BENCHMARK SOLUTIONS TO STEADY NATURAL CONVECTION IN AN INCLINED ENCLOSURE THROUGH INTEGRAL TRANSFORM METHOD

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Abstract. *The Generalized Integral Transform Technique (GITT) is employed in the solution of two-dimensional laminar natural convection within inclined enclosures filled by air (Prandtl number of 0.71), subjected to differentially heated walls and insulated horizontal surfaces. The hybrid nature of the GITT approach allows for the establishment of reference results in the solution of non-linear partial differential systems, as the coupled set of heat and fluid flow equations that govern the steady natural convection problem under consideration. The aim of the present work is to provide reference results to steady-state natural convection in square cavities with Rayleigh numbers equal to* 10^4 *and* 10^5 *for several different inclination angles,* $\alpha = 40^0$, 60^0, 120^0 *and* 140^0*. Numerical values of the mean Nusselt numbers and streamfunctions are presented to all examined situations.*

Key Words: *Natural convection, Integral transform, Hybrid method*

1. INTRODUCTION

The thermally driven square cavity with adiabatic top and bottom walls is one of the most popular test-problems in heat transfer literature. It is important to emphasize that this problem is free of any singularity in the boundary conditions except the presence of the corners of the cavity, which makes it more attractive than other problems, offering a sufficiently complete and complex model for evaluations on accuracy and performance of each individual numerical scheme proposed in heat and fluid flow. Once of the most relevant contributions in this classical problem ($\alpha = 90^0$, vide Fig.1) was given by de Vahl Davis (1983) that utilized the streamfunction-vorticity formulation of the flow equations, and adopted the finite differences method with a false transient scheme to obtain solutions in the Rayleigh number range from 10^3 to 10^6. Non-conservative second order differencing was employed, and Richardson's extrapolation strategy was invoked to generate the final benchmark results.

The importance of the inclined convection problem was firstly pointed out in the review paper of Ostrach (1972). Despite of being an excellent test-problem to check the accuracy of different computational schemes, natural convection inside inclined enclosures occurs in a variety of engineering applications such as storage tanks, solar energy collectors, multilayered walls and double windows.

Talaie and Chen (1985) making use of the finite analytic method (FA), investigated the steady and transient natural convection in inclined enclosures with angles between zero and 90^0. Steady state results were presented for aspect radio of 1, Grashof numbers between 10^4 and 10^6 and Prandtl number of 1. Then, transient solutions were presented for an enclosure with aspect radio of 10, heated for bellow, for Grashof and Prandtl numbers of 10^4 and 1, respectively. The evolution of flow motion in the confined rectangular enclosure was examined.

In a combined experimental and numerical work, Hamady et al. (1989) studied the influence of inclined boundaries and Rayleigh number on the local natural convection heat transfer in an air-filled differentially heated enclosure. The QUICK (quadratic upstream interpolation for convection kinematics) scheme was utilized to attain the numerical solutions. Measurements of local and mean Nusselt numbers were presented at various inclination angles, ranging between 0^0 (heated from above) and 180^0 (heated from bellow), for Rayleigh numbers between 10^4 and 10^6.

Although Rasoul and Prinos (1997) were interested in steady-state results, they made use of the transient natural convection formulation to study the effect of inclination for Rayleigh numbers ranging from 10^3 and 10^6, and Prandtl numbers equal 0.02, 0.71 and 4000, paying attention to their effect on the streamlines, isotherms and local and mean Nusselt numbers. The power law differencing scheme (PLDS) was used for the discretization of the convective terms. The SIMPLE method was used for transforming the continuity equation into Poisson equation and a pressure correction equation was solved. Finally, a tri-diagonal matrix algorithm was applied for solving the algebraic equations.

The Generalized Integral Transform Technique (GITT), reviewed in detail by Cotta (1993), Cotta and Mikhailov (1997) and Cotta (1998), has been progressively established as a powerful tool in benchmarking and engineering applications for linear and nonlinear diffusion and convection-diffusion problems. More specifically it is worth mentioning the integral transform solutions of the 2-D and 3-D Navier-Stokes equations under streamfunction-only formulation for incompressible flow within cavities, and the solutions of laminar and turbulent flows inside regular and irregular ducts. Transient and steady natural convection inside rectangular enclosures, under Boussinesq approximation and with variables properties, were investigated by Leal et al. (1999) and Leal et al. (2000), respectively.

The hybrid nature of GITT allows for the automatic global error control along the solution process, towards an user prescribed accuracy target, making it particularly suitable in obtaining reference results for test-problems, which can then be employed in the validation of purely numerical approaches. The aim of the present work is to furnish some benchmark results through GITT for the natural convection problem inside inclined enclosures, in this case for inclination angles of $\alpha= 40^0$, 60^0, 120^0 and 140^0, $Ra=10^4$ and 10^5, Prandtl number equal 0.71 and aspect ratio of 1.

2. PROBLEM FORMULATION

Steady laminar natural convection of a Newtonian fluid inside a inclined square enclosure is considered. The lateral walls are differentially heated, while the top and the bottom walls are kept insulated, as in Fig 1. The Boussinesq approximation for the buoyancy effect is invoked, and this coupled heat and fluid flow problem is formulated via vorticity transport equation in streamfunction-only formulation, and the associated energy equation, in dimensionless form as:

$$\nabla^4\psi = \frac{1}{\Pr}\left(\frac{\partial\psi}{\partial y}\frac{\partial(\nabla^2\psi)}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial(\nabla^2\psi)}{\partial y}\right)+\mathrm{Ra}\left(\frac{\partial T}{\partial x}\sin\alpha-\frac{\partial T}{\partial y}\cos\alpha\right) \tag{1.a}$$

$$\nabla^2 T = \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} \tag{1.b}$$

with boundary conditions:

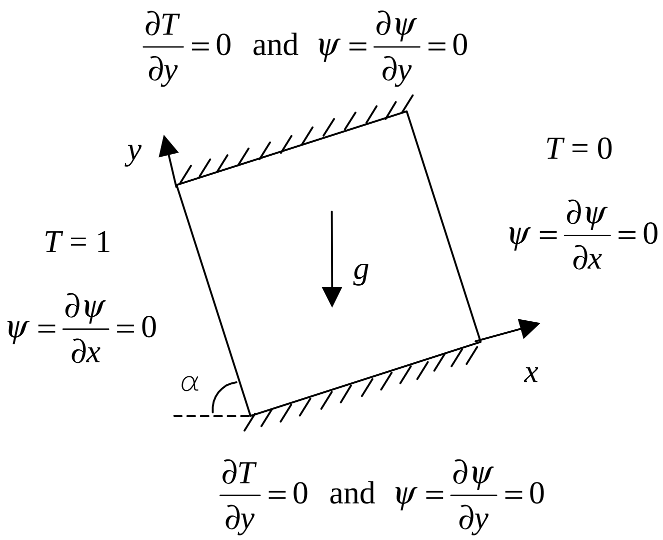


Figure 1- Schematic representation of inclined enclosure.

The remaining dimensionless variables are given by:

$$x=\frac{x_*}{L}\ ;\ \ y=\frac{y_*}{L}\ ;\ \ u=\frac{L}{\alpha}u_*\ ;\ \ v=\frac{L}{\alpha}v_*\ ;\ T=\frac{T_*-T_c}{T_h-T_c}\ ;\ \psi=\frac{\psi_*}{\alpha} \tag{2.a-f}$$

where "*" identifies the dimensional variables, L is the cavity length, while T_h and T_c are the uniform temperatures at hot and cold walls. The Rayleigh and Prandtl numbers are defined, respectively by:

$$\mathrm{Ra}=\frac{g\,\beta\,(T_h-T_c)\,L^3}{\alpha\,\nu}\quad\text{and}\quad \Pr=\frac{\nu}{\alpha} \tag{3.a,b}$$

3. SOLUTION METHODOLOGY

The integral transform approach is based on the eigenfunction expansion of the potentials, in this case, temperature and streamfunction. For this purpose, the boundary conditions on the coordinate variable to be eliminated through integral transformation, are first made homogenous, so as to coincide with the boundary conditions of the eigenvalue problem to be proposed. Thus, a filtering solution for the temperature field is developed, in the form:

$$T(x,y) = T^*(x,y) + T_p(x) \tag{4.a}$$

where the filter T_p is the solution of the pure conduction problem in the cavity, readily obtained as:

$$T_p(x) = 1 - x \tag{4.b}$$

which results in producing a new temperature problem, for T^*, with homogenous boundary conditions, and the final filtered system is rewritten as:

$$\frac{\partial^4\psi}{\partial y^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial x^4} = \frac{1}{\Pr}\left[\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3} + \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial y^2\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^2\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}\right] +$$

$$\mathrm{Ra}\left[\frac{\partial T^*}{\partial x} sin\alpha - sin\alpha\right] - \mathrm{Ra}\frac{\partial T^*}{\partial y} cos\alpha \tag{5.a}$$

$$\frac{\partial^2 T^*}{\partial y^2} + \frac{\partial^2 T^*}{\partial x^2} = \frac{\partial\psi}{\partial y}\frac{\partial T^*}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T^*}{\partial y} + \frac{\partial\psi}{\partial y} \tag{5.b}$$

with now homogeneous boundary conditions:

$$T^* = 0\ ; \quad \psi = \frac{\partial\psi}{\partial x} = 0\ ; \quad x = 0 \tag{5.c-e}$$

$$T^* = 0\ ; \quad \psi = \frac{\partial\psi}{\partial x} = 0\ ; \quad x = 1 \tag{5.f-h}$$

$$\frac{\partial T^*}{\partial y} = 0\ ; \quad \psi = \frac{\partial\psi}{\partial y} = 0\ ; \quad y = 0 \tag{5.i-k}$$

$$\frac{\partial T^*}{\partial y} = 0\ ; \quad \psi = \frac{\partial\psi}{\partial y} = 0\ ; \quad y = 1 \tag{5.l-n}$$

The next step is then the choice of the eigenfunctions for the dependent variables expansions. The "*x*" direction is selected to be eliminated through integral transformation, and the eigenvalue problem of biharmonic-type is adopted for the streamfunction representation.

For the temperature expansion, the classical second order diffusion operator yields a Sturm-Liouville-type problem, readily solved with the appropriate boundary conditions of first kind, at the lateral walls. The respective eigenvalue problems and their solutions are fully discussed in Cotta (1998), Leal et al. (1999) and Leal et al. (2000).

The solution methodology proceeds towards the proposition of the integral transform pair for the potentials, the integral transformation itself and the inversion formula.

For the streamfunction field:

$$\bar{\psi}_i(y) = \int_0^1 \tilde{X}_i(x)\ \psi(x,y)\ dx, \qquad \text{transform} \tag{6.a}$$

$$\psi(x,y) = \sum_{i=1}^{\infty} \tilde{X}_i(x)\ \bar{\psi}_i(y), \qquad \text{inverse} \tag{6.b}$$

and for the temperature field:

$$\bar{T}_m(y) = \int_0^1 \tilde{\phi}_m(x)\ T^*(x,y)\ dx, \qquad \text{transform} \tag{7.a}$$

$$T^*(x,y) = \sum_{m=1}^{\infty} \tilde{\phi}_m(x)\ \bar{T}_m(y), \qquad \text{inverse} \tag{7.b}$$

The integral transformation process is now employed through operation of Eq.(5.a) with $\int_0^1 \tilde{X}_i(x)\,dx$, to find the transformed streamfunction system:

$$\frac{d^4\bar{\psi}_i(y)}{dy^4} = \frac{1}{\Pr}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[A_{ijk}\frac{d\bar{\psi}_j}{dy}\bar{\psi}_k + B_{ijk}\frac{d\bar{\psi}_j}{dy}\frac{d^2\bar{\psi}_k}{dy^2} - C_{ijk}\bar{\psi}_j\frac{d\bar{\psi}_k}{dy} - B_{ijk}\frac{d^3\bar{\psi}_j}{dy^3}\bar{\psi}_k\right] -$$

$$-\mu_i^4\bar{\psi}_i - 2\sum_{j=1}^{\infty}D_{ij}\frac{d^2\bar{\psi}_j}{dy^2} + \mathrm{Ra}\left[\sum_{m=1}^{\infty}E_{im}\bar{T}_m - F_i\right]sin\alpha - \mathrm{Ra}\sum_{m=1}^{\infty}W_{im}\frac{d\bar{T}_m}{dy}cos\alpha \tag{8}$$

Similarly, Eq.(5.b) is operated on with $\int_0^1 \tilde{\phi}_m(x)\,dx$, to yield the transformed temperature problem:

$$\frac{d^2\bar{T}_m(y)}{dy^2} = \beta_m^2\bar{T}_m + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left[Q_{mnj}\bar{T}_n\frac{d\bar{\psi}_j}{dy} - S_{mnj}\frac{d\bar{T}_n}{dy}\bar{\psi}_j\right] - \sum_{j=1}^{\infty}P_{mj}\frac{d\bar{\psi}_j}{dy} \tag{9}$$

Then, the resulting coupled infinite ODE's system for the transformed potentials is described by Eqs. (8 and 9), together with the integral transformed boundary conditions:

$$\bar{\psi}_i(0) = 0\,; \quad \frac{d\bar{\psi}_i(0)}{dy} = 0\,; \quad \frac{d\bar{T}_m(0)}{dy} = 0 \tag{10.a-c}$$

$$\bar{\psi}_i(1) = 0\,; \quad \frac{d\bar{\psi}_i(1)}{dy} = 0\,; \quad \frac{d\bar{T}_m(1)}{dy} = 0 \tag{10.d-f}$$

The related coefficients A_{ijk}, B_{ijk}, C_{ijk}, D_{ij}, E_{im}, F_i, Q_{mnj}, S_{mnj} and P_{mj} were already obtained analytically through Mathematica software system of symbolic manipulation, Wolfram (1991), and automatically generated in Fortran form. More details can be seen in Cotta and Mikhailov (1997), Cotta (1998) and Leal et al. (1999). The new coefficient W_{im}, also obtained analytically through Mathematica software, is defined as:

$$\int_0^1 \tilde{X}_i \, \tilde{\phi}_m \, dx \tag{11}$$

4. COMPUTATIONAL PROCEDURE

A Fortran 77 code was constructed and implemented on a PC Pentium 266-128Mb. The subroutine DBVPFD from the IMSL Library (1989) was employed as the boundary value problem solver, with an automatic local relative error selected to be 10^{-4} (i.e. ±1 in the fourth significant digit). For computational purposes, the expansions were truncated to *NV* and *NT* terms, respectively, streamfunction and temperature fields, towards the user prescribed accuracy target. An inclined air-filled square cavity (Pr= 0.71) was considered with four inclination angles, $\alpha = 40^0$, 60^0, 120^0 and 140^0 for Rayleigh numbers equals to 10^4 and 10^5.

Once the transformed potentials, $\bar{\psi}_i$ and $\bar{T}_m$, have been numerically evaluated under controlled accuracy, the inversion formula, together with the filtering solution, are recalled to provide explicit analytical expressions, in the “y” direction, for the original potentials $\psi(x,y)$ and $T(x,y)$.

The average Nusselt number at the midplane vertical cross-section (x=1/2) and at the hot wall (x=0), Nu_x, is obtained from de Vahl Davis (1983) as:

$$Nu_x = \int_0^1 \left[u(x,y)T(x,y) - \frac{\partial T(x,y)}{\partial x} \right] dy \ , \qquad x=0 \quad \text{or} \quad x=1/2 \tag{12.a}$$

The overall average Nusselt number across the cavity is obtained from:

$$\overline{Nu}_g = \int_0^1 Nu_x \, dx \tag{12.b}$$

The integrations required in Eqs. (12a, b) were numerically performed by making use of the appropriate subroutines in the IMSL Library (1989).

5. RESULTS AND DISCUTION

For benchmarking purposes in natural convection cavity problems, the convergence process controls can be done over the mean Nusselt number values. Note that the numerical quantities of $\overline{Nu}_0$ and $\overline{Nu}_{1/2}$ must converge toward the same value under the same conditions, satisfying the conservation of heat flux through any vertical plane. Therefore, it looks clear that the overall average Nusselt number ($\overline{Nu}_g$) must also converge to the this value.

Tables 1 and 2 present, respectively for Ra=10^4 and 10^5, benchmark results for the streamfunction modulus at the cavity center ($|\psi_{MED}|$ at x= y= 1/2), mean Nusselt number at the hot wall ($\overline{Nu}_0$) and at the vertical mid-plane of the cavity ($\overline{Nu}_{1/2}$), and global Nusselt number ($\overline{Nu}_g$) for the several proposed inclination angles. It can be observed from Table 1 that, as the inclination angle decreases, larger truncations orders are required for full convergence, especially for the temperature field (Nt), since the problem closes in the classical Rayleigh-Bernard problem (hot bottom wall and cold top wall - α= 0^0). The convergence behavior becomes more critical as Rayleigh is increased, vide Table 2. It is then clear that the convergence criterion of equality for all the Nusselt numbers requires larger truncation orders for full convergence, especially due to the value at the hot wall ($\overline{Nu}_0$), because the derived series that defines the Nusselt numbers is different in nature from the original eigenfunction expansions.

Table 1– Benchmark results for Ra= 10^4.

	Inclination angles			
	α=140^0	α=120^0	α=60^0	α=40^0
Nv/Nt	32/32	32/32	40/40	40/60
$\lvert\psi_{MED}\rvert$	1.786	3.002	6.828	7.516
$\overline{Nu}_0$	1.324	1.709	2.469	2.471
$\overline{Nu}_{1/2}$	1.324	1.709	2.469	2.471
$\overline{Nu}_g$	1.324	1.709	2.469	2.471

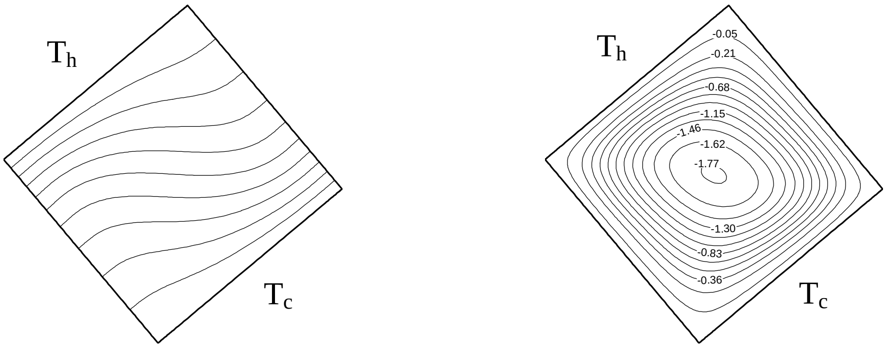
Table 2– Benchmark results for Ra= 10^5.

	Inclination angles			
	α=140^0	α=120^0	α=60^0	α=40^0
Nv/Nt	40/70	40/70	40/70	40/70
$\lvert\psi_{MED}\rvert$	2.027	4.671	15.08	21.03
$\overline{Nu}_0$	1.768	3.027	4.619	4.499
$\overline{Nu}_{1/2}$	1.768	3.027	4.619	4.499
$\overline{Nu}_g$	1.768	3.027	4.619	4.499

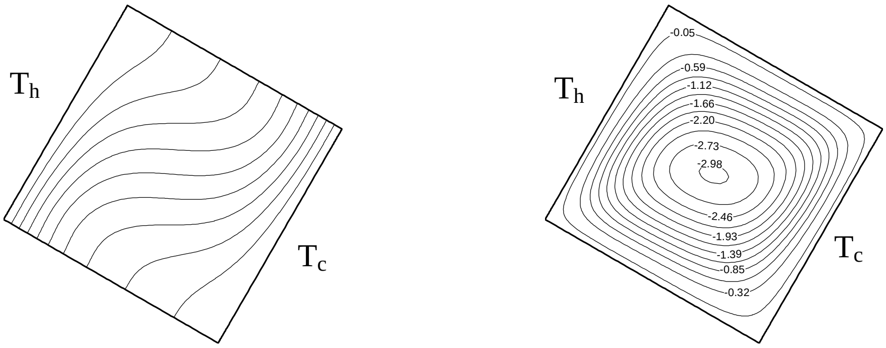
Figures 2 and 3 show, respectively for Ra=10^4 and 10^5, the effect on the isotherms and streamlines in a square cavity for the different inclination angles, namely α= 140^0, 120^0, 60^0 and 40^0.

One can note from Fig. 2 that the flow patterns are similar for all angles with a single cell whose shape changes according to the inclination of the cavity. The isotherms behavior in Fig. 2 indicates similar levels of mean Nusselt numbers, especially for the angles of 60 and 40 degrees.

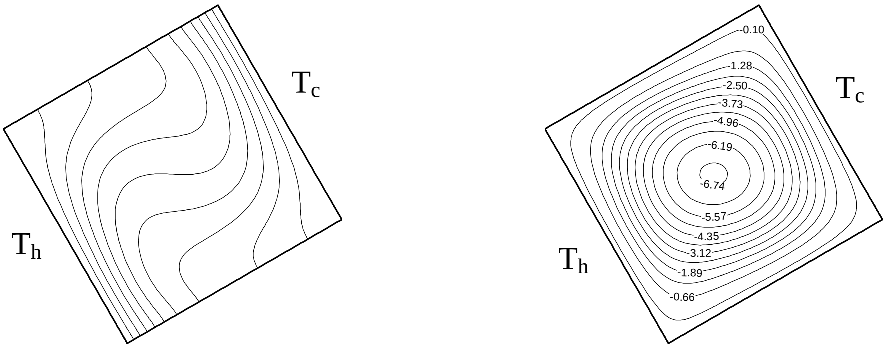
Figure 3 shows two small cells developed within one large rotating flow for α= 140^0. This patterns is maintained up to an inclination angle of 60 degrees. Then, the two cells merge into one and the motion becomes a single dominant central roll cell which has a nearly circular



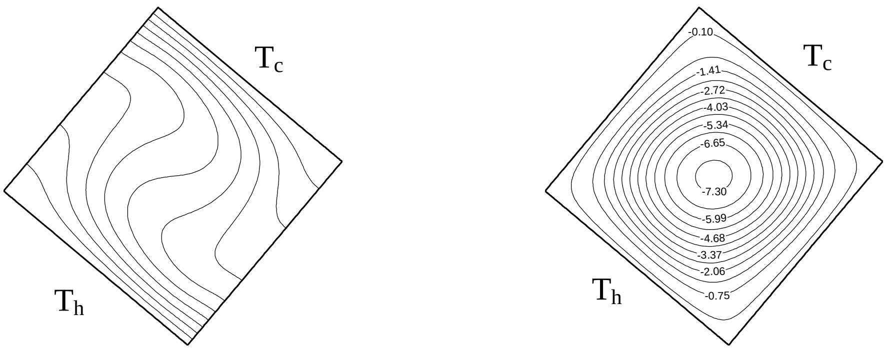
(a)



(b)



(c)



(d)

Figure 2- Isotherms (0.1, (0.1), 0.9) and streamlines for Ra= 10^4.
(a) α = 140^0; (b) α =120^0, (c) α = 60^0 and (d) α = 40^0

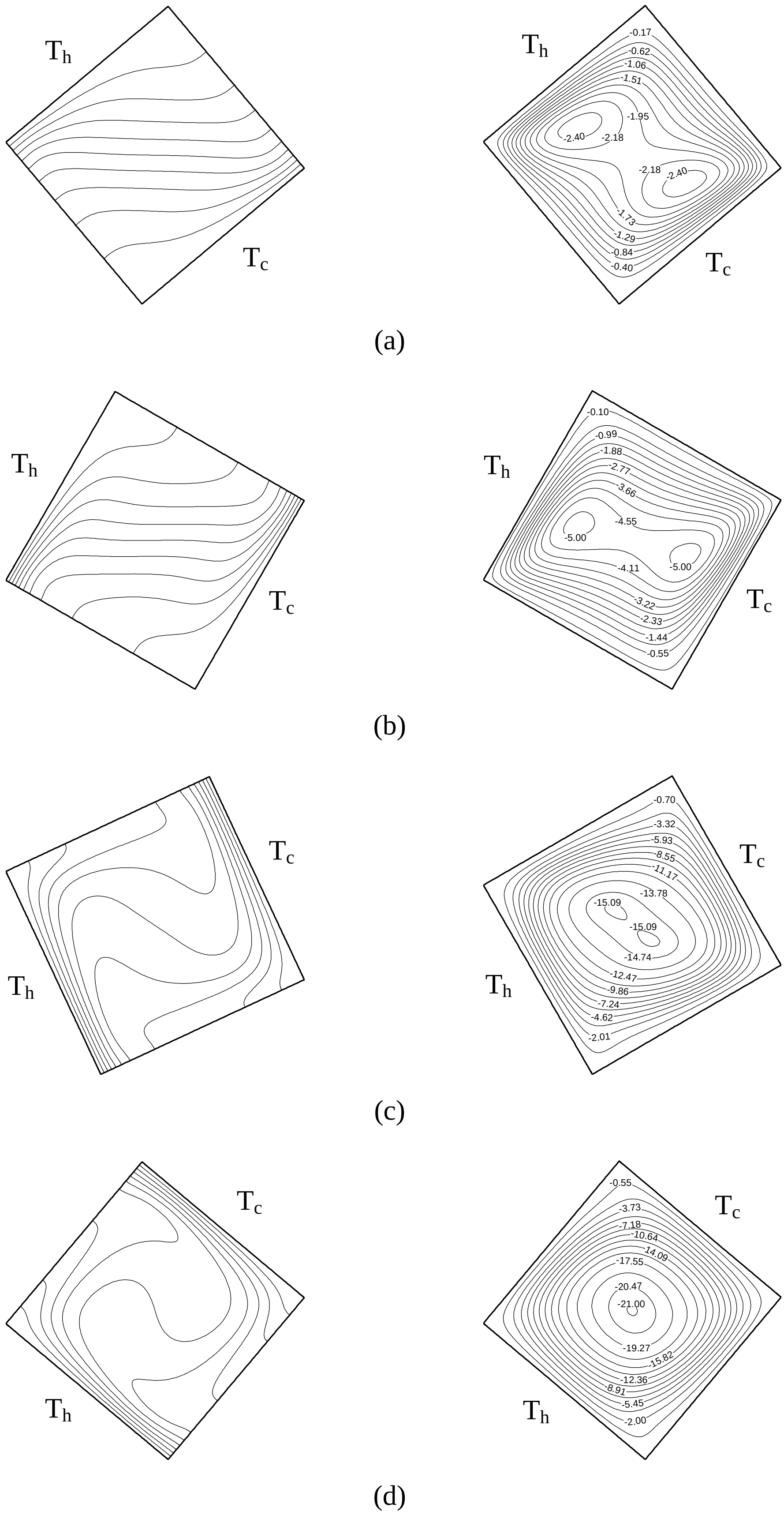


Figure 3- Isotherms (0.1, (0.1), 0.9) and streamlines for Ra= 10^5.
(a) α = 140^0; (b) α =120^0, (c) α = 60^0 and (d) α = 40^0

shape for α= 40^0. This behavior in not observed in the work of Rasoul and Prinos (1997) where at 60 degrees the presence of one cell is already remarked. It takes place in all probability due to the lack of convergence accuracy on their results. The isotherms in Fig. 3 remain perpendicular to the gravitational vector at α= 140^0 and 120^0. In situations of α= 60^0 and 40^0 the temperature stratification in the core region breaks down and the isotherms are no longer orthogonal to the gravitational field.

6. CONCLUSIONS

The Generalized Integral Transform Technique (GITT) was successfully employed to provide some benchmark results to the natural convection problem inside inclined square cavities. The traditional convergence criterion applied for benchmarking purposes in this type of problem is completely obeyed, i.e. the equality for the three calculated values of the mean Nusselt numbers.

Acknowledgments

The authors would like to acknowledge the financial support provided by CAPES/COFECUB, research sponsoring agencies in Brazil and France.

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