CONVERGENCE OF THE GITT APPROXIMATIONS IN HEAT TRANSFER AND FLUID MECHANICS PROBLEMS

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Abstract. *The Generalized Integral Transform Technique (GITT) has used in Heat Transfer and Fluid Mechanics problems for two or more dimensions and has many extensions. The basic idea consist in choosing an appropriate integral transform pair through an associated auxiliary problem, an Sturm-Liouville problem, to be applied in the original partial differential equation governing the involved phenomenon, which results in a denumerable system of coupled ordinary differential equations. The approximation is the truncation of the infinite system in a sufficiently large order and solve it through standard numerical procedures obtaining the so-called complete solution. Then we invoke the inversion formula to construct the potential. In this work we attempt to give an appropriate formulation employing the functional analysis tools. We outline the proof of the convergence of the GITT approximations to the exact solution, defining the error of the approximated potential with respect to the exact solution.*

Keywords: *GITT approximations , C_0-semigroups.*

1. INTRODUCTION.

The General Integral Transform Technique (**GITT**) has been largely applied in the solution of multi-dimensional heat transfer and fluid mechanics problems. A vast literature is available that it is impossible to mention all of them. On the hand the mathematical task concerning the proof of convergence of this approach was briefly discussed by Cotta (1994), but mathematically speaking, in an incomplete manner. In this work we focus our attention to this direction..

In this section we give an illustration in order to introduce the method to solve a problem which appears in Mikhailov and Ösizik (1984). The next section is devoted to establish the GITT with some functional approach. Section 3 contains the mathematical background for the convergence of the GITT approximations. Finally, some conclusions are given, including some possible extensions.

We illustrate the method with a simple problem. Consider a finite region V in R^3. The partial differential equation governing the behavior of the potential T(x,t), depending of the spatial variable x and the time t > 0, is

$$\varphi(t)w(x)\frac{\partial T(x,t)}{\partial t} = \nabla \cdot K(x)\nabla T(x,t) + \left[\gamma(t)w(x) - d(x)\right]T(x,t) + P(x,t), \qquad (1)$$

following Mikhailov and Ösizik (1984), where w(x), K(x) and d(x) are parameters which depend of the spatial variable, while φ (t) and γ (t) are functions of the time, P(x,t) is the source function, with the initial condition

$$T(x,0) = f, \qquad x \in V \qquad (2)$$

and boundary conditions

$$\alpha(x)T(x,t) + \beta(x)K(x)\frac{\partial T(x,t)}{\partial n} = \phi(x,t), \qquad x \in S \qquad (3)$$

where α(x) and β(x) are prescribed boundary coefficients.

Considering the homogeneous version of the equation (1)

$$\varphi(t)w(x)\frac{\partial T_h(x,t)}{\partial t} = \nabla \cdot K(x)\nabla T_h(x,t) + \left[\gamma(t)w(x) - d(x)\right]T_h(x,t) \qquad (4)$$

with the corresponding initial and boundary conditions

$$T_h(x,0) = f(x), \qquad x \in V, \tag{5-1}$$

$$\alpha(x) T_h(x,t) + \beta(x) K(x) \frac{\partial T_h(x,t)}{\partial n} = 0, \qquad x \in S, \quad t > 0. \tag{5-2}$$

After a separation of variables we may write

$$T_h(x,t) = \sum_{i=1}^{\infty} c_i \psi(\mu_i, x) \Gamma_i(t) \tag{6}$$

where

$$\Gamma(t) = \exp\left[-\int_0^t \frac{1}{\varphi(t')} \left(\mu^2 - \gamma(t') \right) dt' \right]$$

Such a solution satisfies the differential equation (4) with the boundary conditions (5-2), through the appropriate determination of the coefficients c_i' s. Therefore, at t = 0

$$T_h(x,0) = f(x) = \sum_{i=1}^{\infty} c_i \psi(\mu_i, x)$$

which is a representation of $f(x)$ in terms of the eigenfunctions ψ_j's, and these are orthogonal with respect to the weighting function w(x) in the region V, *i.e.*

$$\int_V w(x) \psi(\mu_i, x) \psi_j(\mu_j, x) dV = \begin{cases} 0 & for \quad i \neq j; \\ N_i & for \quad i = j \end{cases}$$

where the normalization integral, or simply the norm N_i, is given by

$$N_i = \int_V w(x) \psi^2(\mu_i, x) dV$$

Now, we multiply both sides by the weighting function times $\psi(\mu_j, x)$ and then integrate in the region V, obtaining

$$\int_V w(x) \psi(\mu_j, x) f(x) dV = \sum_{i=1}^{\infty} c_i \int_V w(x) \psi(\mu_i, x) \psi(\mu_j, x) dV = c_i N_i,$$

because the orthogonality property. The coefficients c_i's can be determined,

$$c_j = \frac{1}{N_j} \int_V w(x) \psi(\mu_j, x) f(x) dV = \frac{1}{N_j} \bar{f}_j$$

and the homogeneous potential is

$$T_h(x,t) = \sum_{i=1}^{\infty} \frac{1}{N_i} f_i \psi(\mu_i, x) \exp\left[-\int_0^t \frac{\mu_i^2 - \gamma(t')}{\varphi(t')} dt' \right]$$

completing the formal solution. The norms N_i's and the transformed of the function f(x) can be readily evaluated and $T_h(x,t)$ computed. This result can be interpreted as an expansion of the potential $T_h(x,t)$ in terms of the known eigenfunctions $\psi(\mu_i,x)$, in the form

$$T_h(x,t) = \sum_{i=1}^{\infty} A_i(t) \psi(\mu_i, x)$$

where the expansion functions A_i's, for the present situation, were determined as

$$A_i(t) = \frac{1}{N_i} \bar{f}_i \exp\left[-\int_0^t \frac{\mu_i^2 - \gamma(t')}{\varphi(t')} dt' \right] = \frac{1}{N_i} \overline{T}_i(t)$$

The transformed potential must satisfy the following ordinary differential equation

$$\frac{d\overline{T}_i(t)}{dt} + \frac{1}{\varphi(t)} \left(\mu_i^2 - \gamma(t) \right) \overline{T}_i(t) = 0, \quad for \quad t > 0$$
$$\overline{T}_i(0) = \bar{f}_i, \quad for \quad i = 1,2,\ldots$$

The solution for the non-homogeneous problem might be possible through the appropriate integration of the original partial differential equation to yield an ordinary system for the transformed potential, eliminating all the operators in the x variables, and then substituting into an expansion such as for the homogeneous solution.

2. INTEGRAL TRANSFORM TECHNIQUE

We present the integral transform technique, whose central ideas are exactly the same as above. In order to advance in the task of giving the functional approach of the GITT, consider V a finite domain in R^3, t belongs to the interval $[0,t_{max}]$, K(x), w(x) and d(x) are real functions belonging to $C^1(V)$, φ (t) and γ (t) are piecewise continuous functions of t, P(x,t) belongs to $C(V\times[0,t_{max}])$, and T(x,t) is in $C^2(V\times[0,t_{max}])$. Finally f(x), α(x) and β(x) are functions in PC(V), the set of piecewise continuous functions of x.

Recasting the equation (1) in operator fashion we have

$$LT(x,t) = P(x,t), \quad for \quad x \in V \quad and \quad t > 0$$

where

$$LT(x,t) \equiv \varphi(t)w(x)\frac{\partial T(x,t)}{\partial t} - \nabla \cdot K(x)\nabla T(x,t) - \left[\gamma(t)w(x) - d(x)\right]T(x,t) \qquad (x,t) \in V \times R^{+}$$

being the domain of L the set $L^2(V\times[0,t_{max}])$ and satisfying the initial and the boundary conditions (2) and (3), respectively. Certainly, we can define the following set :

$$W(V) = \left\{u \in L^2\left(V \times [0,t_{max}]\right) \ \ | \ \ \nabla \cdot K(x)\nabla u \in L^2\left(V \times [0,t_{max}]\right)\right\}$$

The norm is defined in the standard form :

$$\|u\|_2 = \sqrt{\int_0^{t_{max}} \int_V \left(u(x,t)\right)^2 dxdt}\,, \quad for \quad u \in L^2\left(V \times [0,t_{max}]\right)$$

Without loss of generality, we can consider K(x) = 1, and the C_0-semigroup theory guarantees that $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_2^2$ is a generator of a C_0-semigroup of linear operators, and the Cauchy problem (1) with the initial and boundary conditions (2)-(3) is well-posed and has a strong solution, see Dautray-Lions (1984-1985), Goldstein (1985) or Pazy (1982).

Now, we assume that the potential T(x,t) can be constructed as an eigenfunction expansion such as

$$T(x,t) = \sum_{i=1}^{\infty} A_i(t)\psi(\mu_i, x)$$

where the eigenfunctions are obtained from the solution of the auxiliary problem, namely a generalized Sturm Liouville problem

$$\nabla \cdot K(x)\nabla \psi(x) + \left[\mu^2 w(x) - d(x)\right]\psi(x) = 0, \quad for \quad x \in V$$

$$\alpha(x)\psi(x) + \beta(x)K(x)\frac{\partial \psi(x)}{\partial n} = 0, \quad for \quad x \in S$$

To obtain a general expression for the expansion coefficients, A_i's, we take moments with respect to the weighting function w(x),

$$\int_V w(x)\psi(\mu_i, x)T(x,t)dV = \sum_{i=1}^{\infty} A_i(t)\int_V w(x)\psi(\mu_i, x)\psi(\mu_j, x)dV = A_i(t)N_i$$

The left hand side of the equation above defines a transformed potential,

$$\overline{T}_i(t) = \int_V w(x)\psi(\mu_i, x)T(x,t)dV$$

while the complete potential T(x,t) can be obtained by the inversion formula

$$T(x,t) = \sum_{i=1}^{\infty} \frac{1}{N_i} \psi(\mu_i, x) \overline{T}_i(t)$$

The last two equations form the integral transform pair (transform and inverse transform, respectively), and its determination is a basic step in the integral transform technique. We can adopt an integral transform pair with a symmetric kernel,

$$\overline{T}_i(t) = \frac{1}{N_i^{1/2}} \int_V w(x) \psi(\mu_i, x) T(x,t) dV$$

$$T(x,t) = \sum_{i=1}^{\infty} \frac{1}{N_i^{1/2}} \psi(\mu_i, x) \overline{T}_i(t)$$

We integrate with respect to x in equation (1), and obtain an ordinary differential system for the transformed potential. We operate on that equation with $\int_V (\psi(\mu_i,x)/N_i^{1/2})dV$, obtaining

$$\varphi(t) \int_V w(x) \frac{\psi(\mu_i,x)}{N_i^{1/2}} \frac{\partial T(x,t)}{\partial t} dV = \int_V \frac{\psi(\mu_i,x)}{N_i^{1/2}} \nabla \cdot K(x) \nabla T(x,t)\, dV \quad +$$

$$+ \gamma(t) \int_V w(x) \frac{\psi(\mu_i,x)}{N_i^{1/2}} T(x,t) dV - \int d(x) \frac{\psi(\mu_i,x)}{N_i^{1/2}} T(x,t) dV + \overline{g}_i^*(t)$$

where,

$$\overline{g}_i^*(t) = \frac{1}{N_i^{1/2}} \int_V \psi(\mu_i, x) P(x,t) dV$$

is a known function of t.

The remaining terms are determined as follows, see Cotta (1994) :

$$\frac{\varphi(t)}{N_i^{1/2}} \int_V w(x) \psi(\mu_i,x) \frac{\partial T(x,t)}{\partial t} dV = \frac{\varphi(t)}{N_i^{1/2}} \sum_{j=1}^{\infty} \frac{1}{N_j^{1/2}} \left(\int_V w(x) \psi(\mu_i,x) \psi(\mu_j,x) dV \right)$$

$$= \varphi(t) \frac{d\overline{T}_i(t)}{dt}$$

for the expression of the left hand side of the equation (1).

$$\frac{\gamma(t)}{N_i^{1/2}} \int_V w(x) \psi(\mu_i,x) T(x,t) dV = \frac{\gamma(t)}{N_i^{1/2}} \sum_{j=1}^{\infty} \frac{1}{N_j^{1/2}} \left(\int_V w(x) \psi(\mu_i,x) \psi(\mu_j,x) dV \right) \overline{T}_j(t)$$

$$= \gamma(t) \overline{T}_i(t)$$

And also

$$\int_V \frac{\psi(\mu_i,x)}{N_i^{1/2}} \nabla\cdot K(x)\nabla T(x,t)dV - \int_V d(x)\frac{\psi(\mu_i,x)}{N_i^{1/2}} T(x,t)dV =$$

$$= \frac{1}{N_i^{1/2}}\int_V \left[\psi(\mu_i,x)\nabla\cdot K(x)\nabla T(x,t) - T(x,t)\nabla\cdot K(x)\nabla\psi(\mu_i,x)\right]dV$$

$$-\frac{\mu_i^2}{N_i^{1/2}}\int_V w(x)\psi(\mu_i,x)T(x,t)dV$$

where the eigenvalue problem was utilized for the substitution

$$d(x)\psi(\mu_I,x) = \nabla\cdot K(x)\nabla\psi(\mu_i,x) + \mu_i^2 w(x)\psi(\mu_i,x)$$

The first volume integral is transformed into a surface integral, and after utilizing Green's formula, we obtain

$$\varphi(t)\frac{d\overline{T}_i(t)}{dt} + \left(\mu_i^2 - \gamma(t)\right)\overline{T}_i(t) = \overline{g}_i(t), \quad for \quad t>0, \quad i=1,2,...$$

a denumerable system of first order linear equations, where

$$\overline{g}_i(t) = \frac{1}{N_i^{1/2}}\int_V \psi(\mu_i,x)P(x,t)dV + \frac{1}{N_i^{1/2}}\int_S K(x)\left[\psi(\mu_i,x)\frac{\partial T(x,t)}{\partial n} - T(x,t)\frac{\partial\psi(\mu_i,x)}{\partial n}\right]dS$$

In this formula, n represents a normal to the boundary S.

3. SURVEY OF THE CONVERGENCE OF THE GITT APPROXIMATIONS

The system of denumerable set of coupled equations must be truncated, as required for computations, with truncation at the N^{th} row and column. Then, by increasing the order of truncation, N, convergence is computationally checked for, providing the desired numerical results at any prescribed accuracy.

For the finite set of coupled equations, derived of the truncation of the denumerable set of coupled equations, i.e.

$$\frac{dy(t)}{dt} = A(t)y(t) + g(t), \quad for \quad t>0$$
$$y(0) = f$$

We obtain the weak solution (in the case of A(t) with constant coefficients) :

$$v(t) = \Psi^*(t)T^{-1}f + \int_0^t \Psi^*(t-t')T^{-1}g(t')dt'$$

where T is the transformation matrix T = { φ [(1)],..., φ [(n)]} with N linearly independent eigenvectors (φ [(1)],..., φ [(n)]), corresponding to the N distinct eigenvalues, and the new dependent variable y(t) = T v(t). The solution to the non-homogeneous problem with constant coefficient matrix, A, can also be expressed in terms of the exponential matrix function, Zwillinger (1997) :

$$y(t) = e^{At} f + \int_0^t e^{A(t-t')} g(t')dt'$$

which is equivalent to defining a fundamental matrix Ψ (t) = e^{At}. The problem is reduced to evaluate the exponential matrix.

Now we define an inner product, denoted by $\langle ..,.. \rangle$ and defined by means of

$$\langle u,v \rangle = \int_{V\times[0,t_{max}]} w(x)u(x,t)v(x,t)dVdt$$

In terms of the correspondent expansions of both functions u(x,t) and v(x,t), we have

$$\begin{aligned}
\langle u,v \rangle &= \int_{V\times[0,t_{max}]} w(x)\left(\sum_{i=1}^{\infty} U_i(t)\frac{\psi(\mu_i,x)}{N_i^{1/2}}\right)\left(\sum_{j=1}^{\infty} V_j(t)\frac{\psi(\mu_j,x)}{N_j^{1/2}}\right)dVdt \\
&= \int_0^{t_{max}} \left(\sum_{i,j=1}^{\infty} U_i(t)V_j(t)\int_V w(x)\frac{\psi(\mu_i,x)}{N_i^{1/2}}\frac{\psi(\mu_j,x)}{N_j^{1/2}}dV\right)dt \\
&= \sum_{i=1}^{\infty}\int_0^{t_{max}} U_i(t)V_i(t)dt
\end{aligned}$$

The norm associated to this inner product satisfies the following relation :

$$\|u\|_V^2 = \sum_{i=1}^{\infty}\int_0^{t_{max}} U_i(t)^2 dt$$

We define the *error function* of the approximated solution of the truncated expansion in terms of eigenfunctions satisfying the associated Sturm-Liouville problem, *i.e.* $T_M(x,t)$, with respect to the solution obtained by the GITT method, and such approximated solution we call GITT *approximation of order* M, i.e.

$$\varepsilon(x,t) = T_M(x,t) - T(x,t) = \sum_{i=M+1}^{\infty} \frac{\psi(\mu_i,x)}{N_i^{1/2}}\overline{T}_i(t)$$

We can use the relation of the norm associated to the inner product defined above.

$$\|\varepsilon\|_V^2 = \sum_{i=M+1}^{\infty} \int_0^{t_{max}} \left(\overline{T}_i(t)\right)^2 dt$$

The problem of the convergence is reduced to prove that the last sum tends to zero, but for the definition of the transformed potential, *i.e.*

$$\overline{T}_i(t) = \int_V w(x)\psi(\mu_i, x)T(x,t)dV$$

the integrand w(x) $\psi(\mu_i,x)T(x,t)$ is a continuous function, the integral is well-defined and the square of the transformed integral too. Furthermore the transformed potential is continuous because we assume that T(x,t) is in $C^2(V\times[0,t_{max}])$, and for such the transformed potential is bounded

$$T_i(t) \leq \max_{0\leq t\leq t_{max}} \left\{\overline{T}_i(t)\right\}, \quad for \quad 0 \leq t \leq t_{max}$$

We can assume weaker conditions but this is a straightforward task. We obtain the following result

$$\|\varepsilon\|_V^2 \leq \sum_{i=M+1}^{\infty} t_{max}\left(\max_{0\leq t\leq t_{max}} \left\{\overline{T}_i(t)\right\}\right)^2 = t_{max} \sum_{i=M+1}^{\infty} \left(\max_{0\leq t\leq t_{max}} \left\{\overline{T}_i(t)\right\}\right)^2 \rightarrow 0, \quad as \quad M \rightarrow \infty$$

in particular if the numeric sequence ($\max_{0<t<tmax}$ $\{T_i(t)\}$ $)_{i=1,2...}$ is square summable. If we consider other functional spaces, such as the space of functions with compact support it is possible to obtain results for problems with domain not finite, but the analysis is a task for the future.

4. CONCLUSIONS

The functional approach gives the appropriate mathematical background for the convergence of the GITT approximations. When the functions involved are smooth. The strong solution of the problem (1) is guaranteed by the C_0-semigroup theory, and the GITT approximations converges to this strong solution in a strong sense. With other assumptions, for example if T(x,t) is only square integrable, we must obtain other results, and this is a straightfoward task.

The rate of convergence depends on the particular nature of the problem, i.e. of the particular conditions of the initial-boundary values problem. Indeed, as the transformed potential is written in terms of the eigenfunctions, and the error function is expressed in terms of these transformed potentials, the velocity of convergence depend of the solution of the auxiliary problem.

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