SHAPE OPTIMIZATION OF A PELTIER HEAT PUMP WITH VARIABLE CROSS-SECTIONAL AREA

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Abstract. *The Peltier effect or thermoelectric cooling is a solid-state method of heat transfer generated at the junction between two semiconductors made of different materials. The difference between the conventional refrigeration and thermoelectric cooling is that a thermoelectric cooling system refrigerates without use of mechanical devices. Heat absorbed at the cold junction is pumped to the hot junction at a rate proportional to electric current passing through the circuit. The aim of this paper is an attempt to answer the question of shape optimization for maximum refrigeration of a convectively cooled thermoelement with variable circular cross-sectional area. The two-point boundary-value initial problem of temperature distribution in the thermoelement is converted to an equivalent two initial-value problem and then numerically integrated. An iterative routine using the Newton's method is employed to calculate a better approximation for the temperature distribution. Solutions to these equations show the effects of various concave and convex geometrical shapes and convective cooling conditions on the overall performances of a Peltier-effect cooling device.*

Keywords: *Thermoelectric cooling, Peltier effect, Thermoelement shape optimization*

1. INTRODUCTION

The method of irreversible thermodynamics promotes a unified treatment of coupled transport phenomena, where energy and mass transfer interactions simultaneously occur between systems that are not in equilibrium. Associated with the simultaneous conduction of thermal and electrical currents through dissimilar conductor materials, the thermoelectric cooling phenomena is one example of such coupled irreversible-flow phenomena. Thermoelectric cooling, sometimes called “Peltier effect”, discovered by the French scientist Jean C. A. Peltier in 1834, represents the release or absorption of a finite heat transfer rate at the junction between two constant – temperature electrical conductors made by different materials (Bejan, 1988).

Since they do not require any moving parts, thermoelectric refrigerators are extremely reliable. Unlike conventional refrigerators, which employ chlorofluorocarbons (CFCs), thermoelectric cooling is an environmentally safe method. Therefore, conceptual design and optimization of thermoelectric cooling systems are the subject of a growing number of

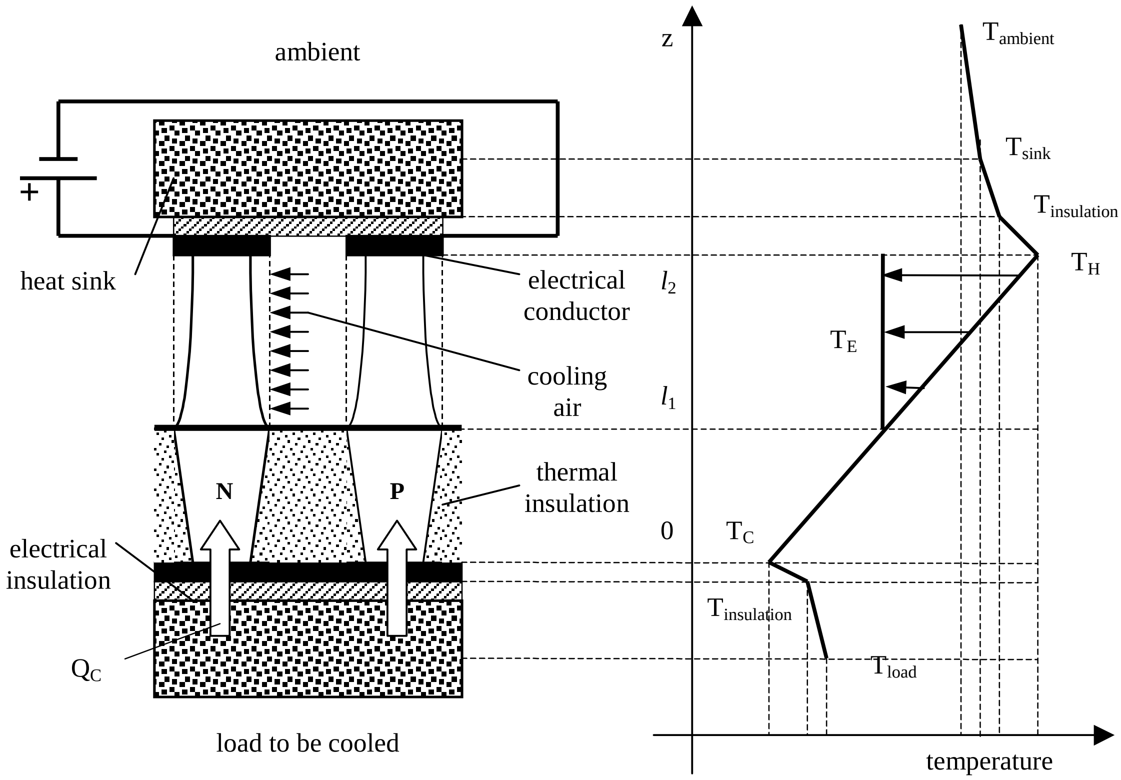


Figure 1 – Schematic view of a solid–state Peltier-effect cooling device with variable circular cross-sectional area arms.

companies that successfully applied them in food service refrigeration for airborne applications, cooled enclosures systems for electronics assemblies and so on.

Early attempts to optimize the thermoelectric devices have been made by Brandt (1962) and Rollinger (1965). Brandt (1962) studied a thermoelectric device with semimetal arms of truncated cone shape and variable properties. He neglected thermal interactions by convection and conduction, between the active thermoelectric material and the ambient, assuming the surroundings to be an evacuated space. Rollinger (1965) studied a convectively cooled thermoelement with variable cross–sectional area. To reduce the differential equation, which describes the steady state one–dimensional temperature distribution through the device arms, to a Bessel equation he assumed special analytical functions for the cross–sectional area and the perimeter of the device arms.

This paper presents a study of the influence of the geometrical shape of active thermoelectric materials on the performance of the solid–state convectively cooled Peltier-effect cooling device.

2. MATHEMATICAL MODEL

The two–legs arrangement in Fig. 1 represents a convectively cooled thermoelectric refrigerator with variable circular cross–sectional area arms. It is assumed that each one of the two arms is completely insulated between $z = 0$ and $z = l_1$, while in thermal interaction by convection with the surroundings (air at T_E) for $l_1 \le z \le l_2$. Temperature of the cold and hot junctions are specified constants, T_C and T_H, respectively. Differential equations for the steady state temperature distribution through each arm can be written:

$$kA(z)\frac{d^2T}{dz^2}+k\frac{dA(z)}{dz}\frac{dT}{dz}+\frac{I^2\rho}{A(z)}+q'=0 \tag{1}$$

with $q' = 0$ for $0 \le z \le l_1$ and $q' = h \cdot p(z) \cdot (T_E - T)$ for $l_1 \le z \le l_2$, where h is the convective heat transfer coefficient between the thermoelectric material and the cooling air, and p(z) is the analytical function describing the variation of the perimeter. In Eq. (1), k is the thermal conductivity of the active thermoelectric material, A(z) is the analytical function describing the variable circular cross–sectional area, I is the electrical current and ρ is the electrical resistivity of thermoelectric material. The non–dimensionalization of Eq. (1) is based on recognizing the following dimensionless variables:

$$\bar{z}=z/r_0 \qquad \bar{r}=r/r_0 \qquad \bar{T}=T/T_C \tag{2}$$

where r_0 is the radius of columnar arms at the cold junction.

Dimensionless form of Eq. (1) is written now:

$$\frac{d^2\bar{T}}{d\bar{z}^2}=-\frac{2}{\bar{r}}\frac{d\bar{r}}{d\bar{z}}\frac{d\bar{T}}{d\bar{z}}+\frac{Bi}{\bar{r}}\bar{T}-\left[\frac{Bi}{\bar{r}}\bar{T}_E+\frac{\rho}{kT_C}\left(\frac{I}{\pi\bar{r}^2r_0}\right)^2\right] \tag{3}$$

where $Bi = 2hr_0/k$ is the Biot number based on the diameter of the variable circular cross-sectional area arms at the cold junction.

Equation (3) is subject to the following boundary conditions:

$$\bar{z}=0 \qquad \bar{T}=1 \tag{4a}$$

$$\bar{z}=\bar{l}_2 \qquad \bar{T}=\bar{T}_H \tag{4b}$$

To study how the shape of columnar arms generally influences the performance of a Peltier-effect cooling device, it is considered the following analytical form of the dimensionless radius $\bar{r}$:

$$\bar{r}(\bar{z})=a\bar{z}^2-2a\bar{l}\bar{z}+1 \qquad 0\le\bar{l}\le\bar{l}_2 \tag{5}$$

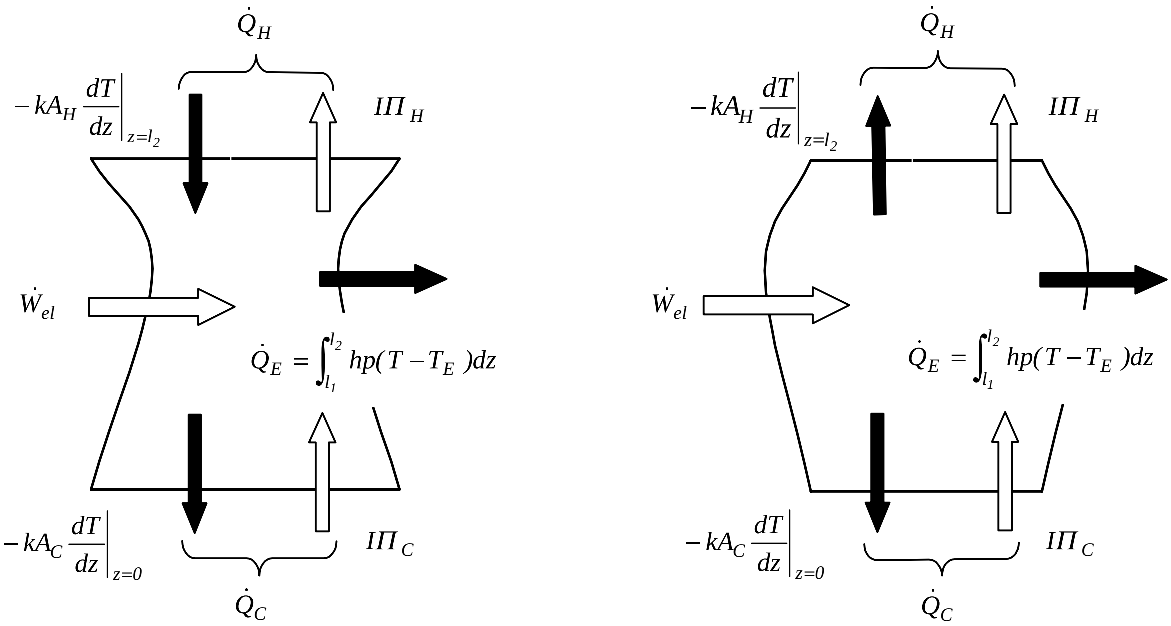
where the a and $\bar{l}$ constants are determined to guarantee positive values of radius $\bar{r}$ and constant-volume column arms:

$$\int_0^{l_2}\pi r^2 dz=\pi r_0^2 l_2 \tag{6}$$

Such a constraint is often justified by the high cost of the thermoelectric material that are employed in manufacture of thermoelectric devices and by the cost associated with the weight of the columnar arms. To select optimally the shape of the columnar arms means to maximize the total heat pumping, $\dot{Q}_C$, when the volume of each column arm is known and fixed.

Solving Eq. (6) it is obtained directly:

$$a = \left| \frac{2(\bar{l} - \bar{l}_2 / 3)}{\bar{l}_2 (\bar{l}_2^2 / 5 + 4\bar{l}^2 / 3 - \bar{l}\bar{l}_2)} \right| \tag{7}$$



(a) (b)

Figure 2 - The first law of thermodynamics for the concave (a) and convex (b) columnar arms of the Peltier-effect cooling device.

The positive values of coefficient a characterize the concave parabolic columnar shape of the thermoelectric active material in Fig. (2a), while the negative ones are specific for the convex parabolic columnar shape arms in Fig. (2b).

The heat balance at the cold junction, $\bar{z} = 0$, gives the expression for the dimensionless heat pumping $\overline{Q}_C$:

$$\overline{Q}_C = \frac{\bar{l}_2}{\overline{T}_H - 1} \left(\frac{I\Pi_C}{\pi k r_0 T_C} - \left.\frac{d\overline{T}}{d\bar{z}}\right|_{\bar{z}=0} \right) \tag{8}$$

where Π_C represents the Peltier coefficient of thermoelectric active material at the cold junction, $\overline{Q} = \dot{Q} / \dot{Q}_0$, and $\dot{Q}_0 = kA_C (T_H - T_C) / l_2$.

The energy balance written at the hot junction, $\bar{z} = \bar{l}_2$, gives the dimensionless heat interaction $\overline{Q}_H$ associated with the isothermal heat sink at T_H:

$$\overline{Q}_H = \frac{\bar{l}_2}{\overline{T}_H - 1} \left(\frac{I\Pi_H}{\pi k r_0 T_C} - \frac{A_H}{A_C} \left.\frac{d\overline{T}}{d\bar{z}}\right|_{\bar{z}=\bar{l}_2} \right) \tag{9}$$

The heat transfer interaction between the columnar arm of the Petier-effect cooling device and the cooling air, $\overline{Q}_E$, reads

$$\overline{Q}_E = \frac{\bar{l}_2 Bi}{\overline{T}_H - 1}\int_{\bar{l}_1}^{\bar{l}_2} \bar{r}(\overline{T} - \overline{T}_E)d\bar{z} \quad (10)$$

Considering each one of the two columnar arms of the solid-state Peltier-effect cooling device in Fig. 1 as a closed system (see Fig. 2), the first law of thermodynamics can be written

$$\overline{W}_{el} = \sum \overline{Q}_j \quad (11)$$

where $\overline{W}_{el}$ is the dimensionless electrical work transfer interaction, and the right side of Eq. (11) represents the algebraic sum of all heat transfer interactions experienced by the system.

The coefficient of performance of the Peltier-effect heat pump is given by

$$COP = \frac{\overline{Q}_C}{\left|\overline{W}_{el}\right|} \quad (12)$$

3. MATHEMATICAL METHOD

The general formulation of the two-point boundary initial problem in Eqs. (3), (4a) and (4b) is:

$$d^2\overline{T}/d\bar{z}^2 = f(\bar{z},\overline{T},\overline{T}');\overline{T}(0) = 1;\overline{T}(\bar{l}_2) = \overline{T}_H \quad (13)$$

where $\overline{T}' = d\overline{T}/d\bar{z}$.

Let us denote an approximation of $\overline{T}'(0)$ by $\overline{Q}^*$, so that the corresponding initial-value problem is given by:

$$d^2\overline{T}_Q/d\bar{z}^2 = f(\bar{z},\overline{T}_Q,\overline{T}_Q');\overline{T}_Q(0) = 1;\overline{T}'(0) = \overline{Q}^* \quad (14)$$

The solution of Eq. (14) is denoted by $\overline{T}_Q$. Now, the objective is to select $\overline{Q}^*$ so that $\overline{T}_Q(\bar{l}_2) = \overline{T}_H$, or $\Phi(\overline{Q}^*) = \overline{T}_Q(\bar{l}_2) - \overline{T}_H = 0$. Newton's formula for the function Φ is

$$\overline{Q}_{n+1}^* = \overline{Q}_n^* - \Phi(\overline{Q}_n^*)/\overline{\Phi}'(\overline{Q}_n^*) \quad (15)$$

where the subscript n+1 indicates the (n+1)th term of the sequence computed by the Eq. (15).

The derivative Φ' is calculated by solving the first variational equation corresponding to Eq. (14)

$$d^2\bar{v}/d\bar{z}^2 = (\partial f/\partial\overline{T}_Q)\bar{v} + (\partial f/\partial\overline{T}_Q')\bar{v}';\bar{v}(0) = 0;\bar{v}'(0) = 1 \quad (16)$$

We recognize Eqs. (14) and (16) as an initial-value problem that can be solved by using the fourth-order Runge-Kutta method. We denote by X the column vector:

$$X = \begin{bmatrix} \overline{T}_Q & \overline{T}'_Q & \overline{v} & \overline{v}' \end{bmatrix}^T \tag{17}$$

Similarly, we denote by F the column vector with components:

$$F = \begin{bmatrix} \overline{T}'_Q & \overline{T}''_Q & \overline{v}' & \overline{v}'' \end{bmatrix}^T \tag{18}$$

The system of differential equations

$$dX / d\overline{z} = F(\overline{z}, X) \tag{19}$$

is accompanied by the auxiliary condition

$$X_{0,n} = \begin{bmatrix} 1 & \overline{Q}_n^* & 0 & 1 \end{bmatrix}^T \tag{20}$$

where $\overline{Q}_o^*$ is a guessed value for the dimensionless heat flux at the cold junction. For $n \geq 1$ $\overline{Q}_n^*$ is calculated based on Eq. (15).

At the end ($\overline{z} = \overline{l}_2$), $\overline{v}(\overline{l}_2)$ will be available, and we have

$$\overline{v}(\overline{l}_2) = \overline{T}'_{Q,n}(\overline{l}_2) = \Phi'(\overline{Q}_n^*) \tag{21}$$

Equation (21) enables us to use the Newton's method in order to calculate a better approximation for $\overline{Q}^*$. When $\left|\overline{T}_{Q,n}(\overline{l}_2) - \overline{T}_H\right| \leq \varepsilon$, where ε is a certain acceptable error, we get an approximate solution of the two-point boundary initial problem (13)

$$\overline{T}(\overline{z}) = \overline{T}_{Q,n}(\overline{z}); 0 \leq \overline{z} \leq \overline{l}_2 \tag{22a}$$

$$\overline{T}'(0) = \overline{Q}_n^* \tag{22b}$$

The calculation of the derivative in Eq. (22b) is one of the more complicated issues in order to evaluate the heat pumping at the cold junction in Eq. (8).

4. NUMERICAL RESULTS

The initial-value problem of Eq. (19) has been solved for a Peltier-effect cooling device made by $Bi_2(Se, Te)_3$ as thermoelectric active material. The properties of $Bi_2(Se, Te)_3$ are those indicated in Table 1, and it is assumed to be identical for p and n elements with a sign change for the Peltier coefficient.

Numerical values of the radius, electrical current, heat sink and load temperatures, and Biot number are:

$$r_0 = 0.002\ m$$
$$l_2 = 0.042\ m$$

$Bi = 0.03; 0.04; 0.05$
$T_C = 260\ K$
$T_H = 310\ K$
$T_E = 275\ K$
$I = 10\ A$

Iterative calculations of temperature distribution and heat pumping at the cold junction have been performed considering the same values of parameter $\bar{l}$ (see Figs. 3a and 3b), for both, the concave and the convex shape of columnar arms of the Peltier cooling device, and assuming $\varepsilon = 10^{-4}$, where ε is an acceptable error for $\left|\overline{T}_{Q,n}(\bar{l}_2) - \overline{T}_H\right| \leq \varepsilon$.

Table 1. Physical properties of various thermoelectric materials.

Thermoelectric material	**k (W·m^{-1}·K^{-1})**	**$\rho \cdot 10^5$ (Ω·m)**	**$\Pi_C \cdot 10^3$ (V)**	**$Z_C \cdot 10^3$ (K^{-1})**
$AgSbTe_2$	0.84	7.5	61.62	0.892
(Zn, Cd) Sb + Sn	1.10	5.9	61.62	0.865
$(Bi, Sb)_2\ Te_3$	1.17	2.42	62.92	2.068
Ag_2Se	0.94	0.78	39.00	3.069
$Bi_2\ (Se, Te)_3$	1.54	1.10	57.46	2.883
PbTe + Na	1.96	1.42	55.12	1.615
Bi_2Te_3 + CuBr	1.92	0.79	52.78	2.717
PbTe + Bi	2.20	1.02	56.16	2.079

The last column in Table 1 contains the numerical values of the figure of merit, $Z_C = \Pi_C^2 / (k\rho T_C^2)$, for different thermoelectric materials.

The results of this calculation, presented in Figs. (3a) - (3f), show an important effect of the geometrical shape and the convective cooling conditions on the temperature distribution along the thermoelectric material. In order to maximize the heat pumping at the cold junction, $\dot{Q}_C$, it is needed to minimize $d\overline{T}/d\bar{z}$, calculated at $\bar{z} = 0$, since it is a positive number in Eq. (8). Numerical results in Figs. (3b) and (3e) show that numerical values of this derivative diminish rapidly when increasing the convective cooling.

Figures (3c) and (3f) show the impact of the geometrical shape on the heat pumping, $\dot{Q}_C$. Graphical results in Fig. (3f) suggest the existence of a optimal convex shape for each value of the Biot number.

In Figs. (3b) and (3e) it is also predicted that the derivative $d\overline{T}/d\bar{z}$, calculated at the hot junction, $\bar{z} = \bar{l}_2$, has positive values for the concave convectively cooled columnar arms, while it has negative values for the convex ones. Therefore, the heat transfer interaction at the hot junction, $\dot{Q}_H$, is larger for the convex shaped columnar arms (see Figs. 2a and 2b). Based on the first law of thermodynamics this predicts also that the electrical work transfer, $\left|\dot{W}_{el}\right|$, is larger for the convex shaped arms.

At the same time, the temperature distributions shown in Figs. (3b) and (3e) indicate that the heat transfer interaction is more intense at the hot junction of a convex shaped arm, than for a concave one. Graphical results presented in Figs. (4b) - (4f) confirm this prediction,

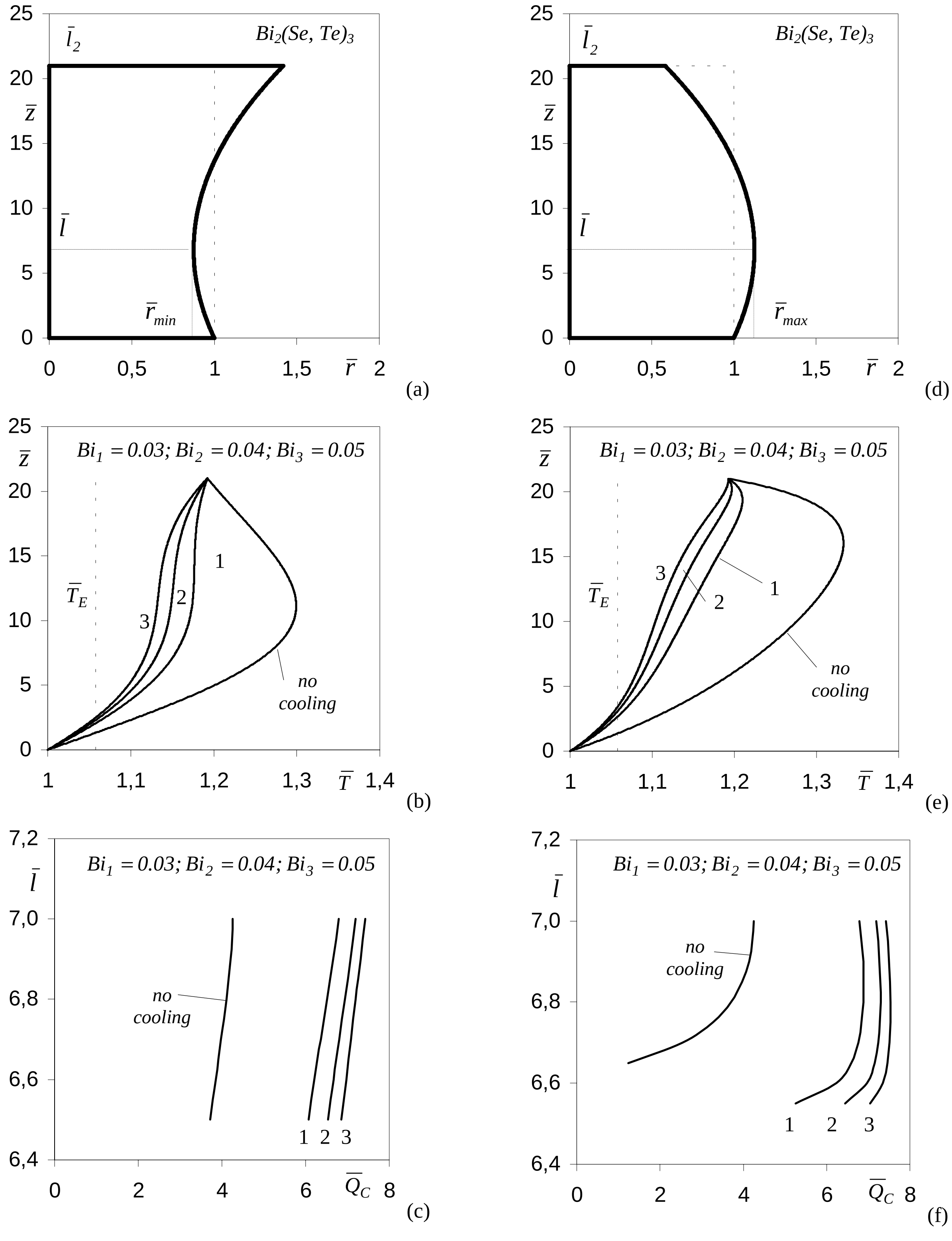


Figure 3 – Dimensionless temperature distribution through the thermoelectric material and the heat pumping variation for parabolic concave and convex arms, subject to a constant-volume constraint ($\bar{r}(\bar{l}) = \bar{r}_{min}$ or $\bar{r}(\bar{l}) = \bar{r}_{max}$).

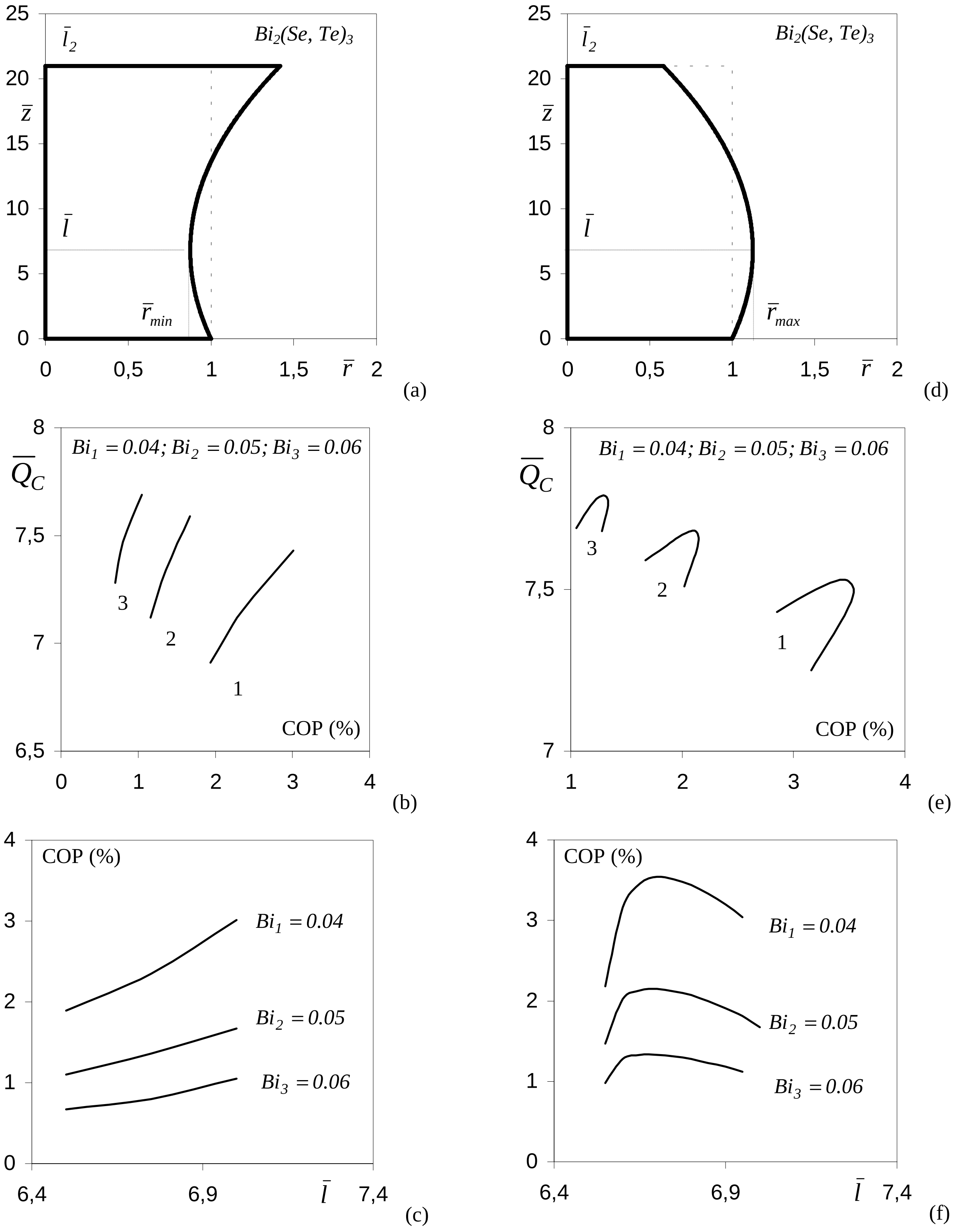


Figure 4 – Variation of the dimensionless heat pumping and the coefficient of performance for parabolic concave and convex columnar arms, subject to a constant-volume constraint.

showing sharp maximum values of the heat pumping at the cold junction, $\dot{Q}_C$, and the coefficient of performance, COP, for the convex shaped columnar arms of a Peltier effect cooling device.

5. CONCLUSIONS

Differential equations, describing the steady state one–dimensional temperature distribution through the thermoelectric active material of a solid–state convectively cooled Peltier-effect cooling device, have been presented and numerically solved.

The results of this study suggest the existence of a optimal convex shape of the columnar arms of the Peltier-effect cooling device and show sharp maximum values of the heat pumping at the cold junction, $\dot{Q}_C$, and the coefficient of performance, COP.

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