is fully-developed and axial conduction is neglected. Results are presented for the Nusselt number and normalized temperature as a function of the Brinkman number, which quantifies the intensity of viscous dissipation, and of numbers accounting for elastic effects, such as the Weissenberg number and the extensibility parameter.*

***Keywords**. Heat transfer, thermal development, viscoelastic, pipe flow, FENE-P model*

1. Introduction

In polymer processing, highly elastic fluids flow under non-isothermal conditions, but such flows may usually be considered as dynamically fully-developed, because of their very high viscosities. Hence, the thermal entry problem in channels and pipes is very frequent and the issue of viscous dissipation is also quite relevant due to the combination of large viscosities and shear rates.

There are very few investigations of the thermal entry flow for viscoelastic fluids; these were recently reviewed by Coelho et al (2003), who derived analytically a solution for the Phan-Thien— Tanner (PTT) model in its simplified version with a linear stress function, under various boundary conditions. Previous investigations by the same group had already provided asymptotic heat transfer solutions in pipe and channel flows for the same fluids (Pinho and Oliveira 2000, Coelho et al 2002).

Another very common rheological constitutive equation for polymeric liquids is the non-linear dumbbell model, such as the FENE-P equation, which was derived for dilute solutions but may be extended to semi-dilute and concentrate solutions following the ideas of the encapsulated dumbbell model, Bird et al (1987). The fully-developed dynamical isothermal solution for the FENE-P model has been derived by Oliveira (2002), but the literature is very scarce regarding its performance in heat transfer problems.

This paper presents a solution of the thermal entry flow in pipes for the FENE-P fluid, for imposed constant wall temperature and wall heat flux and in the presence of viscous dissipation. Although more often multimode constitutive equations are being used, single mode versions can be sufficiently accurate in pure shear flows, as in here.

2. Governing equations

The entry flow problem of very viscous fluids is characterized by fully-developed dynamics and a thermally developing flow. Since the model parameters are assumed to be independent of temperature, the dynamical problem is decoupled from the thermal problem. The pipe is aligned in the *x*-direction with the origin at *x*= 0 and two types of wall boundary conditions are considered: (i) constant wall temperature T_w; and (ii) constant wall heat flux $\dot{q}_w$. At *x*= 0 the thermal boundary condition is applied and for $x<0$, $T=T_0$, the bulk temperature at inlet. The radial coordinate is denoted *r*, with *R* meaning the pipe radius.

The temperature profile is the solution of the following energy conservation equation:

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \tau_{xr}\frac{du}{dr} \tag{1}$$

subject to adequate thermal boundary conditions. Therefore, the solution requires prior knowledge of the velocity and shear stress profiles.

Oliveira (2002) has obtained the fully-developed axial velocity (u) and shear stress (τ_{xr}) profiles for the FENE-P model, which are given in non-dimensional form by Eqs. (2) and (3)

$$u^* = \frac{u}{U} = 2\frac{U_N}{U}\left[1 - r^{*2}\right]\left\{1 + \beta\left[1 + r^{*2}\right]\right\} \tag{2}$$

$$\tau^*_{xr} = \frac{\tau_{xr}}{\eta U/R} = -4\frac{U_N}{U}r^* \tag{3}$$

where

$$\beta = 16\frac{We^2}{a^2L^2}\left(\frac{U_N}{U}\right)^2 \tag{4}$$

In equations (2) to (4), stars denote non-dimensional quantities, U and U_N stand for the cross-section average velocity of the FENE-P fluid and of a Newtonian fluid under the same pressure gradient (i.e., $U_N = -R^2(dp/dx)/(8\eta)$), respectively. *We* is the Weissenberg number ($We \equiv \lambda U/R$), r^* is the normalised radius ($r^* \equiv r/R$) and a, L^2 and η are constitutive parameters, explained below.

The non-linear differential rheological constitutive equation gives the stress tensor (τ) as a function of a configuration tensor (**A**), as follows

$$\tau = \frac{\eta}{\lambda}\left(f\mathbf{A} - a\mathbf{I}\right) \tag{5}$$

$$\overset{\nabla}{\mathbf{A}} = -\frac{1}{\lambda}\left(f\mathbf{A} - a\mathbf{I}\right) \tag{6}$$

Above, η stands for the zero-shear rate polymer viscosity, λ is the relaxation time and f is the spring force function given by a Warner expression which, after introducing Peterlin's approximation, may be written as:

$$f \equiv f(tr\mathbf{A}) = \frac{L^2}{L^2 - tr\mathbf{A}} \tag{7}$$

L^2 is the extensibility parameter that also appears via a, which is defined by $a = 1\Big/\left(1 - 3/L^2\right)$, i.e., there are three independent constitutive parameters: η, λ and L^2.

The interesting finding regarding the dynamic solution (Eqs. 2 and 4) is its similarity with the dynamical solution for a simplified PTT fluid with a linear stress coefficient, derived by Oliveira and Pinho (1999). The solution for the PTT fluid, is also given by Eq. (2), but with β defined differently as

$$\beta = 16\varepsilon We^2\left(\frac{U_N}{U}\right)^2 \tag{8}$$

where ε is the elongational parameter of the PTT model. From equality of Eqs. (4) and (8) we can see that

$$\varepsilon = \frac{1}{a^2L^2} = \frac{\left(1 - 3/L^2\right)^2}{L^2} \tag{9}$$

Note also that the cubic equations giving U_N/U for the FENE-P and PTT models are identical, provided this transformation is carried out.

The shear stress profiles for the linear PTT and the FENE-P fluids are a result of identical lateral momentum equations

$$0=-\frac{dp}{dx}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{xr}\right) \tag{10}$$

and can also be matched using Eq. (9).

Thus, we end up with the same energy equation to be solved for the temperature distribution $T(r,x)$, using the following boundary conditions: at inlet: $T(r,x=0)=T_0$; at the symmetry axis: $\partial T(r=0,x)/\partial r=0$; at the wall ($r = R$): either $T(r,x)=T_w$, or $k\,\partial T(r,x)/\partial r=\dot{q}_w$.

Therefore, the main conclusion is that the solution of the thermal entry flow for the FENE-P fluid is exactly the same as that for the simplified PTT model with linear stress coefficient given by Coelho et al (2003), provided that the substitution embodied in Eq. (9) is made. Note here the use of β to represent what in Coelho et al (2003) was denoted by *a*, because *a* now has a special meaning in the context of the FENE-P equation (see after Eq. (7) above).

Before proceeding, it is important to realize that the main differences between the FENE-P model and the simplified PTT model with linear stress coefficient lie on their differing transient responses in shear and extensional flows; as seen above, the steady state responses can be made to collapse with appropriate algebraic transformations. Since the fluids under analysis are usually very viscous, the dynamic flow develops very quickly compared with the thermal flow and, consequently, the thermal entry flow is indeed a steady state flow from a rheological point of view, hence the thermal solutions for the two models are basically ruled by the same final equations with appropriate choice of parameter β (Eqs. 4 or 8).

3. Method of solution

The solution to this entry flow problem is carried out with the non-dimensional form of the energy equation (Eq. 11) and is based on the separation solution methods described in Mikhailov and Özisik (1994), leading to an eigenvalue problem.

$$u^*\frac{\partial\theta}{\partial x^*}=\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial\theta}{\partial r^*}\right)+Br\tau_{xr}^*\frac{\partial u^*}{\partial r^*} \tag{11}$$

The normalised boundary conditions are: at inlet $\theta\left(r^*,0\right)=0$; at symmetry axis $\partial\theta\left(r^*,x^*\right)/\partial r^*=0$; at the wall $(1-m)\theta\left(1,x^*\right)+m\,\partial\theta\left(r^*,x^*\right)/\partial r^*=(1-m)+m/2$. Here, the use of index *m* allows the generalization of the wall boundary conditions, with *m*=0 for imposed wall temperature and *m*=1 for imposed constant heat flux.

In Eq. (11) $x^*=x\alpha/\left(R^2\overline{U}\right)$ where α is the thermal diffusivity ($\alpha\equiv k/\left(\rho c_p\right)$) and θ is defined according to the imposed wall condition: Eq. (12) for given wall temperature T_w and Eq. (13) for given wall heat flux $\dot{q}_w$.

$$\theta\equiv\frac{T-T_0}{T_w-T_0} \tag{12}$$

$$\theta\equiv\frac{T-T_0}{\dot{q}_w D/k} \tag{13}$$

The Brinkman number, a measure of viscous dissipation effects, is also defined differently: as $Br=\eta U^2/\left[k\left(T_w-T_0\right)\right]$, for the given wall temperature case, or $Br=\eta U^2/\left[\dot{q}_w D\right]$, for the given wall heat flux case. The pipe diameter is denoted *D*.

The solution to Eq. (11) is the sum of a general solution of the corresponding homogeneous equation (*Br*= 0) and of a particular solution for $Br\neq 0$. For the solution of the homogeneous equation, the separation of variables is applied, by which $\theta\left(x^*,r^*\right)=\Psi\left(r^*\right)\phi\left(x^*\right)$ and leading to the following differential equations

$$\frac{d\phi\left(x^*\right)}{dx^*}+\mu^2\phi\left(x^*\right)=0 \tag{14}$$

$$\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{d\Psi\left(\mu,r^*\right)}{dr^*}\right)+\mu^2u^*\Psi\left(\mu,r^*\right)=0 \tag{15}$$

subject to boundary conditions

$$r^* = 0, \frac{d\Psi}{dr^*}\left(\mu, r^*\right) = 0 \tag{16}$$

$$r^* = 1, (1-m)\Psi(\mu,1) + mr^* \frac{d\Psi\left(\mu, r^*\right)}{dr^*} = 0 \tag{17}$$

Eq. (14) is easily integrated and contributes to the final solution of $\theta\left(r^*, x^*\right)$ with a decaying exponential function of the longitudinal coordinate $e^{-\mu^2 x^*}$. It depends on the eigenvalues μ^2, that are determined numerically as part of the solution to Eq. (15). The determination of the eigenvalues μ^2, and of the corresponding eigenfunctions, was accomplished by means of the freeware Fortran code SLEDGE (Pruess and Fulton, Netlib, cited by Pryce, 1993), which was adequately modified. Full details of the solution procedure followed are to be found in Coelho et al (2003) and the codes are available from the internet (at http://www.dem.uminho.pt/people/ftp/research/sturmptt.html).

4. Results and discussion

Given the similarity of the transport equations and of the dynamical solution for the thermal entry flow problem for the PTT and the FENE-P models, the results of Coelho et al (2003) apply here provided the substitution in Eq. (9) is performed. Now, it is adequate to present here results for at least one case in order to judge, in a quantitative way the effects of molecular extensibility and viscous dissipation. The case of pipe heating is selected, which corresponds to a prescribed positive wall heat flux, $\dot{q}_w > 0$, and thus $Br > 0$ (see definition above).

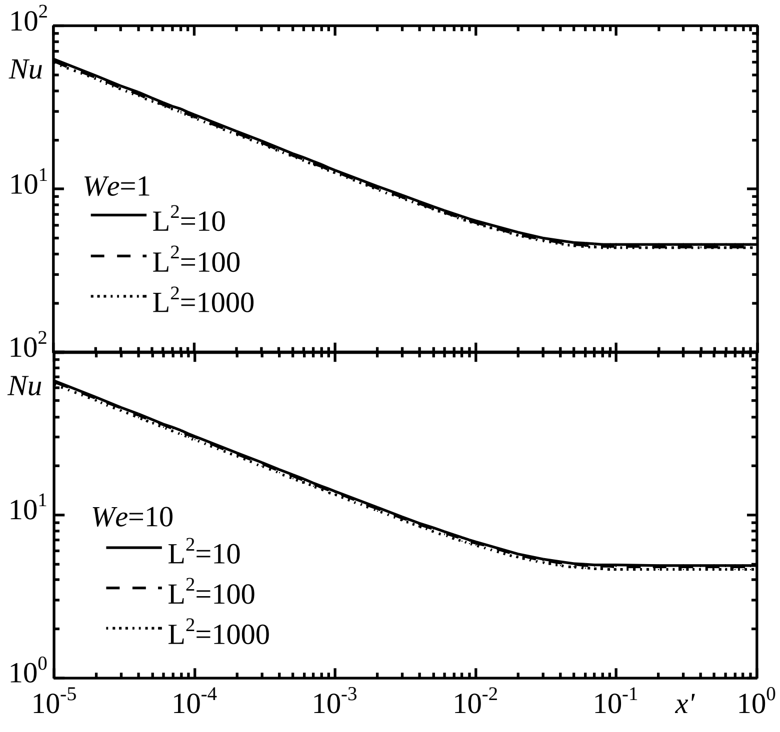

Figure 1- Effect of viscoelasticity, measured by *We* and L^2, on the Nusselt number variation in the absence of viscous dissipation (Br= 0).

In Figure 1**,** the longitudinal variation of the Nusselt number is plotted as a function of *We* and L^2 for negligible viscous dissipation. The Nusselt number increases with fluid elasticity, but variations are small, not exceeding 20% under thermally fully-developed conditions (large *x'*): for Newtonian fluids Nu_{fd} = 4.364 whereas for *We*= 10 and L^2 = 10, Nu_{fd} = 4.908. At very high elasticity, for instance when $We^2/(aL)^2$ = 100 (as when *We*=100 and L^2=100), Nu_{fd} = 5.053. The normalised coordinate x' is defined as $x' = x/(DPe)$ where *Pe* is the Peclet number, $Pe \equiv UD/\alpha$.

For imposed wall heat flux the Nusselt number is defined by

$$Nu = \frac{hD}{k} = \frac{\dot{q}_w D}{k(T_w - T_b)} \qquad (18)$$

with *h* and *k* representing the heat transfer coefficient and thermal conductivity, respectively.

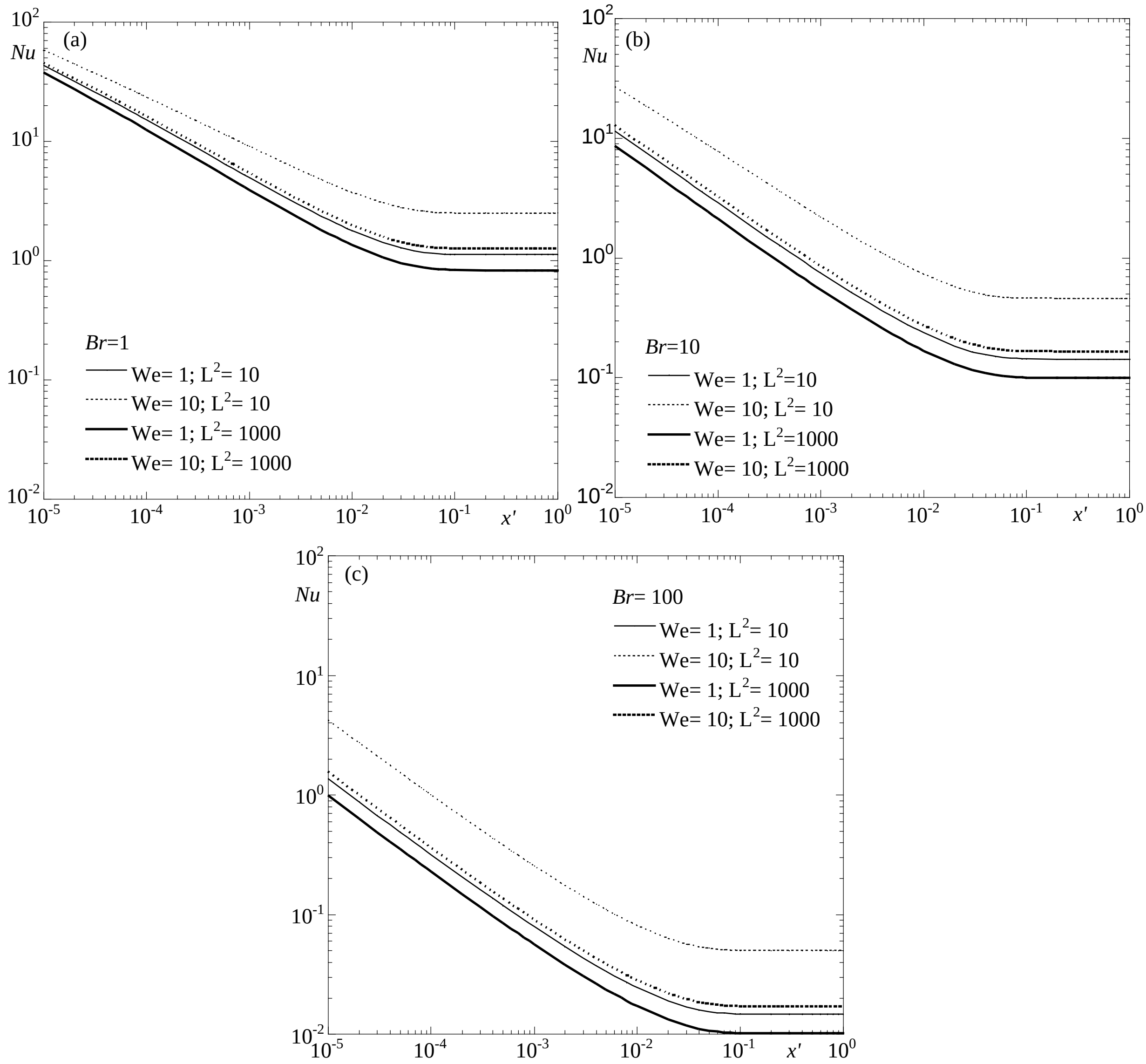

Figure 2- Effect of viscoelasticity, measured by *We* and L^2, on the Nusselt number variation for various Brinkman numbers, and for wall heating ($\dot{q}_w > 0$): (a) *Br*= 1; (b) *Br*= 10; (c) *Br*= 100.

In Figure 2, the combined effects of viscous dissipation and fluid elasticity are assessed. Viscous dissipation decreases the Nusselt number, because the fluid closer to the wall gets warmer, i.e. the wall temperature increases faster than the bulk temperature of the fluid and, therefore, for the same wall heat flux the heat transfer coefficient need not be as high. In contrast, fluid elasticity, measured by the Weissenberg number, tends to increase the Nusselt number because of the distortion in the velocity profile: as *We* increases, shear-thinning intensifies and the velocity profile becomes flatter, leading to higher shear rates in the wall region and improved wall heat transfer, thus reducing the temperature variation across the duct and increasing the Nusselt number. This effect is intensified by lowering the extensibility parameter, L^2.

The difference between the behaviours of the PTT and FENE-P fluids in this pure shear flow lies on the effect of ε and L^2. Whereas in the former an increase in both ε and *We* made the velocity profile flatter, and thus increased the Nusselt number, for the latter fluid L^2 and *We* gave opposite effects with *We* increasing shear-thinning and the Nusselt number and higher values of L^2 decreasing shear-thinning and *Nu* (in fact, Eq. (9) yields $\varepsilon \approx 1/L^2$ for large L^2).

5. Conclusions

We have described a method to obtain thermal entry flow results for the viscoelastic FENE-P model. Nusselt number variations have been given for particular sets of model parameters and one boundary condition (imposed wall heat flux), but the reader may easily use the codes supplied and apply the equivalence between parameters of the PTT and FENE-P models, established by Eq. (9), to obtain results for other conditions. Other thermal results, and full details, are presented in Coelho et al (2003).

In actual engineering applications the thermal properties cannot be considered independent of the temperature. A sensitivity study of this effect for the PTT fluid has recently been carried out by Nóbrega et al (2004), where guidelines on the required corrections can be found. A final comment on the fact of whether the present results could be obtained from solution of the governing equations for a generalized Newtonian fluid possessing an appropriate viscosity function. While the assertion is correct, it should be realized that the viscosity prescription would be rather complex; in fact it would have to take into account the effect of the normal stress τ_{xx} variation across the pipe, as in the PTT and FENE-P models, and hence in a sense "elasticity" would be indirectly present.

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