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ON INCREASING THE EFFICIENCY OF A DISCRETE ORDINATES METHOD FOR MULTISLAB PROBLEMS IN RADIATIVE TRANSFER

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Abstract. *In this article, we report recent improvements in a two-component method for solving both conservative and non-conservative discrete ordinates radiative heat transfer problems defined on a multislab domain irradiated from one side with a beam of radiation. The beam is composed by a monodirectional (singular) stream and by a continuous (regular) distribution in angle. Specifically, we have increased the computational efficiency of our two-component method by reducing by one-half the required memory and the number of systems for the determination of the coupling coefficients in the auxiliary equations of the spectral nodal method used for the solution of a discrete ordinates version of the original multislab problem. We illustrate the increased efficiency of our two-component method with numerical results for a model problem in shortwave radiative transfer and we conclude this article with a discussion.*

Keywords*. radiative transfer, discrete ordinates, mixed beams, multislab problems, computational efficiency.*

1. Introduction

In a recent work (de Abreu, 2003), we describe a two-component method for solving both conservative and non-conservative discrete ordinates (S_N) radiative heat transfer problems defined on a multislab domain irradiated from one side with a beam of radiation. The beam is composed by a monodirectional (singular) stream and by a continuous (regular) distribution in angle. Our two-component method starts with a variant to the singular-regular Chandrasekhar procedure (1950) for the decomposition of the target problem into an uncollided problem with one-sided singular boundary conditions and a diffusive problem with regular boundary conditions. Solution to the uncollided problem is fairly easily obtained but, solution to the diffusive problem is not so for the most part. Then, we have considered a standard S_N approximation (Lewis and Miller Jr., 1993) to the diffusive problem and solved it with an improved spectral nodal method free from spatial truncation error (de Abreu, 2003; 2004a). In addition, we have used the slab-geometry equivalence between S_N and spherical harmonics (P_N) formulations (Duderstadt and Martin, 1979) to generate an angularly continuous approximation to the solution of the diffusive problem. At last, we compose uncollided and diffuse solutions for giving an approximate solution to the target problem.

In this article, we report some improvements in our two-component method. Specifically, we have increased its computational efficiency by reducing by one-half the storage and the number of systems for the determination of the coupling coefficients in the auxiliary equations of the spectral nodal method used here for the solution of our S_N version of the diffusive problem. Increase in computational efficiency is achieved by means of periodic relations involving the aforementioned coupling coefficients. We illustrate the increased efficiency of our two-component method with numerical results for a model problem in shortwave radiative transfer.

2. Target problem and analysis

In this section, we set down and analyze the target problem that represents the class of radiative transfer problems dealt with in this article. Since most of the related discussion can be found in previous work (de Abreu, 2004a,b), presentation here will be cursory. We consider the equation of transfer with arbitrary (Legendre) order of anisotropic scattering of the form

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=S(\tau,\mu),\ \tau\in\Omega\equiv[\tau_0,\tau_R],\ -1\le\mu\le 1, \tag{1}$$

where τ is the optical variable defined on a multislab domain Ω with no reemitting boundaries denoted by τ_0 (left) and τ_R (right), respectively; μ is the cosine of the polar angle defined by the direction of the propagating radiation and the positive τ-axis. The quantity $I(\tau,\mu)$ is the frequency-integrated intensity of the radiation field in direction μ at optical depth τ and $S(\tau,\mu)$ is the scattering source function given by

$$S(\tau,\mu)=\frac{\varpi(\tau)}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\beta_\ell(\tau)P_\ell(\mu)\int_{-1}^{1}d\mu'\,P_\ell(\mu')I(\tau,\mu'). \tag{2}$$

The quantity $\varpi(\tau)$ is the single scattering albedo at depth τ; $(2\ell+1)\beta_\ell(\tau)$ is the ℓth-order component of the Legendre expansion of the scattering phase function and $P_\ell(\mu)$ denotes the ℓth-degree Legendre polynomial. We assume that the multislab domain Ω consists of R contiguous and disjoint layers of homogeneous material each, i.e. the quantities $\varpi(\tau)$ and $\beta_\ell(\tau)$, for all ℓ, are piecewise constant functions of τ on Ω. Equation (1) is subject to the boundary conditions

$$I(\tau_0,\mu) = I_0\delta(\mu-\mu_0)+\gamma_0(\mu), \mu>0, \mu_0>0, \tag{3.1}$$

$$I(\tau_R,-\mu) = 0, \mu>0, \tag{3.2}$$

where I_0 is a nonnegative real; μ_0 is the cosine of the polar angle defining the direction of incidence of the monodirectional component of the beam of radiation upon the left boundary of the multislab domain Ω; the symbol δ is to denote a Dirac distribution and $\gamma_0(\mu)$, $\mu > 0$, is a nonnegative function of μ representing the angularly continuous part of the incident beam of radiation. Equations (1-3) define the (mathematical) target problem representing the class of radiative transfer problems dealt with in this article.

Following a decomposition technique introduced by Chandrasekhar (1950) in solving a basic problem in radiative transfer in planetary atmospheres, we decompose the target problem (1-3) into the uncollided problem

$$\mu\frac{\partial}{\partial\tau}I^u(\tau,\mu)+I^u(\tau,\mu)=0, \tau\in\Omega, -1\le\mu\le 1, \tag{4}$$

with the left singular boundary conditions

$$I^u(\tau_0,\mu) = I_0\delta(\mu-\mu_0);\ I^u(\tau_R,-\mu) = 0, \mu>0, \mu_0>0, \tag{5}$$

and the diffusive problem

$$\begin{aligned}&\mu\frac{\partial}{\partial\tau}I^d(\tau,\mu)+I^d(\tau,\mu)=\\&\frac{\varpi(\tau)}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\beta_\ell(\tau)P_\ell(\mu)\int_{-1}^{1}d\mu'\,P_\ell(\mu')I^d(\tau,\mu')+s^u(\tau,\mu),\ \tau\in\Omega, -1\le\mu\le 1,\end{aligned} \tag{6}$$

with the regular boundary conditions

$$I^d(\tau_0,\mu) = \gamma_0(\mu);\ I^d(\tau_R,-\mu) = 0, \mu>0, \tag{7}$$

so that

$$I(\tau,\mu) = I^u(\tau,\mu)+I^d(\tau,\mu), \tau_0\le\tau\le\tau_R, -1\le\mu\le 1.$$

The quantity

$$s^u(\tau,\mu) \equiv \frac{\varpi(\tau)}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\beta_\ell(\tau)P_\ell(\mu)\int_{-1}^{1}d\mu'\,P_\ell(\mu')I^u(\tau,\mu') \tag{8}$$

in Eq. (6) is a depth-dependent anisotropic source given in terms of the solution $I^u(\tau,\mu)$ to the uncollided problem (4-5). Solution to the uncollided problem (4-5) is fairly easily obtained and has the closed form

$$\begin{cases}I^u(\tau,\mu) = I_0\delta(\mu-\mu_0)\exp\left[-\dfrac{1}{\mu}(\tau-\tau_0)\right], \tau\in\Omega, \mu>0, \mu_0>0,\\ I^u(\tau,\mu) = 0, \tau\in\Omega, \mu<0.\end{cases} \tag{9}$$

We substitute the closed form solution (9) into the source (8) to yield

$$s^u(\tau,\mu) = I_0 \exp\left[-\frac{1}{\mu_0}(\tau-\tau_0)\right]\frac{\varpi(\tau)}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\beta_\ell(\tau)P_\ell(\mu)P_\ell(\mu_0). \tag{10}$$

We now decompose the multislab domain Ω into its R contiguous and disjoint homogeneous subdomains (layers) and we define the local (layer-level) diffusive equations

$$\mu\frac{\partial}{\partial\tau}I_r^d(\tau,\mu) + I_r^d(\tau,\mu) = \frac{\varpi_r}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\beta_{\ell,r}P_\ell(\mu)\int_{-1}^{1}d\mu' P_\ell(\mu')I_r^d(\tau,\mu') + s_r^u(\tau,\mu), \tag{11}$$
$$\tau_{r-1} \le \tau \le \tau_r,\ r = 1:R, -1 \le \mu \le 1,$$

with

$$I_1^d(\tau_0,\mu) = \gamma_0(\mu);\ I_R^d(\tau_R,-\mu) = 0,\ \mu > 0,$$

and with intensity continuity conditions at layer interfaces, i.e.

$$I_j^d(\tau_j,\mu) = I_{j+1}^d(\tau_j,\mu),\ -1 \le \mu \le 1,\ j = 1:R-1, \tag{12}$$

where τ_j , j = 1 : R-1, is to denote the jth layer interface. We consider standard S_N approximations to the local diffusive Eqs. (11) in the form

$$\mu_m\frac{d}{d\tau}I_{r,m}^d(\tau) + I_{r,m}^d(\tau) = \frac{\varpi_r}{2}\sum_{\ell=0}^{L_r}(2\ell+1)\beta_{\ell,r}P_\ell(\mu_m)\sum_{n=1}^{N}\omega_n P_\ell(\mu_n)I_{r,n}^d(\tau) \tag{13}$$
$$+ s_{r,m}^u(\tau),\ m = 1:N,\ \tau_{r-1} \le \tau \le \tau_r,\ r = 1:R,$$

where $I_{r,m}^d(\tau) \cong I_r^d(\tau,\mu_m)$, $s_{r,m}^u(\tau) \cong s_r^u(\tau,\mu_m)$. Solution to the S_N Eqs. (13) can be expressed in terms of a homogeneous solution and of a particular solution in the vector form

$$\mathbf{I}_r^d(\tau) = \sum_{i=1}^{N}\alpha_{r,i}\mathbf{I}_{r,i}^d(\tau) + \mathbf{I}_{r,p}^d(\tau),\ \tau_{r-1} \le \tau \le \tau_r, \tag{14}$$

where

$$\mathbf{I}_r^d(\tau) \equiv [I_{r,1}^d(\tau), I_{r,2}^d(\tau), \ldots, I_{r,N}^d(\tau)]^T; \tag{15}$$

$\alpha_{r,i}$, i = 1 : N, are (open) scalars depending upon the boundary conditions;

$$\mathbf{I}_{r,i}^d(\tau) \equiv [I_{r,i,1}^d(\tau), I_{r,i,2}^d(\tau), \ldots, I_{r,i,N}^d(\tau)]^T, i = 1:N, \tau_{r-1} \le \tau \le \tau_r, \tag{16}$$

are the elements of a vector basis for the null space of the local S_N radiative transfer operator

$$\left[\mu_m\frac{d}{d\tau}+1\right](\bullet) - \frac{\varpi_r}{2}\sum_{\ell=0}^{L_r}(2\ell+1)\beta_{\ell,r}P_\ell(\mu_m)\sum_{n=1}^{N}\omega_n P_\ell(\mu_n)(\bullet),\ m = 1:N, \tag{17}$$

and

$$\mathbf{I}_{r,p}^d(\tau) \equiv [I_{r,p,1}^d(\tau), I_{r,p,2}^d(\tau), \ldots, I_{r,p,N}^d(\tau)]^T, \tau_{r-1} \le \tau \le \tau_r. \tag{18}$$

The entries of vector (16) are either exponential functions given by

$$I^{d}_{r,i,m}(\tau) = a_{r,m}(\nu_{r,i})\exp\left(\frac{\tau - \tau_{r,i}}{\nu_{r,i}}\right), \tau_{r-1} \le \tau \le \tau_r, i = 1:N,\ m = 1:N, \tag{19}$$

or first-degree polynomials of the form

$$\frac{(\tau_r - \tau)}{\Delta\tau_r} + \frac{\mu_m}{\Delta\tau_r(1-\beta_{1,r})}, m = 1:N, \tag{20}$$

and

$$\frac{(\tau - \tau_{r-1})}{\Delta\tau_r} - \frac{\mu_m}{\Delta\tau_r(1-\beta_{1,r})}, m = 1:N, \tag{21}$$

with $\Delta\tau_r \equiv \tau_r - \tau_{r-1}$ and $|\beta_{1,r}| < 1$. The quantities $\tau_{r,i}$, $i = 1 : N$, in (19) are appropriate optical depths and $\nu_{r,i}$ and $a_{r,m}(\nu_{r,i})$, are the separation constants and the angular components of the elementary solutions (19), respectively. We use polynomials (20-21) as elementary solutions of the homogeneous version of Eqs. (13) for conservative layers (Chandrasekhar, 1950; de Abreu, 2004a). A numerical scheme for determining the separation constants and angular components is fully described in a work by the author (de Abreu, 1998). The entries of the solution vector (18) are given by the exponential functions

$$I^{d}_{r,p,m}(\tau) = f_{r,m}\exp\left(-\frac{\tau}{\mu_0}\right), \tau_{r-1} \le \tau \le \tau_r, m = 1:N. \tag{22}$$

The determination of the constants $f_{r,m}$, $m = 1 : N$, in the exponential functions (22) is reported in detail by Siewert (2000). At this point, we have been completed with all background material needed for a brief description of the two-component method.

3. A two-component method for multislab radiative transfer problems

The method described in this section is a conjugation of basic relations from more general results in the theory of radiation transport and spectral nodal methods recently developed. The approximate solution to the target problem proposed here is a distribution on τ and μ of the form

$$I_N(\tau,\mu) = I^u(\tau,\mu) + I^d_{N-1}(\tau,\mu), \tau_0 \le \tau \le \tau_R, -1 \le \mu \le 1, \tag{23}$$

where the second term on the right side denotes the spherical harmonics (P_{N-1}) approximation (Duderstadt and Martin, 1979; Lewis and Miller Jr., 1993) to the solution of the local diffusive Eqs. (11), which is given by

$$I^d_{N-1}(\tau,\mu) = \sum_{\ell=0}^{N-1}\frac{(2\ell+1)}{2}\phi^d_{r,\ell}(\tau)P_\ell(\mu), \tau_{r-1} \le \tau \le \tau_r, -1 \le \mu \le 1, r = 1:R, \tag{24}$$

The quantities

$$\phi^d_{r,\ell}(\tau) = \sum_{t=1}^{N}\omega_t P_\ell(\mu_t) I^d_{r,t}(\tau), \tau_{r-1} \le \tau \le \tau_r, r = 1:R, \tag{25}$$

are the P_{N-1} angular moments of the diffuse component of the intensity.

As the name implies, our two-component method has two ingredients: a numerical component and an analytical component. The numerical component is to provide layer-average

$$\bar{I}^d_{r,m} \equiv \frac{1}{\Delta\tau_r}\int_{\tau_{r-1}}^{\tau_r} I^d_{r,m}(\tau)\,d\tau, m = 1:N, r = 1:R, \tag{26}$$

and layer-edge values for the entries of the S_N solution vector (15) *without having to determine the scalars* $\alpha_{r,i}$, r = 1 : R, i = 1 : N. The numerical component is thus suited to radiative transfer problems where the quantities of interest are, for example, the angular distribution of radiation leaving the multislab domain and angle-integrated layer-edge quantities such as radiative heat fluxes (Chandrasekhar, 1950; Thomas and Stamnes, 1999). The analytical component of our two-component method is to reconstruct the approximate solution (24) by solving a system of linear algebraic equations for the scalars $\alpha_{r,i}$ in the S_N solution (15). Inputs to the system are layer-edge values supplied by the numerical component. The analytical component is to be applied when the intensity of the radiation field $I_N(\tau,\mu)$ at any depth τ and direction μ is sought. We briefly describe either component.

The numerical component of our two-component method is a numerical method designed for solving the S_N diffusive problem (13) with no optical truncation error. It is an extension to anisotropic scattering of arbitrary order and depth-dependent anisotropic sources of the spectral Green's function (SGF) method for neutron transport problems (Barros and Larsen, 1990). For this reason, it is referred to as the extended spectral Green's function (ESGF) method. The ESGF method has two main ingredients: one is standard and the other is nonstandard. The standard ingredient is the derivation of radiative balance equations on each layer of the multislab domain Ω, i.e.,

$$\begin{aligned}&\frac{\mu_m}{\Delta\tau_r}(I^d_{r,m} - I^d_{r-1,m}) + \bar{I}^d_{r,m} = \\ &\frac{\varpi_r}{2}\sum_{\ell=0}^{L_r}(2\ell+1)\beta_{\ell,r}P_\ell(\mu_m)\sum_{n=1}^{N}\omega_n P_\ell(\mu_n)\bar{I}^d_{r,n} + \bar{s}^u_{r,m}\,,\ r = 1:R\,,m = 1:N,\end{aligned} \tag{27}$$

where

$$\bar{s}^u_{r,m} \equiv \frac{1}{\Delta\tau_r}\int_{\tau_{r-1}}^{\tau_r} d\tau\, s^u_{r,m}(\tau) = s^0_{r,m}\frac{\mu_0}{\Delta\tau_r}\left[\exp\left(-\frac{\tau_{r-1}}{\mu_0}\right) - \exp\left(-\frac{\tau_r}{\mu_0}\right)\right], \tag{28}$$

is the discretized source term. The nonstandard ingredient is to set in the ESGF auxiliary equations

$$\bar{I}^d_{r,m} = \sum_{u=1}^{N/2}\theta_{r,m,u}I^d_{r-1,u} + \sum_{u=N/2+1}^{N}\theta_{r,m,u}I^d_{r,u} + g_{r,m}, r = 1:R\,,m = 1:N, \tag{29}$$

where the layer-dependent coefficients $\theta_{r,m,u}$ and $g_{r,m}$ are determined so that the analytical solution (14) does satisfy the ESGF auxiliary Eqs. (29), for arbitrary scalars $\alpha_{r,i}$ and for the entries of vector (18) given by the exponential functions (22). We suitably defer the discussion of the ESGF auxiliary Eqs. (29) to the next section. Equations (27) and (29) constitute the system of discretized equations of the ESGF method. Solution methods for this system are discussed elsewhere (Barros and Larsen, 1990).

The analytical component of our two-component method is a local (layer-level) analytical reconstruction scheme of the approximate solution (24). It is based upon solving a local system of N linear algebraic equations whose unknowns are the scalars $\alpha_{r,i}$, r fixed, i = 1 : N. Inputs to the system are the layer-edge intensities that are incident upon the layer of interest (de Abreu and Barros, 1994). These layer-edge intensities are supplied by the ESGF method. More details can be found in a recent work by the author (de Abreu, 2004a).

4. Increasing the efficiency with periodic relations

The coefficients $\theta_{r,m,u}$ and $g_{r,m}$, r = 1 : R, m = 1 : N, u = 1 : N, in the ESGF Eqs. (29) follow from a standing condition — the open form (14) satisfies the ESGF auxiliary Eqs. (29) for arbitrary scalars $\alpha_{r,i}$, r = 1 : R, i = 1 : N, and also for arbitrary constants $f_{r,m}$, r = 1 : R, m = 1 : N, in the exponential functions (22). From this condition (de Abreu, 2003; 2004a), the coefficients $g_{r,m}$, r = 1 : R, m = 1 : N, can be found to be given by

$$\begin{aligned}g_{r,m} = &\frac{\mu_0 f_{r,m}}{\Delta\tau_r}\left[\exp\left(-\frac{\tau_{r-1}}{\mu_0}\right) - \exp\left(-\frac{\tau_r}{\mu_0}\right)\right] - \\ &\left[\exp\left(-\frac{\tau_{r-1}}{\mu_0}\right)\sum_{u=1}^{N/2}\theta_{r,m,u}f_{r,u} + \exp\left(-\frac{\tau_r}{\mu_0}\right)\sum_{u=N/2+1}^{N}\theta_{r,m,u}f_{r,u}\right], m = 1:N,\end{aligned} \tag{30}$$

and the N coefficients $\theta_{r,m,u}$ (r and m fixed, u varying from 1 to N) are found to satisfy the system of N linear algebraic equations

$$\frac{\nu_{r,j} a_{r,m}(\nu_{r,j})}{\Delta\tau_r}\left[\exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right) - \exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\right] =$$
$$\exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=1}^{N/2}\theta_{r,m,u} a_{r,u}(\nu_{r,j}) + \exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=N/2+1}^{N}\theta_{r,m,u} a_{r,u}(\nu_{r,j}), \tag{31}$$
$$j = 1 : N,$$

for a non-conservative layer ($0 \le \varpi_r < 1$), and the system

$$\left[\frac{1}{2} + \frac{\mu_m}{\Delta\tau_r(1-\beta_{1,r})}\right] = \sum_{u=1}^{N/2}\theta_{r,m,u}\left[1 + \frac{\mu_u}{\Delta\tau_r(1-\beta_{1,r})}\right] + \sum_{u=N/2+1}^{N}\theta_{r,m,u}\left[\frac{\mu_u}{\Delta\tau_r(1-\beta_{1,r})}\right], \tag{32a}$$

$$\left[\frac{1}{2} - \frac{\mu_m}{\Delta\tau_r(1-\beta_{1,r})}\right] = \sum_{u=1}^{N/2}\theta_{r,m,u}\left[-\frac{\mu_u}{\Delta\tau_r(1-\beta_{1,r})}\right] + \sum_{u=N/2+1}^{N}\theta_{r,m,u}\left[1 - \frac{\mu_u}{\Delta\tau_r(1-\beta_{1,r})}\right] \tag{32b}$$

and

$$a_{r,m}(\nu_{r,j})\frac{\nu_{r,j}}{\Delta\tau_r}\left[\exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right) - \exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\right] =$$
$$\left[\exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=1}^{N/2}\theta_{r,m,u} a_{r,u}(\nu_{r,j}) + \exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=N/2+1}^{N}\theta_{r,m,u} a_{r,u}(\nu_{r,j})\right], j = 3 : N, \tag{32c}$$

for a conservative one ($\varpi_r = 1$). Upon substitution of the exponential solutions (19) into the homogeneous version of Eqs. (13) and from a parity analysis of the resulting equations (Siewert, 2000; de Abreu, 2004a,b), it is not difficult to show that the constants $\nu_{r,i}$ appear in ± pairs and that the angular components satisfy the relation $a_{r,m}(\nu_{r,i}) = a_{r,-m}(\nu_{r,-i})$, for all r, m and i, where the lowercase subscripts –m and –i are to denote the discrete direction $-\mu_m$ and the separation constant $-\nu_{r,i}$, respectively. Let us perform a parity analysis of the systems (31) and (32) with the help of the above results. We begin with system (31) for non-conservative layers. Let m vary only from 1 to N/2 in (31), so that we may licitly define a system for fixed r and (m+N/2). Using the above relation for the angular components and considering the parity of the separation constants, we can write the system for fixed r and (m+N/2) in the form

$$\frac{\nu_{r,-j} a_{r,m+N/2}(\nu_{r,-j})}{\Delta\tau_r}\left[\exp\left(\frac{\tau_r - \tau_{r,-j}}{\nu_{r,-j}}\right) - \exp\left(\frac{\tau_{r-1} - \tau_{r,-j}}{\nu_{r,-j}}\right)\right] =$$
$$\exp\left(\frac{\tau_{r-1} - \tau_{r,-j}}{\nu_{r,-j}}\right)\sum_{u=1}^{N/2}\theta_{r,m+N/2,u} a_{r,u}(\nu_{r,-j}) + \exp\left(\frac{\tau_r - \tau_{r,-j}}{\nu_{r,-j}}\right)\sum_{u=N/2+1}^{N}\theta_{r,m+N/2,u} a_{r,u}(\nu_{r,-j}), \tag{33}$$
$$j = 1 : N.$$

The optical depths $\tau_{r,j}$, j = 1 : N, are chosen (de Abreu, 2004a,b) so that

$$\frac{\tau_{r-1} - \tau_{r,-j}}{\nu_{r,-j}} = \frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}, \tag{34}$$

implying that

$$\frac{\tau_r - \tau_{r,-j}}{\nu_{r,-j}} = \frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}. \tag{35}$$

Upon substitution of (34) and (35) into the system (33) and noting that $a_{r,m+N/2}(\nu_{r,-j}) = a_{r,m}(\nu_{r,j})$ and $a_{r,u}(\nu_{r,-j}) = a_{r,-u}(\nu_{r,j})$, we can write the system (33) in the form

$$\frac{\nu_{r,j} a_{r,m}(\nu_{r,j})}{\Delta\tau_r}\left[\exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right) - \exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\right] =$$
$$\exp\left(\frac{\tau_{r-1} - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=1}^{N/2}\theta_{r,m+N/2,u+N/2}\, a_{r,u}(\nu_{r,j}) + \exp\left(\frac{\tau_r - \tau_{r,j}}{\nu_{r,j}}\right)\sum_{u=N/2+1}^{N}\theta_{r,m+N/2,u-N/2}\, a_{r,u}(\nu_{r,j}), \tag{36}$$
$$j = 1 : N.$$

Termwise comparison of systems (31) and (36) leads us to the periodic relations of period $N/2$

$$\theta_{r,m,u} = \theta_{r,m+N/2,u+N/2},\ m = 1 : N/2,\ u = 1 : N/2, \tag{37}$$

and

$$\theta_{r,m,u} = \theta_{r,m+N/2,u-N/2},\ m = 1 : N/2,\ u = N/2+1 : N, \tag{38}$$

for the coupling coefficients in the ESGF auxiliary Eqs. (29). The periodic relations (37) and (38) show that the coupling coefficients $\theta_{r,m,u}$, for m = N/2+1 : N, correspond to those for m = 1 : N/2. Since the coefficients $\theta_{r,m,u}$ are solutions to systems of linear algebraic equations for fixed r and m, the N/2 systems associated with m = N/2+1 : N do not need to be solved, for their solutions are corresponding solutions to systems associated with m = 1 : N/2. Therefore, the periodic relations (37) and (38) have a doubling attractive feature. First, they reduce computer memory requirements, for half of the coefficients $\theta_{r,m,u}$ (m = N/2+1 : N) does not need to be stored. Second, they save computer execution time, since we do not have to solve systems for determining the coefficients $\theta_{r,m,u}$ for m = N/2+1 : N. So, the periodic relations (37) and (38) are likely to increase the computational efficiency of our two-component method. The same periodic relations can be obtained from termwise inspection of systems (32) and the conservative counterpart of system (36).

5. A test problem

We illustrate the increased efficiency of our two-component method with numerical results for a test problem relevant to the transfer of shortwave radiation in a vertically heterogeneous atmosphere. We should notice that the numerical results reported here come from the execution of our FORTRAN program on an IBM-compatible PC (1.4 GHz-clock Intel Pentium 4 processor and 256 Mbytes of RAM) running on GNU/Linux, version 0.2. The executable file has been generated with the g77 GNU Fortran package, release 2.95. The execution (CPU) times reported here were generated with the TIME GNU internal routine, option –S.

Our test problem is based on a six-layer model for a stratified atmosphere described in a work of Devaux *et al.* (1979). Each of the six layers has the same scattering law but the single scattering albedo is allowed to be different in each layer. The optical thickness $\Delta\tau_r$ and single scattering albedo ϖ_r for each layer are provided in Tab. (1). The scattering law is approximated by the L = 8 scattering phase function data given in Tab. (2). The atmosphere is illuminated with a mixed beam having a normally incident component and a linearly anisotropic diffuse component. The boundary data for this six-layer model problem are $\tau_0 = 0$, $\tau_6 = 21$, $I_0 = 0.5$, $\mu_0 = 1$ and $\gamma_0(\mu) = \mu$, $\mu > 0$.

Table 1. Optical thickness and single scattering albedo.

r	$\Delta\tau_r$	ϖ_r
1	1.0	1.0
2	2.0	0.70
3	3.0	0.75
4	4.0	0.80
5	5.0	0.85
6	6.0	0.90

In Tab. (3), we present layer-edge results for converged S_{200} downwelling (q^+) and upwelling (q^-) radiative heat fluxes (Chandrasekhar, 1950; Thomas and Stamnes, 1999). Since no approximation has been introduced in the derivation of the periodic relations reported in the previous section, the numerical results in Tab. (3) are the same as those tabulated in a work under consideration for publication (de Abreu, 2003). In Tab. (4), we show computer memory and execution times for the runs with (case 1) and without (case 2) the periodic relations (37) and (38). It is apparent from the savings in computer memory and execution time that the periodic relations increased the computational efficiency of our two-component computer code. The reduction in execution time is relatively modest, as compared to the memory one, because a considerable fraction of the CPU time is used up for computing the separation constants and the angular components in the exponentials (19).

Table 2. Scattering phase function data.

ℓ	$(2\ell+1)\beta_\ell$
0	1
1	2.00916
2	1.56339
3	0.67407
4	0.22215
5	0.04725
6	0.00671
7	0.00068
8	0.00005

Table 3. Numerical results for radiative heat fluxes.

	$\tau_0 = 0$	$\tau_1 = 1$	$\tau_2 = 3$	$\tau_3 = 6$	$\tau_4 = 10$	$\tau_5 = 15$	$\tau_6 = 21$
$q^{+}_{200}(\tau_j)$	5.235972	4.671763	1.481298	0.309292	4.7609E-02[a]	6.4262E-03	8.3764E-04
$q^{-}_{200}(\tau_j)$	0.994869	0.430660	0.168251	0.046167	9.4260E-03	1.7057E-03	0

[a] Should be read as 4.7609×10^{-2}.

Table 4. Computer memory and execution time.

	Memory (kbyte)	CPU (second)
Case 1	184.7	115.5
Case 2	316.7	146.4

6. Concluding remarks

We conclude this article by noting that the periodic relations (37) and (38) increased the computational efficiency of the two-component method *without degrading* its numerical accuracy. The periodic relations (37) and (38) are exact in the following sense: if the S_N Eqs. (13) were to describe exactly the transport processes for the diffuse component of the radiation in the multislab medium, then the two-component method would generate exact solutions for the diffuse component of the intensity of the radiation field, *with or without* the periodic relations (37) and (38). The periodic relations (37) and (38) do neither improve nor degrade the numerical results generated by our two-component method. We note also that the periodic relations (37) and (38) are in close connection to the concept of discrete Green's functions and response matrices for boundary layer sources (Barros and Larsen, 1990; de Abreu, 2004b). We will explore this topic further in our continued research, and we intend to report on our findings in a forthcoming article.

7. References

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