MASS TRANSFER MODELING IN OBLATE SPHEROIDAL SOLIDS

João E. F. Carmo
Universidade Federal da Paraiba (UFPB), CCT, Departamento de Física, CEP 58109-970, Campina Grande-PB, Brasil
Antonio G. B. Lima - E-mail: gilson@dem.ufpb.br
Universidade Federal da Paraiba (UFPB), CCT, Departamento de Engenharia Mecânica, CEP 58109-970, Campina Grande-PB, Brasil

Abstract. *A numerical solution of the diffusion equation to describe mass transfer inside oblate spheroids, considering diffusion coefficient and convective boundary condition, is presented. The diffusion equation in oblate spheroidal coordinate system was used, for a bidimensional case, and the finite-volume method was employed to discretize the basic equation. The equation was solved iteratively using the Gauss-Seidel method. As application plots are presented for several aspect ratios. The effects of the Fourier number, Biot number and the aspect ratio of the body on the drying rate and moisture content during the process are presented. To investigate the effect of the aspect ratio only, different results of the average moisture content, are showed. The results shown that the model is consistent and it may be used to solve other cases as those that include disk and sphere and/or those with variable properties under small modifications.*

Keywords: *Mass transfer, Ellipsoid, Numerical, Oblate spheroid*

1. INTRODUCTION

Determination of the mass transfer rate from a particle of arbitrary shape to na infinitely extend fluid, is of considerable importance in certain engineering application such drying, wetting, heating and cooling.

Fundamental solution of these problem for sphere, cylinder and plate have been provided by Crank (1992) , Luikov (1968), Kakaç & Yener (1993), Carslaw & Jaeger (1959). Sphere and infinity cylindrical however represent special case in that axis symmetry in two direction and their orientation relative to the free stream is without effect on the mass transfer rate inside it. There are many situations, however, where the particles are not spherical or cylindrical. In this connection, for example, rice, wheat, banana, silkworm cocoon may be classified as prolate spheroids and as oblate spheroid lentil. Numerical and analytical solutions of the diffusion equation for prolate spheroids has been reported by Haji-Sheikh & Sparrow (1966), Lima *et al.* (1997), Lima & Nebra (1999a); Lima & Nebra (1999b); Lima & Nebra (1999c) and Lima & Nebra (2000). All particles cited may therefore be taken to be axi-symmetric around a axis, so that diffusion around this axis can be analyzed using diffusion equation applied to some axi-symmetric body of revolution, in particular ellipsoids. It is therefore of interest to study as extension of these results, particle of the shapes, such as prolate and oblate spheroids, then exist great analytical difficulties to solve the diffusion problem.

In the present investigation two– dimensional shapes were studied with the following objectives: a) To present a numerical model to describe moisture transport inside the oblate spheroidal solids which will depend only of the particle dimensions.

b) to obtain convective mass transfer data for sphere and to compare the results with those in the literature, to validate the numerical model.

c) to determine convective mass transfer data for other shapes in various aspect rations (oblate spheroids)

2. MATHEMATICAL MODELLING

2.1 Analytical formulation

The Fick's second law in the Cartesian coordinate system is given by:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial M}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\,\frac{\partial M}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\,\frac{\partial M}{\partial z}\right) \quad (1)$$

where M is the moisture content and D is the diffusion coefficient. The Equation (1) is an appropriate equation to predict the mass diffusion in bodies with rectangular shape such as plate and parallelepipeds. Then, to predict the diffusion phenomenon in oblate spheroids, it is necessary to convert this equation for an appropriate coordinate system. In this case the oblate spheroidal coordinate system. The Figure 1 shows the body with oblate spheroidal geometry.

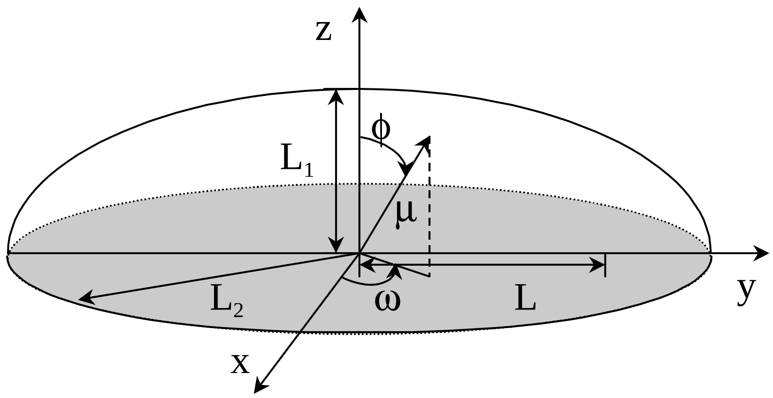


Figure 1 - Characteristics of an oblate spheroid

The relationships between the Cartesian (x, y ,z) and oblate spheroidal coordinate systems (μ, ϕ ,ω) are given by (Stratton *et al.*, 1941; Flammer, 1957; Abramowitz & Stegun, 1972):

$$x = L\cosh\mu\,\mathrm{sen}\phi\,\cos\omega; \qquad y = L\cosh\mu\,\mathrm{sen}\phi\,\mathrm{sen}\omega; \qquad z = L\,\mathrm{senh}\mu\,\cos\phi \quad (2)$$

where $L = \left(L_2^2 - L_1^2\right)^{1/2}$.

Defining $\xi = senh\mu$, $\eta = cos\phi$ and $\zeta = cos\omega$, calculating the metric coefficients and laplacian in the new coordinate system, due to symmetry around z-axis and D constant, the Eq. (1) may be written as follows:

$$\frac{\partial M}{\partial t} = \left[\frac{1}{L^2 \left(\xi^2 + \eta^2\right)} \frac{\partial}{\partial \xi} \left(\left(\xi^2 + 1\right) D \frac{\partial M}{\partial \xi} \right) + \frac{1}{L^2 \left(\xi^2 + \eta^2\right)} \frac{\partial}{\partial \eta} \left(\left(1 - \eta^2\right) D \frac{\partial M}{\partial \eta} \right) \right] \quad (3)$$

with the following boundary conditions:

- Free surface: the diffusive flux is equal to the convective flux at the surface of the prolate spheroid

$$\frac{D}{L} \sqrt{\frac{\left(\xi^2 + 1\right)}{\left(\xi^2 + \eta^2\right)}} \frac{\partial M}{\partial \xi} \Big|_{\xi = \xi_f} + h_m \left[M\left(\xi = \xi_f, \eta, t\right) - M_e \right] = 0 \text{ with } \xi_f = L_1/L \text{ in the surface.} \quad (4)$$

where h_m is the mass transfer coefficient and M_e, the equilibrium moisture content.

- Planes of symmetry: the angular and radial gradients of moisture content are equals to zero at the planes of symmetry.

$$\frac{\partial M\left(\xi, \eta = 1, t\right)}{\partial \eta} = 0 \qquad \frac{\partial M\left(\xi, \eta = 0, t\right)}{\partial \eta} = 0 \qquad \frac{\partial M\left(\xi = 0, \eta, t\right)}{\partial \xi} = 0 \quad (5)$$

- Constant initial conditions in the interior of the solid

$$M(\xi; \eta; 0) = M_o \quad (6)$$

The average moisture content of the body was calculated as follows (Whitaker, 1980):

$$\overline{M} = \frac{1}{V} \int_v M dV \quad (7)$$

where

$$dV = \frac{L^3 \left(\xi^2 + \eta^2\right)}{\sqrt{1 - \zeta^2}} d\xi \, d\eta \, d\zeta \quad (8)$$

and V is the overall volume in the domain in study ($0 \le \xi \le L_1/L$ and $0 \le \eta \le 1$).

Defining the dimensionless parameters .

$$\eta^* = \eta \qquad \xi^* = \xi \qquad t_m^* = \frac{Dt}{L^2} \qquad V^* = \frac{V}{L^3} \qquad M^* = \frac{M - M_e}{M_o - M_e} \qquad Bi = \frac{h_m L}{D} \quad (9)$$

Calculating the derivatives and putting in the Eq. (3), we have:

$$\frac{\partial M^*}{\partial t_m^*}=\frac{1}{(\xi^*)^2+(\eta^*)^2}\left[\frac{\partial}{\partial \xi^*}\left((\xi^*)^2+1\right)\frac{\partial M^*}{\partial t_m^*}\right]+\frac{1}{(\xi^*)^2+(\eta^*)^2}\left[\frac{\partial}{\partial \eta^*}\left(1-(\eta^*)^2\right)\frac{\partial M^*}{\partial \eta^*}\right] \quad (10)$$

Based in the dimensionless parameters the initial, symmetry and boundary conditions are given by:

$$M^*\left(\xi^*,\eta^*,t_m^*=0\right)=1; \quad M_f^*\left(\xi^*=\xi_f;\eta^*;t_m^*\right)=-\frac{1}{Bi}\sqrt{\frac{\left(\xi^{*2}+1\right)}{\left(\xi^{*2}+\eta^{*2}\right)}}\frac{\partial M^*}{\partial \xi^*}\Bigg|_{\xi=\xi_f} \quad (11)$$

$$\frac{\partial M^*\left(\xi^*,\eta^*=1,t_m^*\right)}{\partial \eta^*}=0;\frac{\partial M^*\left(\xi^*,\eta^*=0,t_m^*\right)}{\partial \eta^*}=0; \quad \frac{\partial M^*\left(\xi^*=0,\eta^*,t_m^*\right)}{\partial \xi^*}=0 \quad (12)$$

The average concentration of the body was calculated as follows:

$$\overline{M}^*=\frac{1}{V^*}\int_{v^*} M^* dV^* \quad (13)$$

Therefore,

$$\overline{M}^*=\frac{1}{\int_0^1\int_1^{\frac{L_1}{L}}\left(\xi^{*2}+\eta^{*2}\right)d\xi^* d\eta^*}\int_0^1\int_1^{\frac{L_1}{L}} M^*\left(\xi^*,\eta^*\right)\left(\xi^{*2}+\eta^{*2}\right)d\xi^* d\eta^* \quad (14)$$

In this equation, V^* is the total volume in the domain considered for the new coordinate system.

2.2 Numerical formulation

Due to the symmetry existing in the body, the computational domain showed in Fig. 2 was adopted, where the nodal points (P, N, S, W, E) and lines of ξ and η constants are also presented.

It is possible to verify the existence of the symmetry in the plane that contain the points (x=0, y=0, z=0) and (x=0, y=0, z = L), in particular y≥0 and z ≥0.

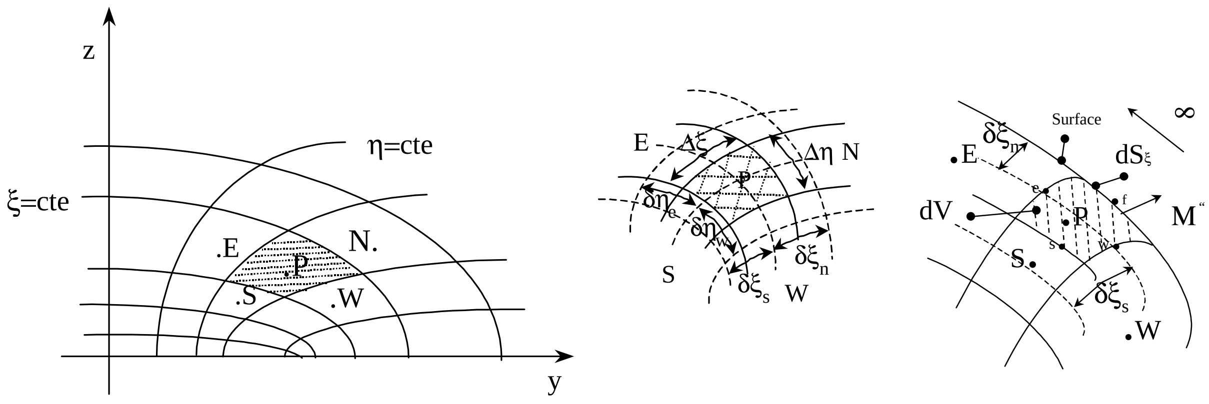


Figure 2 - Schematic of the control volume used in this work

The numerical solution of the problem utilizing the finite-volume method is obtained by the integrating the Eq. (10) in the volume and time. For an implicit formulation and practice B the discretized equation is given by (Maliska,1995; Patankar, 1980):

$$A_P M_P^* = A_E M_E^* + A_W M_W^* + A_N M_N^* + A_S M_S^* + A_P^0 M_P^0 \tag{15}$$

where:

$$A_E = \frac{\left(1-\left(\eta_e^*\right)^2\right)}{\delta\eta_e^*}\Delta\xi;\quad A_W = \frac{\left(1-\left(\eta_w^*\right)^2\right)}{\delta\eta_w^*}\Delta\xi^*;\quad A_P = A_E + A_W + A_N + A_S + A_P^0 + SC$$

$$A_N = \frac{\left(\left(\xi_n^*\right)^2+1\right)}{\delta\eta_n^*}\Delta\eta^* \quad ;\quad A_S = \frac{\left(\left(\xi_s^*\right)^2+1\right)}{\delta\xi_s^*}\Delta\eta^* \quad A_P^0 = \frac{\left(\left(\xi_P^*\right)^2+\left(\eta_P^*\right)^2\right)}{\Delta t_m^*}\Delta\eta^*\Delta\xi^*$$

$$SC = \frac{\Delta\eta^*}{\left[\dfrac{1}{Bi\sqrt{\left[\left(\xi_f^*\right)^2+\left(\eta_P^*\right)^2\right]}\sqrt{\left[\left(\xi_f^*\right)^2+1\right]}}+\dfrac{\delta\xi_n^*}{\left[\left(\xi_f^*\right)^2+1\right]}\right]}$$

The quantity *SC* is a source term which contains the moisture content at the surface that is added to the nodal points preceding the boundary points; at these nodal points coefficient A_N is equal to zero. For the other nodal points SC is equal to zero and A_N is given by the expression presented above. The calculation started with the given initial condition and stopped when the following convergence criterious was satisfied at each point of the computational domain

$$\left|M^{*n-1} - M^{*n}\right| \le 10^{-7};\qquad \frac{\sum(A_K)}{A_P} \le 1 \qquad \textit{For all equations} \tag{16}$$

where n represent the n th. iteration at each step time. To solve the set equations generated by the Eq. (15) it was utilized a computational code developed by Lima (1999) and modified for oblate spheroidal bodies.

3. RESULTS AND DISCUSSIONS

A computational code utilizing the Microsoft Fortran Power Station was used to obtain moisture content profile and the average moisture content for each case as a function of Bi and Fo numbers changing from circular disk ($L_2/L_1 \rightarrow \infty$) to sphere ($L_2/L_1$=1.00) particular cases of oblate spheroids. Numerical solutions of the mass diffusion equation for an oblate spheroid solid has been determined for various values of the input parameters using an uniform grid size 20x20 points and Δt=20s permitting an independence of the grid size and time step. The diffusion coefficient used in this work was 1.22×10^{-9} m^2/s. In all the cases was used L_2=1.0.

To validate the numerical methodology used in the present work, the Fig. 3 illustrate a comparison between numerical results obtained by the authors and analytical results given by Haji-Sheikh (1986) for Bi infinity. As shown the good agreement it was obtained.

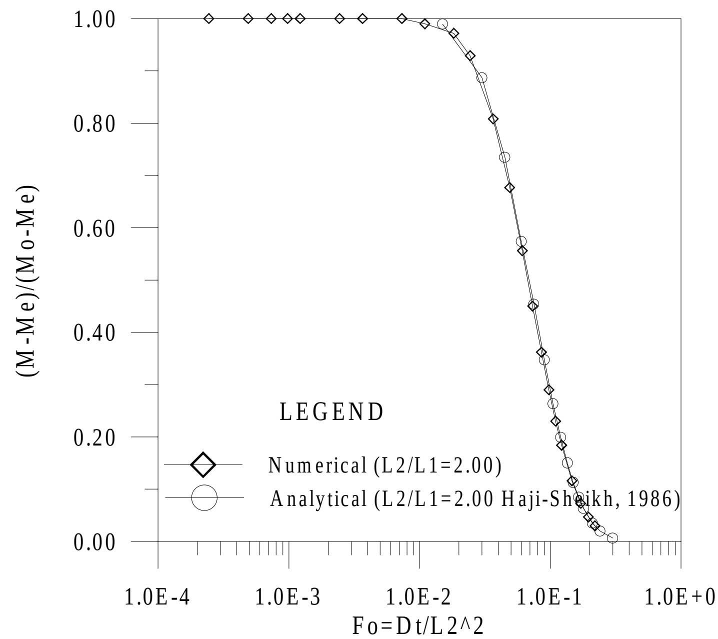


Figure 3 - Comparison between numerical (this work) and analytical (Haji-Sheikh, 1986) results of the dimensionless moisture content in the center of an oblate spheroid with aspect ratio L_2/L_1=2.00. Bi infinity

The Figures 4a-c illustrate the effect of Fourier number in the drying kinetic of the solid for fixed values of other parameters. The showed curves are for different Biot number and body dimensions. The analysis of the curves indicated that the average moisture content decrease with the increase of the Fourier number for any Biot number. For fixed values of aspect ratio, the increase of the Bi increase the drying rate of the product and consequently permitting the reduction of the Fourier number and the drying overall time too.

The Figures 5a-b shown the effect of increase of Fo and Bi in the moisture content in the center of oblate spheroids for L_2/L_1=5.0 and 1.43, respectively. The Figures 6a-b shown the same effect in the focal point of the solid. The Figures 7a-b show the moisture content inside the solid as a function of angular and radial coordinates for Fo=0.01098, L_2/L_1=5.0 and several Biot numbers.

From an examination of these figures it is possible to conclude that for Bi≥10.0 the behavior of moisture content in the center and focal point are approximately equals, when L_2/L_1=1.43, but this effect is different when L_2/L_1 increase. It is also observed that, for small Biot number(Bi≤0.05) both the center and focus are dried slowly, this effect is augmented with the increase of the aspect ratio. In contrast, for higher Biot numbers the focus dry faster with the increase of L_2/L_1.

In conection, it can be seen that the difference between the moisture content in the center and focal point increases with the increase of the aspect ratio L_2/L_1 for fixed value of Biot number. This behavior also occurs at any Biot number. This is due to the localization of the focal point. When L_2/L_1 increase the focal point converge to surface and the solid approximate to a circular disk. In contrast, with the decrease of aspect ratio the focal point converge to the center of oblate spheroid and its converge to sphere.

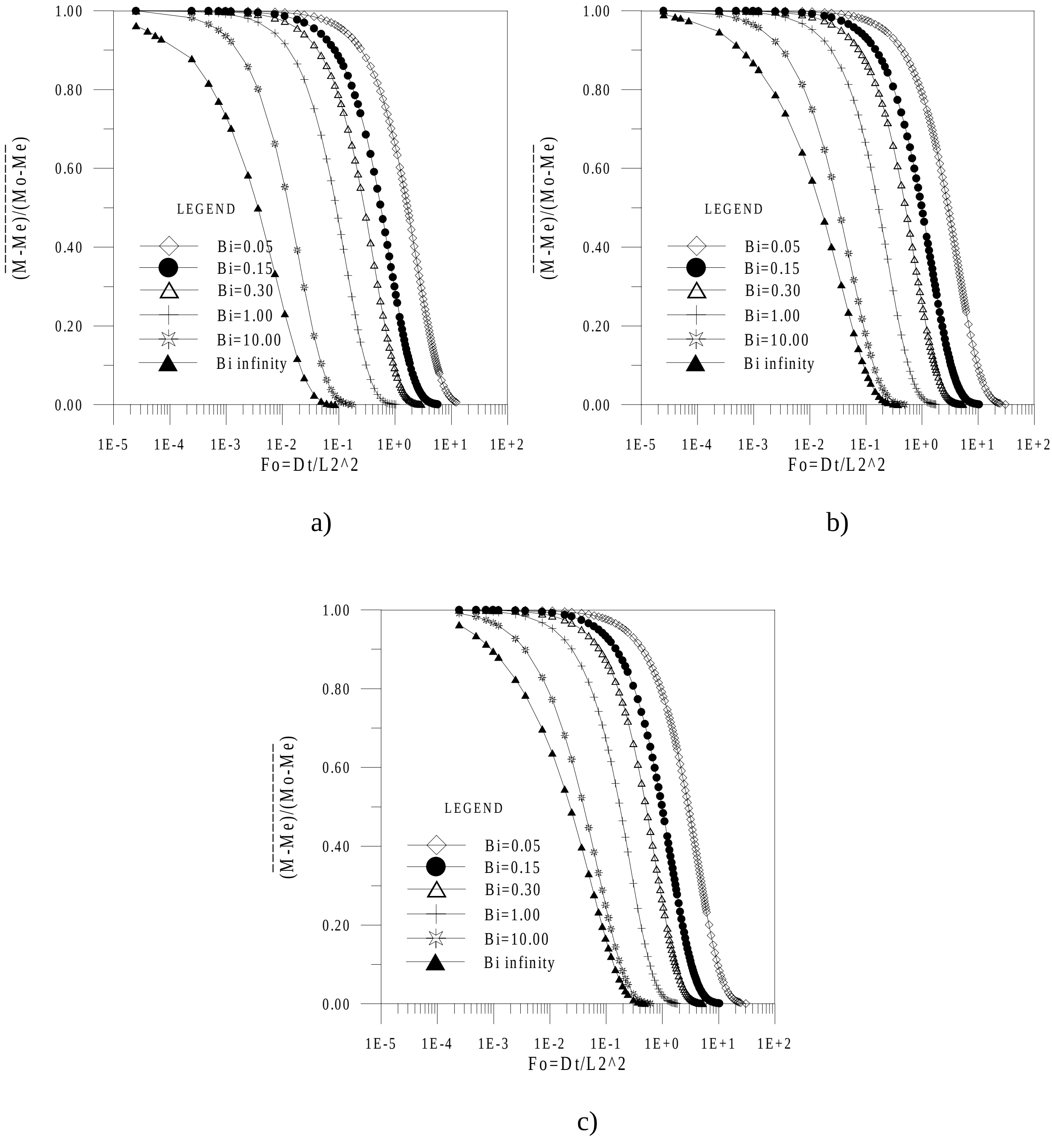
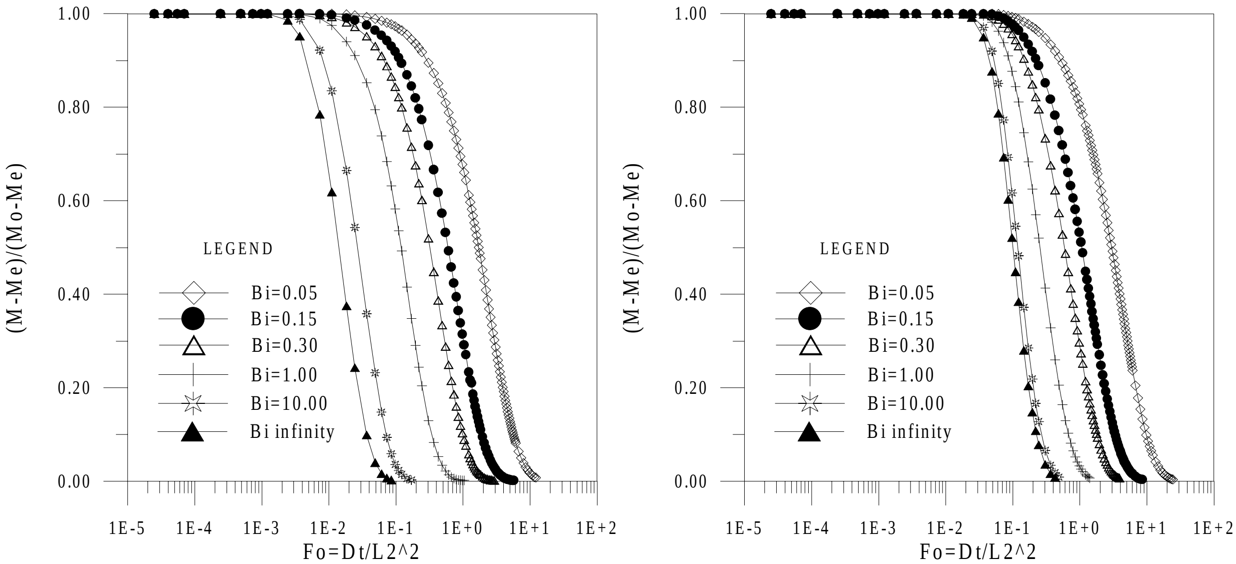
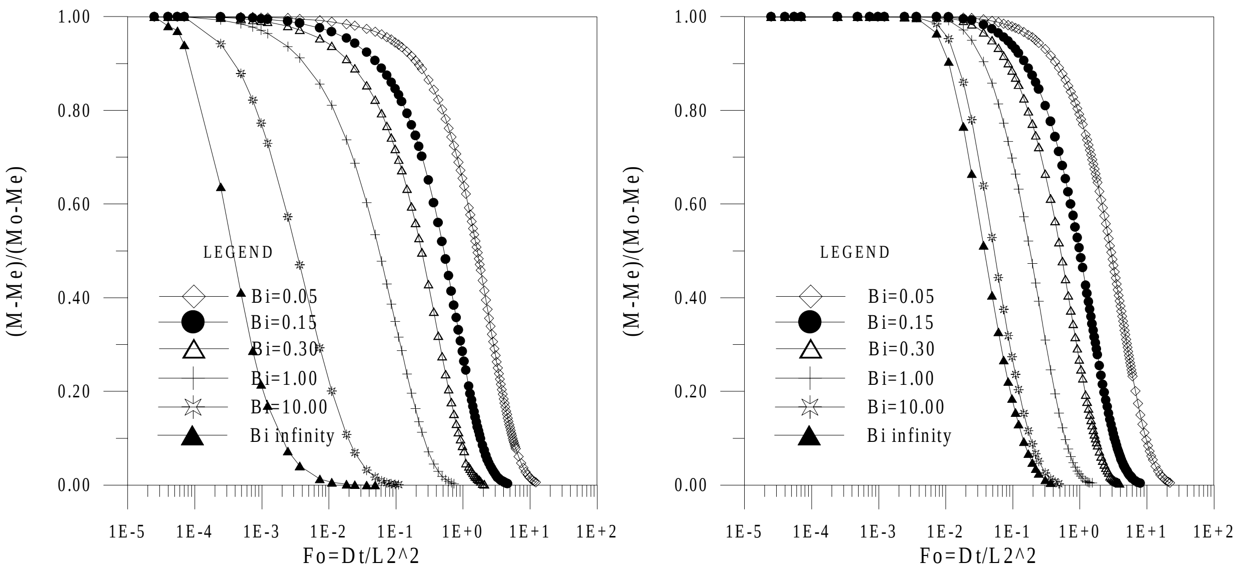


Figure 6 - Average moisture content of a oblate spheroid solid as a function of Fourier number for à) L_2/L_1=5.0, b) L_2/L_1=2.0, c) L_2/L_1=1.43 and several Biot numbers.



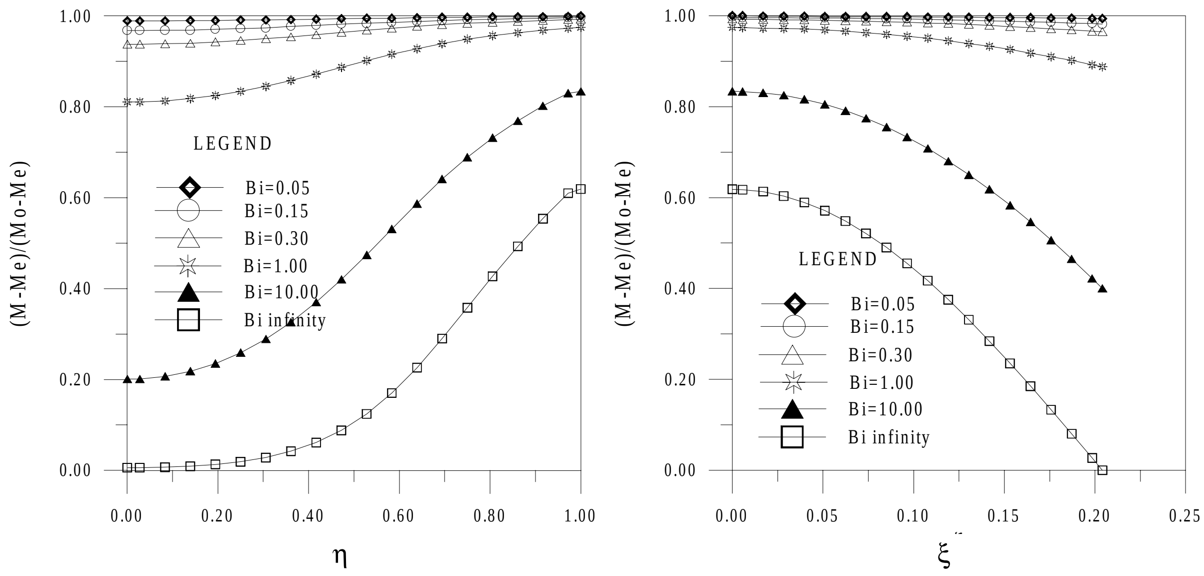
(a) $L_2/L_1=5.0$ (b) $L_2/L_1=1.43$

Figure 5 - Moisture content in the center of a prolate spheroid solid as a function of Fourier number and several Biot numbers



(a) $L_2/L_1=5.0$ (b) $L_2/L_1=1.43$

Figure 6 - Moisture content in the focal point of a prolate spheroid solid as a function of Fourier number and several Biot numbers



(a) ξ=0.0 (0≤y≤L) (b) η=1.0 (0≤z≤L_1)

Figure 7 - Moisture content as a function of (a) angular and (b) radial coordinates of a prolate spheroid solid for Fo=0.01098, L_2/L_1=5.0 and several Biot numbers

In the graphs shown, we can observe that significant difference exists among the drying curves, principally for intermediary values of Fourier number. Particular attention may be directed to the behavior of the drying kinetic for the body with L_2/L_1=5.0 and Bi infinity: the results obtained for the moisture content distribution in the interior of the solid indicate that moisture diffusion occurs initially faster closed to the focal point, decreasing with the drying time in direction to the center of the solid. This behavior is due to the geometrical form of the body.

4. CONCLUSIONS

From the analysis of the results obtained, the conclusions may be summarized as follows:

⇒ the average moisture content decrease with the increase of the Fourier number for any Biot number and aspect ratio, for the fixed values of aspect ratio,

⇒ the increase of the Bi increase the drying rate of the product permitting the reduction in the Fourier number;

⇒ body with highest aspect ratio dry first for any Biot number due to relationships between the area and volume of the solid is smallest;

⇒ the difference between the moisture content in the center and focal point increase with the increase of the aspect ratio L_2/L_1 for any Biot number;

⇒ the model may be used in many physical problems of mass transfer as diffusion in circular disk, sphere and oblate spheroids and also in cases which includes variable diffusion coefficients and other boundary conditions with small modifications.

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