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NUMERICAL SOLUTION FOR TRANSIENT THERMAL INTERACTION BETWEEN A LAMINAR BOUNDARY LAYER FLOW AND A FLAT PLATE

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Abstract. *The present work reports an analytical-numerical method that deals with the approximate solution of the boundary layer equations for transient convective heat transfer of a Newtonian fluid flowing over a plate, in laminar regime. The transients are caused by the applied uniform wall heat flux on the upper face of the plate. The first step in the solution procedure is the utilization of the Integral Method where the Karman-Pohlhausen approximation is used to obtain polynomial forms for the steady velocity and transient temperature fields in the fluid and the solid. From the polynomial forms of the velocity and temperature fields, the integral form of the energy equations in the fluid and in the solid are obtained. By coupling the thermal boundary layer and the interface temperature a non-linear partial differential equation for the temperature interface is established, and solved by means of a finite-difference method. For illustration of the proposed analytical-numerical approach, a typical application dealing with air heating on a flat plate is considered.This work provides new data on the unsteady behaviour of convective heat transfer, in addition to other previous theoretical and experimental studies.*

Keywords: *external forced convection, transient convection, integral method, heat transfer coefficient.*

Nomenclature

a	thermal diffusivity	$m^2.s^{-1}$
C	constant	
h	convective heat transfer coefficient	$W.m^{-2}.K^{-1}$
Pr	Prandtl number	
t	time variable	s
T	fluid temperature	K
U	velocity component parallel to the plate	$m.s^{-1}$
V	velocity component perpendicular to the plate	$m.s^{-1}$

Greek symbols

δ	dynamic boundary layer thickness	m
δ_t	thermal boundary layer thickness	m
ϕ	heat flux density	$W.m^{-2}$
λ	thermal conductivity	$W.m^{-1}.K^{-1}$
ν	kinematic viscosity	$m^2.s^{-1}$
θ	temperature difference ($T-T_\infty$)	K

1. Introduction

The evaluation of the heat transfer coefficients between a solid and a fluid is necessary for the control and the dimensioning of thermal systems used in various energy processes. The solution of a heat transfer problem between a fluid and a solid often requires the knowledge of the heat transfer coefficient, which incorporates flow features and thermal properties of both media. In the absence of complementary data, there is a practical trend to extrapolate the available correlations for steady-state situations to transient regime as well. However, in many cases, especially when the boundary conditions are time-dependent, this solution path seems to be inadequate and a fully unsteady approach is needed, (Wang et al., 1977, Chung and Kassemi, 1980, Petit et al., 1981, Pierson and Padet, 1985, Remy et al., 1995, Polidori et al., 1998, Rebay and Padet, 1999, Polidori et al., 1999, Rebay et al., 2000, Polidori and Padet, 2002, Padet and De Lorenzo, 2002, Lachi et al, 2004).

While the integral and differential methods have been widely employed in the solution of steady-state external convection problems, and are well-documented even in various textbooks, like Schilichting (1968), besides Ozisik (1980) and Kakaç (1993) in heat conduction analysis, much less information is readily available in its use within transient situations caused by temporal fluctuations of either wall and/or fluid boundary conditions.

The aim of this work is related to this so-called conjugated problem by considering the transient thermal interaction between a laminar boundary layer flow and a semi-infinite flat plate, from a theoretical viewpoint. The unsteady behavior of the analyzed system is due to the generation of an impulsive wall heat flux on the upper face of the flat surface. Results for the time and space evolution of surface temperature, heat flux and deduced convective heat exchange coefficient will be presented.

For illustration of the proposed analytical-numerical approach, a typical application dealing with air heating is considered, where we present the time and space temperature evolutions in the fluid and solid.

2. Mathematical modelling

We consider a semi-infinite flat plate, initially at an equilibrium temperature which is located in a parallel free air stream whose velocity and temperature values are respectively U_∞ and T_∞. The flow is laminar, and assumed to be incompressible. Its thermal properties are considered as constant. The generated dynamical boundary layer is assumed to be steady and independent of the temperature field. At a given time, a uniform heat flux step ϕ_0 is suddenly applied on the upper surface of the plate and continuously maintained as illustrated in Fig. 1. The plate with a finite thickness is absolutely insulated at the bottom. The schematic representation of the physical problem is shown in Fig. 1.

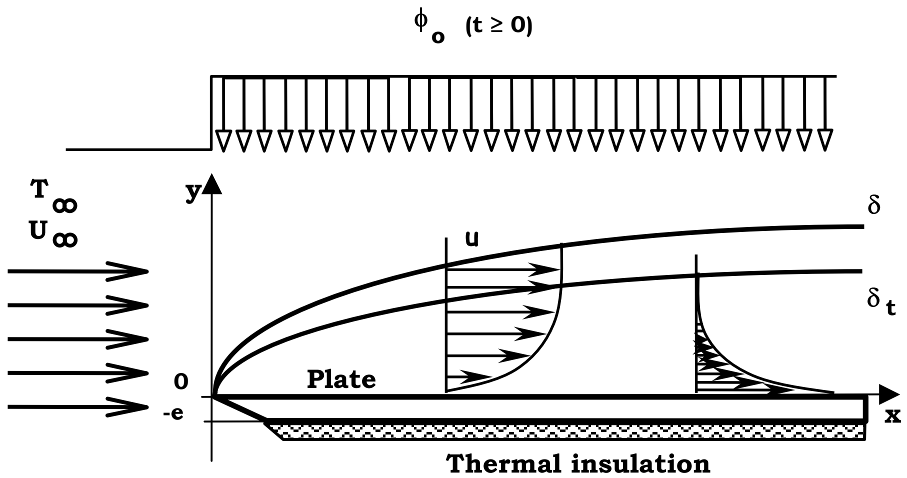

Figure 1 – Schematic representation of the studied physical problem.

2.1. Governing equations

Under steady-state flow but transient heat transfer conditions, the mass, momentum and energy conservation equations in boundary layer formulation are given by:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{1}$$

$$U\ \frac{\partial U}{\partial x}+V\ \frac{\partial U}{\partial y}=\nu\ \frac{\partial^2 U}{\partial y^2} \tag{2}$$

$$\frac{\partial T}{\partial t}+U\ \frac{\partial T}{\partial x}+V\ \frac{\partial T}{\partial y}=a_f\ \frac{\partial^2 T}{\partial y^2} \tag{3}$$

The formulation of the problem for the unsteady temperature distribution in the solid plate, after neglecting longitudinal heat conduction, is given by:

$$\frac{\partial^2 T_s}{\partial y^2} = \frac{1}{a_s}\frac{\partial T_s}{\partial t} \tag{4}$$

The initial and boundary conditions to be satisfied by the velocity and temperature profiles within the fluid are as follows:

$$\begin{aligned}
&U(y,0) = U_\infty \;\; ; \;\; T(y,x,0) = T_\infty \\
&U(0,x) = 0 \quad , \quad x > 0 \\
&U(\delta,x) = U_\infty \; , \quad x > 0 \\
&T(\delta_t,x) = T_\infty \;\; , \quad x > 0 \\
&T(0,x,t) = T_s(0,x,t) \;\; , \quad x > 0
\end{aligned} \tag{5a-e}$$

For the solid plate:

$$T_s(y,x,0) = T_\infty \;\; , \quad -e<y<0 \tag{6}$$

$$\left.\frac{\partial T_s}{\partial y}\right|_{y=0} = 0 \tag{7}$$

with the heat balance at the solid-fluid interface (y = 0):

$$\phi_o = \lambda_s \left.\frac{\partial T_s}{\partial y}\right|_{y=0} - \lambda_f \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{8}$$

The following temperature differences are defined for convenience:

$\theta = (T - T_\infty)$, $\theta_s = (T_s - T_\infty)$ and $\theta_p = (T_p - T_\infty)$. (9)

where T_p is the temperature at the fluid-wall interface. The problem can be represented by the time dependent semi-integral form of the energy equation, written in dimensionless form. In the fluid boundary layer we integrate the energy equation over y from y = 0 to y = δ_t:

$$\frac{\partial}{\partial t}\int_0^{\delta_t} \theta \, dy + \frac{\partial}{\partial x}\int_0^{\delta_t} U\,\theta \, dy = -a_f \left.\frac{\partial \theta}{\partial y}\right|_{y=0} \tag{10}$$

The integral method can also be used for solving the unsteady conduction in the solid medium:

$$\int_{-e}^{0} \frac{\partial^2 \theta_s}{\partial y^2}\, dy = \frac{1}{a_s}\int_{-e}^{0} \frac{\partial \theta_s}{\partial t}\, dy \tag{11}$$

The solution methodology applied to Eq. (10) is based on the third and second order polynomial Karman-Pohlhausen approximations for, respectively, velocity and temperature fields. The assumed velocity profile is given by:

$$U = U_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \tag{12}$$

where $\delta = C\sqrt{x}$, $C = 5\sqrt{\frac{\nu}{U_\infty}}$

Substitution of the boundary conditions in the fluid temperature profile gives the form:

$$\theta_f = \theta_P \left[1 - 2\frac{y}{\delta_t} + \left(\frac{y}{\delta_t}\right)^2 \right] \tag{13}$$

By applying the boundary conditions within the plate, the polynomial temperature distribution becomes:

$$\theta_s = \theta_p + \left(Q - 2\lambda^+ \frac{\theta_p}{\delta_t} \right) y + \left(\frac{Q}{2\,e} - \frac{\lambda^+}{e}\frac{\theta_p}{\delta_t} \right) y^2 \tag{14}$$

where $Q = \frac{\phi_0}{\lambda_f}$, $\lambda^+ = \frac{\lambda_f}{\lambda_s}$

Note that the above coefficients are also functions of the time variable, t. Substituting the profiles (12) and (13) into the integral energy equation, Eq. (10), yields the following partial differential equation:

$$\begin{aligned} &\frac{\delta_t}{3}\frac{\partial\theta_p}{\partial t} + \frac{\theta_p}{3}\frac{\partial\delta_t}{\partial t} + \left[\frac{U_\infty\sqrt{Re_x}\,\delta_t\theta_p}{15\,x} - \frac{U_\infty\,Re_x\,\delta_t^2\theta_p}{250\,x^2} \right]\frac{\partial\delta_t}{\partial x} - \frac{U_\infty\sqrt{Re_x}\,\delta_t^2\theta_p}{60\,x^2} \\ &\left[\frac{U_\infty\sqrt{Re_x}\,\delta_t^2}{30\,x} - \frac{U_\infty\,Re_x\,\delta_t^3}{750\,x^2} \right]\frac{\partial\theta_p}{\partial x} + \frac{U_\infty\,Re_x\,\theta_p\delta_t^3}{750\,x^3} = 2\,a_f\,\frac{\theta_p}{\delta_t} \end{aligned} \tag{15}$$

Similarly, the substitution of Eq. (14) into (11) yields the following differential equation for the unknown interface temperature θ_p (x,t) and thermal boundary layer thickness, δ_t :

$$\frac{\partial}{\partial t}\left(\frac{\theta_p}{\delta_t} + \frac{3\theta_p}{2e\lambda^+} \right) = -\frac{3a_s}{e^2}\left(\frac{\theta_p}{\delta_t} + \frac{3\theta_p}{2\,e\,\lambda^+} \right) + \frac{9a_s\theta_p}{2e^3\lambda^+} + \frac{3a_s\phi_o}{2e^2\lambda_f} \tag{16}$$

The solution of Eq. (16) above may be written integro-differential form as follows:

$$\frac{\theta_p}{\delta_t} = \frac{Q}{2}\left(1 - \exp(-R\,t)\right) - \frac{3}{2K}\theta_p + \frac{3}{2K}\,R\,\exp(-R\,t)\int_0^t \theta_p\,\exp(R\xi)\,d\xi \tag{17}$$

where $R = -3\frac{a_s}{e^2}$, $K = e\,\lambda^+$

Thus, the proposed physical problem has been reduced to the coupled one-dimensional partial differential equations, Eqs. (15) and (16), with two unknown variables δ_t and θ_p. Note that the boundary layer thickness is in fact a function of time, t, as well.

For convenience we now substitute $G = {\theta_p}/{\delta_t}$ given by Eq. (17) into Eq. (15). We obtain the final form of the energy equation (15), whose solution gives the interfacial temperature distribution θ_p (x,t) in terms of a single dependent variable. After this substitution the resulting equation is then given as:

$$\left(\frac{2G^4\,\theta_p}{3} + \frac{G^3\,\theta_p^2}{2\,K} \right) \frac{\partial \theta_p}{\partial t}$$
$$+ \frac{3\,U_\infty}{8\,C\sqrt{x}} \left(G^3\theta_p^2 + \frac{G^2\theta_p^3}{K} \right) \frac{\partial \theta_p}{\partial x} - \frac{U_\infty}{4\,C^3 x^{3/2}} \left(\frac{G\theta_p^4}{6} + \frac{\theta_p^5}{5K} \right) \frac{\partial \theta_p}{\partial x} \tag{18}$$
$$+ \frac{3\,U_\infty\,R}{4\,CK\sqrt{x}} \left(-\frac{G^2\theta_p^3}{2} + \frac{\theta_p^5}{15\,C^2 x} \right) exp(-R\,t) \int_0^t \frac{\partial \theta_p}{\partial x} exp(R\xi)\,d\xi$$
$$= 2\,a_f G^6 - \frac{RG^4\,\theta_p^2}{3} + \frac{U_\infty G^3\theta_p^3}{16\,C\,x^{3/2}} + \frac{QRG^4\theta_p^2}{6} - \frac{U_\infty G\theta_p^5}{80\,C^3 x^{5/2}}$$

This equation in terms of the unknown surface temperature is solved by a finite-difference method with an explicit numerical scheme, which leads to a 6th-order polynomial form (see Eq. 20 below). The knowledge of θ_p (x,t) from numerical solution of the above-mentioned equation, permits to deduce the convective exchange coefficient by making use of the heat flux distribution ϕ_f(x,t) at the interface:

$$h(x,t) = \frac{\phi_f(x,t)}{\theta_p(x,t)} = 2\lambda_f \frac{G}{\theta_p(x,t)} \tag{19}$$

3. Numerical computation and results

In this paper we will focus on the illustrative example that is summarized as: the fluid used is air, initially at the ambient temperature. The free stream velocity is taken as U_∞ = 1 m/s and the applied heat flux is ϕ_O = 100 W/m^2. The thickness of the plate is 7 mm and its thermal diffusivity is 1.67 10^{-4} m^2/s.

Eq. (18) was solved by use of the finite difference method with an explicit upwind numerical scheme. The integration time step was kept constant and equal to Δt = 0.005 *s*, while the space step varied from Δx = 2.10^{-4} *m* near the plate leading edge, to Δx = 10^{-3} *m* beyond the abscissa *x* = 1 *cm*.

Using the subscript *j* to denote time, and subscript *i* to represent the *x* location, the numerical representation used for the discretized form of Eq. (18) is given by:

$$A_6\,\theta_p^6 + A_5\,\theta_p^5 + A_4\,\theta_p^4 + A_3\,\theta_p^3 + A_2\,\theta_p^2 + A_1\,\theta_p + A_0 = 0 \tag{20}$$

A computer Borland Turbo Pascal program was written to solve Eq. (18) for each mesh node (*i*+1, *j*+1), by means of an iterative scanning process applied to the closed interval where positive roots exist. Such intervals are identified with the aid of the Sturm rule and after bracketing the root of the equation we use a hybrid algorithm with the Newton-Raphson and Bisection schemes. The root returned is then further refined until its accuracy is known to within ± 10^{-6}.

Figure 2 shows the interface temperature distribution as a function of time and distance. It can be observed, when time increases, that the space distributions of the interfacial temperature for different times are monotonously increasing. For large time values the temperature profile tends to the steady state solution.

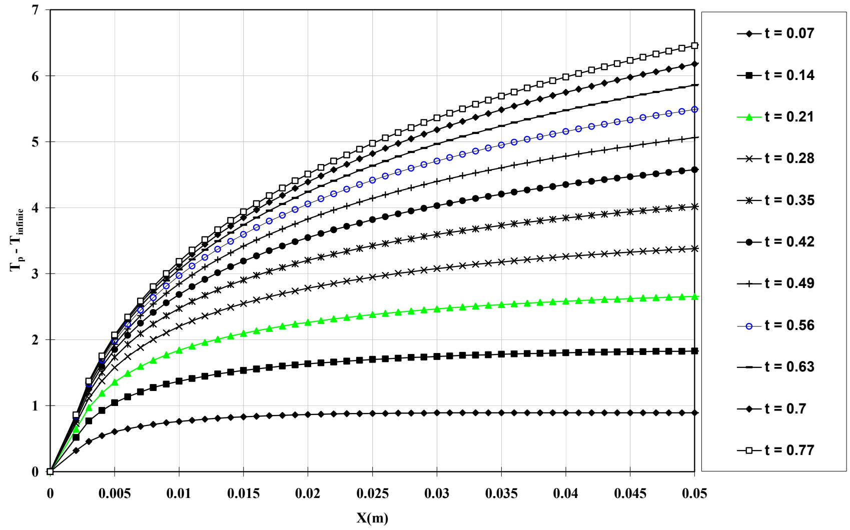

Figure 2 – Solid/fluid interfacial temperature evolutions.

The transient evolutions of the local convective heat transfer coefficient deduced from Eq. (19) are shown in Figure 3. During the first stages of the process, it can be seen that the heat transfer to the fluid is entirely governed by the conduction phenomenon in the solid. For small values of time the solid plate absorbs more energy than the fluid, therefore the heat flux evolution at the interface is lower than the interfacial temperature one. Consequently, the convective heat transfer curves decrease rapidly at small values of time. The very short transitional stages observed here are due mainly to the large temperature variations within the thermal boundary layer, which just starts to be formed. At large times the heat transfer coefficient tends to its steady state value.

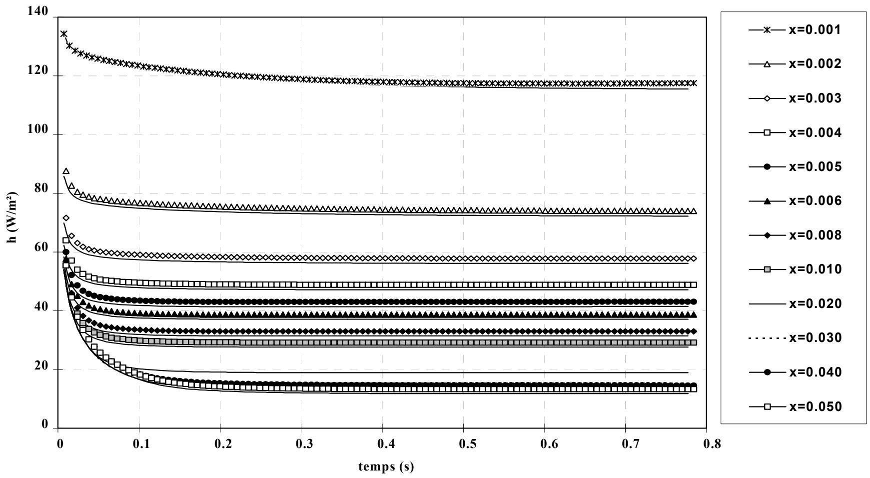

Figure 3 – Transient convective heat transfer coefficient, at different locations x.

In order to see the transient evolutions inside the wall and the fluid boundary layer, we sketched on Figures 4 and 5 the temporal temperature evolutions inside the two media for the fixed time values 0.035 and 0.35 s, and for some selected x positions. These evolutions confirm that with the inclusion of the wall thickness into the modeling, the convection heat transfer phenomenon is delayed and the time required for establishment of the steady-state is a longer one, for in this case, the wall absorbs a portion of the incident energy. These profiles in terms of the y coordinate, plotted for five different abscissa x, were assumed parabolic, and as the time variable increases they gradually take in the solid a linear form characteristic of the final steady regime.

Figures 6 and 7 show the spatial distribution of the heat flux deduced from the temperature derivatives at the interface, for different times. For any given position along the interface we always have the algebraic sum of the two heat flux distributions equal to the impulsive uniform heat flux ϕ_0, as required by the interface thermal balance.

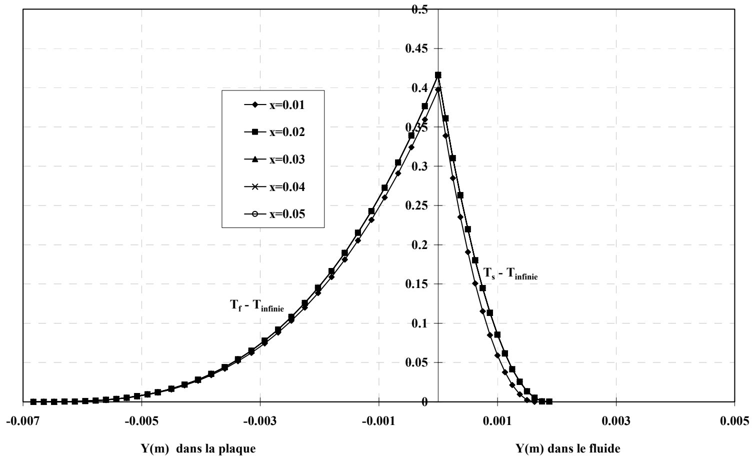

Figure 4 – The solid and the fluid temperature evolutions inside the solid and the fluid for the fixed time, t=0.035 s.

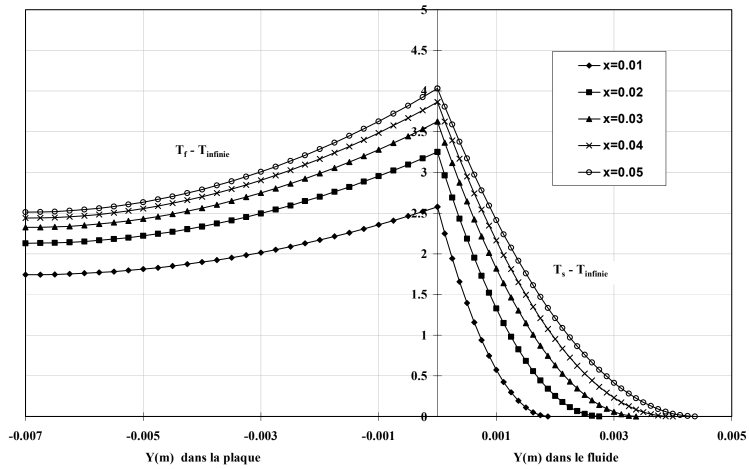

Figure 5 – The solid and the fluid temperature evolutions for the fixed time value 0.35 s.

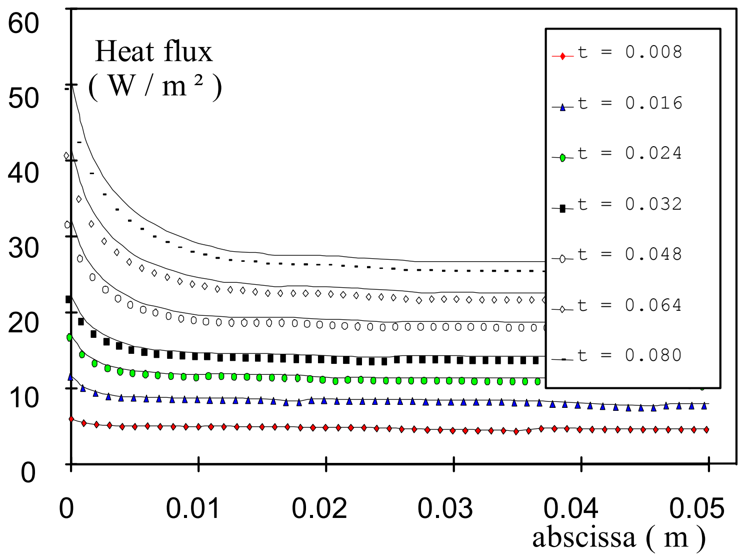

Figure 6 – Spatial heat flux distributions (at the fluid interface).

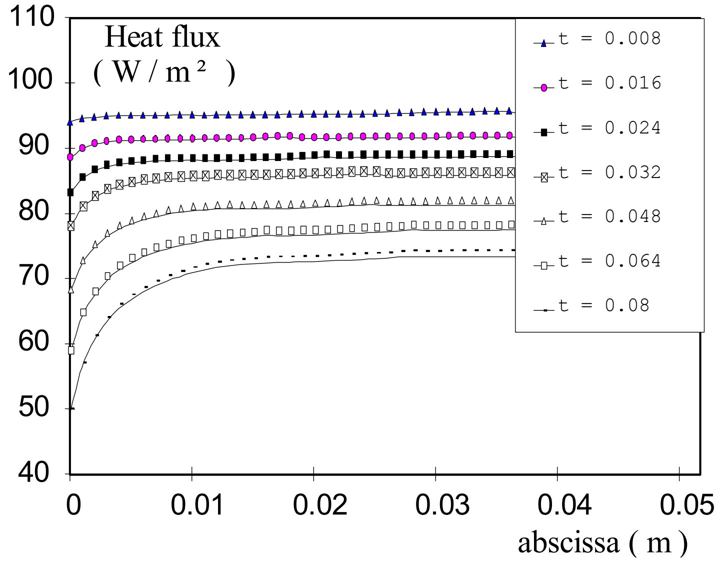

Figure 7 – Spatial heat flux distributions (at the plate interface).

4. Conclusion

An analytical-numerical approach has been employed in order to characterize the time and space evolution of the convective exchanges in the laminar boundary layer of a finite flat plate suddenly heated on the upper surface. The analysis has been developed for a finite thickness plate with air cooling. The transient evolutions of the temperatures inside the plate and within the fluid are presented and critically discussed. This analysis shows the effect and the difficulties of the thermal coupling induced by the presence of a plane wall with non-negligible thickness.

5. References

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