TRANSIENT ISOTHERMAL PSEUDOPLASTICITY OF BLOOD BY A STABLE FINITE ELEMENT METHOD

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Abstract. *In this work, we analyse transient pseudoplastic isothermal blood flow by a stabilised mixed finite element method which accomodates same order interpolations for all the variables present, and the algorithm used to solve the resultant system resolves for a large range of the power index. Convergence analysis and numerical isothermal results are presented.*

Keywords: *Finite elements, Pulsating flow, Pseudoplasticity, Blood flow, Stabilised Formulation*

1. INTRODUCTION

Blood is known to be a suspension of several cells in a fluid called plasma. But it is the elastic property of the red cells that makes blood to behaves like a non-Newtonian fluid depending on where it is flowing. Blood behaves differently when flowing in large or in microvessels. These are two extreme cases. While in large vessels blood can be modelled as a Newtonian fluid for high shear rates, in microvessels erythrocytes pass one by one deforming themselves and pulling plasma. For medium and small vessels non-Newtonian effects begin to appear.

Rheological measurements demonstrated that the increase in apparent viscosity at low shear rates is associated with the formation of red cell aggregations mantained by adhesion processes involving proteins. In some cases, the temperature effects in viscosity are of importance once fluidity increases with temperature while adhesion forces decrease. Recently some investigations with magnetic resonance imaging have shown pseudoplastic effects not only in medium and small vessels but also in large vessels when blood was kept at small shear rates.

Pseudoplastic blood behaviour can be modelled by power-law relations. When these relations are introduced in the momentum equations, non linear systems are formed and several numerical difficulties arise in addition to the incompressibility contraint. The purely nonlinear character of these constitutive relations precludes the applications of Newton-like methods to solve efficiently for high values of the power index. In this work, in order to overcome these difficulties we analyse transient pseudoplastic isothermal blood flow by a stabilised mixed finite element method which accomodates same order interpolations for all the variables present, and the algorithm used to solve the resultant system resolves for a large range of the power index. Convergence analysis and numerical isothermal results are presented.

2. THE MATHEMATICAL MODEL

Define $\Omega \subset R^d$, with $d = 2$ for simplicity, as an open bounded domain occupied by the fluid material, with boundary $\Gamma = \Gamma_e \cup \Gamma_w \cup \Gamma_o$, with subscripts ${}_{e,w}$ and ${}_o$ related to entrance, wall and exit of a finite legth of the blood vessel. Consider a finite period Y of time $t \in [0, Y]$. By considering the Hellinger-Reissner principle, the transient pseudoplastic isothermal blood flow that we will consider can be modelled by the following problem. From now on, it will be ommited the independent variables $\mathbf{x}$ and t from the arguments of the dependent variables.

Problem T: Find the velocity $\mathbf{u} = \mathbf{u}(t) : \Omega \times [0, Y] \to \mathbf{R^2}$, the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^T(t), \boldsymbol{\sigma} : \Omega \times [0, Y] \to R^2 \times R^2$ such that:

$$m_b \dot{\mathbf{u}} - \text{div } \boldsymbol{\sigma} + \mathbf{f} = 0 \qquad \text{in} \qquad \Omega \times [0, Y] \;, \tag{1}$$

$$\mathbf{B}\mathbf{u} = \nabla \mathbf{u}^s = A(\boldsymbol{\sigma}, \theta) \qquad \text{in} \qquad \Omega \times [0, Y] \;, \tag{2}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0, Y] \;, \tag{3}$$

$$\nabla \mathbf{u}(t)^s = \frac{(\nabla \mathbf{u} + \nabla \mathbf{u}^T)}{2} \qquad \text{in} \qquad \Omega \times [0, Y] \;, \tag{4}$$

with boundary conditions:

$$\mathbf{u} = \overline{\mathbf{u}}(t) \qquad \text{on} \qquad \Gamma_e \times [0, Y] \;, \tag{5}$$

$$\mathbf{u} = 0 \qquad \text{on} \qquad \Gamma_w \times [0, Y] \;, \tag{6}$$

$$\nabla \mathbf{u} = 0 \qquad \text{on} \qquad \Gamma_o \times [0, Y] \;, \tag{7}$$

and initial condition

$$\mathbf{u} = \overline{\mathbf{u}}(0) \qquad \text{on} \qquad \Gamma_e \;, \tag{8}$$

$$\mathbf{u} = 0 \qquad \text{in} \qquad \Omega \cap (\Gamma_w \cup \Gamma_o) \;, \tag{9}$$

where $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial t$, m_b is the density, $A(\sigma, \theta)$ is a function of the temperature and of the deviatoric part of the stress tensor $\sigma_D = \sigma - \frac{1}{2}\text{tr}\sigma\mathbf{I} = \sigma - \frac{1}{2}p\mathbf{I}$, p being the pressure. $A(.,.)$ is subjected to the constraint $\text{tr}A = A : \mathbf{I} = 0$ since div $\mathbf{u} = 0$ in Ω.

At the entrance section of the vessel, it is prescribed an estimated pulse $\overline{\mathbf{u}}(t)$ that will be considered a sinusoidal function of the time,

$$\overline{\mathbf{u}}(t) = \omega[1 - \sin(2\pi t / Y)], \tag{10}$$

with w being the amplitude of the flow wave.

3. RHEOLOGICAL MODEL FOR BLOOD

Blood is a complex material composed of different kinds of small and large molecule and cell suspended in a newtonian fluid called plasma, making its rheological behaviour a complex task. Its apparent viscosity is affected by, for example, shear rate, temperature and chemical composition (see Karam F., 2000 and references therein). At high shear rate, blood may be considered as having constant viscosity around 3.5 cP, depending on chemical contents. For low shear non-Newtonian behaviour is important. Several non-Newtonian effects have been reported for blood flow and several constitutive models have been proposed. One of the first of those Models was the Casson's one, fitting Cokelet's viscometric data. It is valid for a small range of low shear flow in very thin vessels and considers an yield stress (very low for real blood flow).

More recently, pure pseudoplastic behaviour has been detected even for low shear rate in large vessel diameters, in which experimental data were adjusted by a power law relation (Mo *et al.*, 1991).

In the present work, a power law model for blood in medium-small vessels under medium-low shear rate will be considered together with the concept of an apparent fluidity instead of the apparent viscosity. This choose has been made with the objective of generating adequate formulations and efficient algorithms, next presented here, to be easily parallelised to solve for large systems in simulating vascular flow problems.

Then, defining an apparent viscosity $A_a^{-1}(.)$ and an apparent fluidity $A_a(\mathbf{S})$ we may write

$$\mathbf{S} = A^{-1}(\boldsymbol{\epsilon}(\mathbf{u})) = A_a^{-1}(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\epsilon}(\mathbf{u}) \quad or \quad \boldsymbol{\epsilon}(\mathbf{u}) = A(\mathbf{S}) = A_a(\mathbf{S})\mathbf{S}. \tag{11}$$

The corresponding power-law (Ostwald-de-Waele) model may be written as

$$\mathbf{S} = A^{-1}(\boldsymbol{\epsilon}(\mathbf{u})) = A_a^{-1}(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\epsilon}(\mathbf{u}) = K''|\boldsymbol{\epsilon}(\mathbf{u})|^{s-2}\epsilon(\mathbf{u}), \tag{12}$$

or

$$B\mathbf{u} = \boldsymbol{\epsilon}(\mathbf{u}) = A(\mathbf{S}) = A(\mathbf{S})\mathbf{S} = K'|\mathbf{S}|^{n-2}\mathbf{S} \tag{13}$$

with $1/n + 1/s = 1$, where n and s are the power law indices (dimensionless) K' and K'' are the consistency parameters, with dimensions depending on n and s, respectively. Newtonian fluids have $n = s = 2$, pseudoplastic fluids correspond to $s < 2$ and $n > 2$ and dilatant fluids to $s > 2$ and $n < 2$, and

$$|\boldsymbol{\epsilon}(\mathbf{u})| = (\nabla^s\mathbf{u} : \nabla^s\mathbf{u})^{\frac{1}{2}}, \quad |\mathbf{S}| = (\mathbf{S} : \mathbf{S})^{\frac{1}{2}} \tag{14}$$

The temperature dependence on the apparent viscosity or fluidity may be considered by writing

$$\epsilon(\mathbf{u}) = A_a(\mathbf{S}, \theta)\mathbf{S} = \nu(\theta)A_a(\mathbf{S})\mathbf{S}. \tag{15}$$

or

$$\mathbf{S} = A_a^{-1}(\epsilon(\mathbf{u}), \theta)\epsilon(\mathbf{u}) = \nu^{-1}(\theta)A_a^{-1}(\epsilon(\mathbf{u}))\epsilon(\mathbf{u}) \tag{16}$$

For $\nu^{-1}(\theta)$ an Arrhenius law may be included.

Here, preparing the ground for general laws as (15), we shall consider the isothermal case of blood at 37^oC.

4. VARIATIONAL FORMULATION

To present the proposed variational formulation for this problem, let U and V be spaces for stresses and velocities, respectively, V^* the dual space of V, and $U_T \subset U$ the space of null trace as defined below:

$$U = \{\boldsymbol{\tau} = [\tau_{ij}] \; ; \; \tau_{ij} = \tau_{ji} \in L^2(\Omega), i, j = 1, 2\}, \quad U_T = \{\mathbf{T} \in U; \mathrm{tr}\mathbf{T} = 0\} \tag{17}$$

$$V = \{\mathbf{v} = \{v_i\} \; ; \; v_i \in H_0^1(\Omega) \; , \; i = 1, 2\} \; . \tag{18}$$

and equipped with the usual $L^2(\Omega)$ and $H^1(\Omega)$ norms, respectively. Define also the product spaces $\bar{U} = U_T \times U$ and $\bar{V} = U_T \times V$.

Now we present the following formulation for the transient Problem T:

Problem $\overline{\mathbf{M}}^t$: For each $t \in [0, \infty)$, find $\{\{\mathbf{S}, \boldsymbol{\sigma}\}, \{\boldsymbol{\lambda}, \mathbf{u}\}\} \in \bar{U} \times \bar{V}$ such that:

$$c(\dot{\mathbf{u}}, \mathbf{v}) + \left(\bar{A}(\{\mathbf{S}, \boldsymbol{\sigma}\}), \{\mathbf{T}, \boldsymbol{\tau}\}\right) + \bar{B}(\{\mathbf{T}, \boldsymbol{\tau}\}, \{\boldsymbol{\lambda}, \mathbf{u}\}) = 0 \; , \qquad \forall \, \{\mathbf{T}, \boldsymbol{\tau}\} \in \bar{U} \tag{19}$$

$$\bar{B}(\{\mathbf{S}, \boldsymbol{\sigma}\}, \{\boldsymbol{\mu}, \mathbf{v}\}) = G(\{\boldsymbol{\mu}, \mathbf{v}\}) \; , \quad \forall \, \{\boldsymbol{\mu}, \mathbf{v}\} \in \bar{V} \tag{20}$$

where

$$\left(\bar{A}(\{\mathbf{S}, \boldsymbol{\sigma}\}), \{\mathbf{T}, \boldsymbol{\tau}\}\right) = (A(\mathbf{S}), \mathbf{T}) + \delta_1(\boldsymbol{\sigma}_D - \mathbf{S}, \boldsymbol{\tau}_D - \mathbf{T}) \; , \tag{21}$$

$$\bar{B}(\{\mathbf{T}, \boldsymbol{\tau}\}, \{\boldsymbol{\mu}, \mathbf{v}\}) = -(\boldsymbol{\tau}_D - \mathbf{T}, \boldsymbol{\mu}) + (\nabla \mathbf{v}^s, \boldsymbol{\tau}) \; , \tag{22}$$

$$G(\{\boldsymbol{\mu}, \mathbf{v}\}) = (\mathbf{f}, \mathbf{v}) \; , \qquad c(\dot{\mathbf{u}}, \mathbf{v}) = m_b(\dot{\mathbf{u}}, \mathbf{v}). \tag{23}$$

For infinite time, the limiting state of this problem corresponds to the solution of the stationary problem:

Problem $\overline{\mathbf{M}}$: Find $\{\{\mathbf{S}, \boldsymbol{\sigma}\}, \{\boldsymbol{\lambda}, \mathbf{u}\}\} \in \bar{U} \times \bar{V}$ such that:

$$\left(\bar{A}(\{\mathbf{S}, \boldsymbol{\sigma}\}), \{\mathbf{T}, \boldsymbol{\tau}\}\right) + \bar{B}(\{\mathbf{T}, \boldsymbol{\tau}\}, \{\boldsymbol{\lambda}, \mathbf{u}\}) = 0 \; , \quad \forall \, \{\mathbf{T}, \boldsymbol{\tau}\} \in \bar{U}, \tag{24}$$

$$\bar{B}(\{\mathbf{S}, \boldsymbol{\sigma}\}, \{\boldsymbol{\mu}, \mathbf{v}\}) = G(\{\boldsymbol{\mu}, \mathbf{v}\}) \; , \quad \forall \, \{\boldsymbol{\mu}, \mathbf{v}\} \in \bar{V}, \tag{25}$$

that was studied by Sánchez (1993) in creep analysis, Karam F. (1996) and Karam F. *et al.* (2000) in the thermally coupled problem context. Adapting those results, it is possible to ensure the asymptotic behaviour of the solution of **Problem** $\overline{\mathbf{M}}^t$ by the following:

Theorem 1: Let $\{\{\mathbf{S}(t), \boldsymbol{\sigma}(t)\}, \{\boldsymbol{\lambda}(t), \mathbf{u}(t)\}\}$ and $\{\{\mathbf{S}^\infty, \boldsymbol{\sigma}^\infty\}, \{\boldsymbol{\lambda}^\infty, \mathbf{u}^\infty\}\} \in \bar{U} \times \bar{V}$, be the solution of **Problem** $\overline{\mathbf{M}}^t$ and **Problem** $\overline{\mathbf{M}}$, respectively. Then:

$$\lim_{t \to \infty} \|\mathbf{S}(t) - \mathbf{S}^\infty\|_U = 0 \; , \qquad \lim_{t \to \infty} \|\boldsymbol{\sigma}(t) - (\boldsymbol{\sigma}^\infty + C\mathbf{I})\|_U = 0 \; , \tag{26}$$

$$\lim_{t \to \infty} \|\boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\infty\|_U = 0 \; , \qquad \lim_{t \to \infty} \|\mathbf{u}(t) - \mathbf{u}^\infty\|_V = 0 \; , \tag{27}$$

where $C \in \mathbf{R}$, represents a constant hydrostatic pressure.

5. FINITE ELEMENT APPROXIMATIONS

To introduce equal order finite element approximations for **Problem** $\overline{\mathbf{M}}^t$, let $\Omega \subset \mathbf{R^2}$ be a polygonal domain discretized by a uniform mesh of N_e finite elements such that:

$$\bar{\Omega} = \bigcup_{e=1}^{N_e} \bar{\Omega}^e \qquad \text{and} \qquad \Omega^e \cap \Omega^f = \emptyset \ , \quad e \neq f \ ,$$

where Ω^e is the interior of element e, and $\bar{\Omega}^e$ its closure. Let $Q_h^l(\Omega)$ the space of class C^{-1} , constructed by finite element interpolation polinomials of order $l \geq 0$, with h denoting the mesh parameter, $h = \max h_e$, $e = 1, 2, \ldots, N_e$, with h_e the diameter of element e . Let $S_h^k(\Omega) = Q_h^k(\Omega) \cap H_0^1(\Omega)$ be the space of class C^0 constructed by finite element interpolation polinomials of order k which are zero valued on the boundary of $\bar{\Omega}$.

Define the approximations for U and V as $U_h^l = (Q_h^l)^2 \subset U$ and $V_h^k = (S_h^k)^2 \subset V$, respectively. Let $U_{Th}^l = U_h^l \cap U_T$; $U_{0h}^l = U_h^l \cap U_0$ y $Q_{0h}^l = Q_h^l \cap L_0^2(\Omega)$.

Define the approximation for **Problem** $\overline{\mathbf{M}}^t$,in $\bar{U}_h = U_{Th} \times U_h^l$ and $\bar{V}_h = U_{Th} \times V_h^k$, as the following consistent formulation.

Problem $\overline{\mathbf{PG}}_h^t$: For each $t \in [0, \infty)$, find $\{\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}, \{\boldsymbol{\lambda}_h, \mathbf{u}_h\}\} \in \bar{U}_h \times \bar{V}_h$, such that:

$$c(\dot{\mathbf{u}}_h, \mathbf{v}_h) - \Big(\bar{A}_h(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) + \bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\boldsymbol{\lambda}_h, \mathbf{u}_h\}) =$$

$$= F_h(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}), \quad \forall \{\mathbf{T}_h, \boldsymbol{\tau}_h\} \in \bar{U}_h, \tag{28}$$

$$\bar{B}(\{\mathbf{S}_h, \boldsymbol{\sigma}_h(t)\}, \{\boldsymbol{\mu}_h, \mathbf{v}_h\}) = G(\{\boldsymbol{\mu}_h, \mathbf{v}_h\}) \ , \quad \forall \{\boldsymbol{\mu}_h, \mathbf{v}_h\} \in \bar{V}_h \ , \tag{29}$$

where

$$\Big(\bar{A}_h(\{\mathbf{S}_h, \boldsymbol{\sigma}_h(t)\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) = \Big(\bar{A}(\{\mathbf{S}_h, \boldsymbol{\sigma}_h(t)\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) + \frac{\delta_2 h^2}{\theta}(\mathrm{div}\ \boldsymbol{\sigma}_h, \mathrm{div}\ \boldsymbol{\tau}_h)_h \ , \tag{30}$$

$$F_h(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}) = \frac{\delta_2 h^2}{\theta}(\mathbf{f}, \mathrm{div}\ \boldsymbol{\tau}_h)_h \ , \tag{31}$$

The following corresponding stationary formulation for **Problem** $\overline{\mathbf{PG}}_h^t$ is important for blood flowing in deep capillares, where pulsating and transient effects has been lost:

Problem $\overline{\mathbf{PG}}_h$: Find $\{\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}, \{\boldsymbol{\lambda}_h, \mathbf{u}_h\}\} \in \bar{U_h^l} \times \bar{V_h^k}$ such that:

$$\Big(\bar{A}_h(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) + \bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\boldsymbol{\lambda}_h, \mathbf{u}_h\}) = F_h(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}) \ , \quad \forall \{\mathbf{T}_h, \boldsymbol{\tau}_h\} \in \bar{U_h^l} \tag{32}$$

$$\bar{B}(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}, \{\boldsymbol{\mu}_h, \mathbf{v}_h\}) = G(\{\boldsymbol{\mu}_h, \mathbf{v}_h\}) \ , \quad \forall \{\boldsymbol{\mu}_h, \mathbf{v}_h\} \in \bar{V_h^k} \ , \tag{33}$$

where:

$$\Big(\bar{A}_h(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) = \Big(\bar{A}(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) + \frac{\delta_2 h^2}{\theta}(\mathrm{div}\ \boldsymbol{\sigma}_h, \mathrm{div}\ \boldsymbol{\tau}_h)_h \tag{34}$$

$$\bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\boldsymbol{\mu}_h, \mathbf{v}_h\}) = (\boldsymbol{\tau}_{Dh} - \mathbf{T}_h, \boldsymbol{\mu}_h) - (\nabla \mathbf{v}_h^s, \boldsymbol{\tau}_h) \ , \tag{35}$$

$$F_h(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}) = -\frac{\delta_2 h^2}{\theta}(\mathbf{f}, \mathrm{div}\ \boldsymbol{\tau}_h)_h\ , \tag{36}$$

$$G(\{\boldsymbol{\mu}_h, \mathbf{v}_h\}) = -(\mathbf{f}, \mathbf{v}_h)\ , \tag{37}$$

where $(\cdot,\cdot)_h$ is a scalar product dependent on the mesh.

The following result shows that when $t \to \infty$, the solution of **Problem** $\overline{\mathbf{PG}}_h^t$ tends to the solution of the stationary **Problem** $\overline{\mathbf{PG}}_h$ unless globally constant pressure mode. It may be pointed out that p_h is defined in Q_h rather than Q_{0h} and consequentely, doing the descomposition $q_h = \bar{q}_h + q_h^*$, the $\bar{p}_h$ part includes not only the sectionally constant with zero global mean mode, but also the globally constant mode.

Theorem 2: Let $\{\{\mathbf{S}_h(t), \boldsymbol{\sigma}_h(t)\}, \{\boldsymbol{\lambda}_h(t), \mathbf{u}_h(t)\}\}$ and $\{\{\mathbf{S}_h^\infty, \boldsymbol{\sigma}_h^\infty\}, \{\boldsymbol{\lambda}_h^\infty, \mathbf{u}_h^\infty\}\}$ the solutions of **Problem** $\overline{\mathbf{PG}}_h^t$ and **Problem** $\overline{\mathbf{PG}}_h$, respectively. Then, for $k \geq 2$,

$$\lim_{t\to\infty} \|\mathbf{S}_h(t) - \mathbf{S}_h^\infty\|_U = 0\ , \qquad \lim_{t\to\infty} \|\boldsymbol{\sigma}_h(t) - (\boldsymbol{\sigma}_h^\infty + c_h\mathbf{I})\|_U = 0\ , \tag{38}$$

$$\lim_{t\to\infty} \|\boldsymbol{\lambda}_h(t) - \boldsymbol{\lambda}_h^\infty\|_U = 0\ , \qquad \lim_{t\to\infty} \|\mathbf{u}_h(t) - \mathbf{u}_h^\infty\|_V = 0\ . \tag{39}$$

6. NUMERICAL ANALYSIS TO THE COMPLETELY DISCRETE PROBLEM

Let $[0, T]$ be the time interval, $\Delta t = Y/N$ the integration interval, $t_m = m\Delta t$ and $\mathbf{u}^m = \mathbf{u}(t_m)$. Using the implicit Euler scheme, by defining the Euler operator as

$$\partial_t \mathbf{u}^m = \frac{1}{\Delta t}(\mathbf{u}^m - \mathbf{u}^{m-1})\ , \tag{40}$$

to approximate $\dot{\mathbf{u}}(t_m)$, we have

Problem $\overline{\mathbf{PG}}_h^d(t)$: For each $t_m, m = 1, 2, \ldots$, find $\{\{\mathbf{S}_h^m, \boldsymbol{\sigma}_h^m\}, \{\boldsymbol{\lambda}_h^m, \mathbf{u}_h^m\}\} \in \bar{U}_h \times \bar{V}_h$, such that:

$$c(\partial_t \mathbf{u}_h^m, \mathbf{v}_h) + \Big(\bar{A}_h(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) = \bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\boldsymbol{\lambda}_h, \mathbf{u}_h\}) - F_h(\{\mathbf{T}_h, \boldsymbol{\tau}_h\})\ ,$$

$$\forall\, \{\mathbf{T}_h, \boldsymbol{\tau}_h\} \in \bar{U}_h^l \tag{41}$$

$$\bar{B}(\{\mathbf{S}_h, \boldsymbol{\sigma}_h\}, \{\boldsymbol{\mu}_h, \mathbf{v}_h\}) = G(\{\boldsymbol{\mu}_h, \mathbf{v}_h\})\ , \quad \forall\, \{\boldsymbol{\mu}_h, \mathbf{v}_h\} \in \bar{V}_h^k\ . \tag{42}$$

At $t = t_m$, defining an elliptic projection $\Big\{\{\overline{\mathbf{S}}_h^m, \overline{\boldsymbol{\sigma}}_h^m\}, \{\overline{\boldsymbol{\lambda}}_h^m, \overline{\mathbf{u}}_h^m\}\Big\}$ and for $\boldsymbol{\rho}_u = \mathbf{u} - \overline{\mathbf{u}}$ and so on, the corresponding error equation is

$$\begin{aligned} c(\partial_t \beta_u^m, \mathbf{v}_h) + (\bar{A}_h \Big(\{\mathbf{S}_h^m, \boldsymbol{\sigma}_h^m\}) - \bar{A}_h(\{\bar{\mathbf{S}}_h^m, \bar{\boldsymbol{\sigma}}_h^m\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) - \\ \bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\beta_\lambda^m, \beta_u^m\}) = c(\dot{\mathbf{u}}(t_m) - \partial_t \bar{\mathbf{u}}_h(t_m), \boldsymbol{\tau}_h) + \\ \Big(\bar{A}_h(\{\mathbf{S}^m, \boldsymbol{\sigma}^m\}) - \bar{A}_h(\{\bar{\mathbf{S}}_h^m, \bar{\boldsymbol{\sigma}}_h^m\}), \{\mathbf{T}_h, \boldsymbol{\tau}_h\}\Big) - \\ \bar{B}(\{\mathbf{T}_h, \boldsymbol{\tau}_h\}, \{\boldsymbol{\rho}_\lambda^m, \boldsymbol{\rho}_u^m\}), \quad \forall\, \{\mathbf{T}_h, \boldsymbol{\tau}_h\} \in \bar{U}_h. \end{aligned} \tag{43}$$

And the following estimates are obtained:

$$\|\boldsymbol{\sigma}(t_m) - \boldsymbol{\sigma}_h^m\|_{h,U} \leq C\left(h^k \sup_{s\leq t_m} \phi^m(s) + \Delta t \sup_{t_j\leq t_m} \int_{t_{j-1}}^{t_j} (\|\ddot{\mathbf{S}}\|_U + \|\ddot{\boldsymbol{\sigma}}_D\|_U) ds\right) \tag{44}$$

$$\|\mathbf{S}(t_m) - \mathbf{S}_h^m\|_U \le C\left(h^k \sup_{s\le t_m} \phi^m(s) + \Delta t \sup_{t_j\le t_m} \int_{t_{j-1}}^{t_j} (\|\ddot{\mathbf{S}}\|_U + \|\ddot{\boldsymbol{\sigma}}_D\|_U) ds \right) , \qquad (45)$$

$$\|\boldsymbol{\lambda}(t_m) - \boldsymbol{\lambda}_h^m\|_U \le C\left(h^k \sup_{s\le t_m} \phi^m(s) + \Delta t \sup_{t_j\le t_m} \int_{t_{j-1}}^{t_j} (\|\ddot{\mathbf{S}}\|_U + \|\ddot{\boldsymbol{\sigma}}_D\|_U) ds \right) , \qquad (46)$$

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_V \le C\left(h^k \sup_{s\le t_m} \phi^m(s) + \Delta t \sup_{t_j\le t_m} \int_{t_{j-1}}^{t_j} (\|\ddot{\mathbf{S}}\|_U + \|\ddot{\boldsymbol{\sigma}}_D\|_U) ds \right) , \qquad (47)$$

where $\phi^m(s) = \phi(s(t_m))$ given by

$$\phi(s) = (\|\dot{\boldsymbol{\rho}}_u\|_V + \|\boldsymbol{\rho}_\sigma\|_{h,U} + \|\boldsymbol{\rho}_S\|_U + \|\boldsymbol{\rho}_\lambda\|_U + \|\boldsymbol{\rho}_u\|_V)^2 . \qquad (48)$$

7. NUMERICAL RESULTS

Two situations of blood flowing in circular vessels of medium size diameter are presented. The amplitude of the pulsating function was considered 1.0 cm/s and the period Y=1.0s, with K=4cP (medium whole blood) in a 0.2cm radius, r, and L=3cm length vessel and m_b=1.056kg/cm^3 for both cases. The first one considering a Newtonian behaviour and the other including the pseudoplastic effect for hematocrit at the medium level of the normal range. Figures 1 and 2 show velocity profiles at L/4 along the period for the entrance problem for n=2 and n=4, respectively. Figures 3 and 4 show developed velocity profiles for the two cases. Figures 5 shows the maximum shear for both n=2 and n=4.

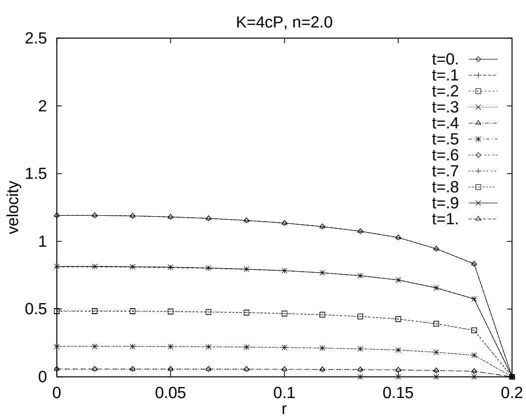

Figure 1: n=2.0 at L/3

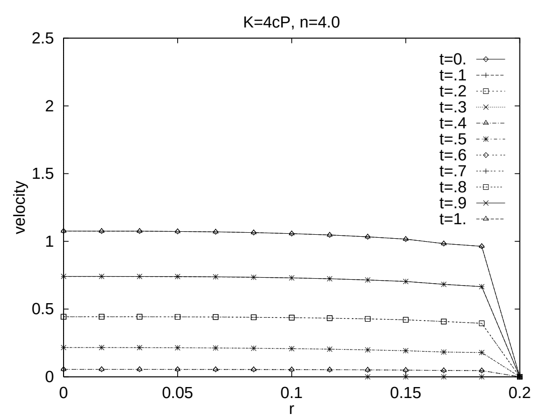

Figure 2: n=4.0 at L/3

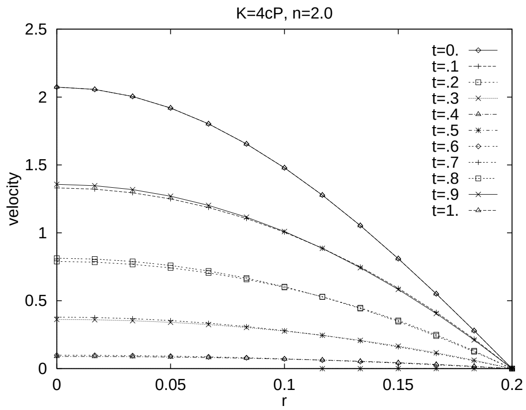

Figure 3: n=2.0 (developed)

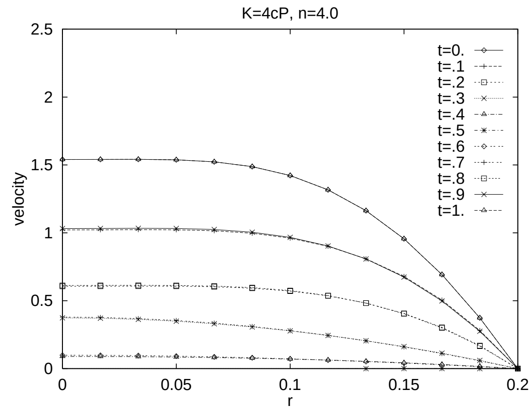

Figure 4: n=4.0 (developed)

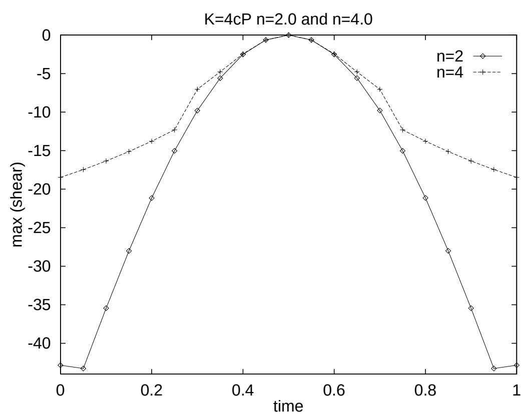

Figure 5: maximum shear at the entrance

8. CONCLUSIONS

In this work, it was presented a procedure to solve pulsating blood flow problems considering its pseudoplastic effect. This procedure is based on introducing an Euler approximation for the time into a stabilised mixed finite element formulation in four variables. Taking advantage of the good properties of the formulation for the stationary problem, that was studied by Sánchez (1993), and Sánchez and Karam F. (1999), that solves for a large range of the power-law index (Karam F., 1996), it was possible to generate a decaying estimate for the transient problem by adapting the estimate results for the creep analysis of Sánchez (1993) to the oscillating case. The resultant sistem for each time step has been solved using a Uzawa algorithm based in Gurreiro *et al.* (1991). With this algorithm applied, the mixed formulation is isolated and the constitutive equation can now be solved for each element. This fact, added to the discontinuous stress interpolations used, makes easy the application of parallelisation algorithms to speed up computations.

Numerical results were presented, considering blood in normal hematocrit range flowing in a medium vessel, comparing the pseudoplastic effect (n=4) with a Newtonian consideration. From the velocity and shear results obtained, it can be seen that the method presented here represents quite well the expected results for the pulsating function prescribed at the entrance of the vessel and non-Newtonian effects were obtained for the blood parameters that are in shape accordance with the large vessel experiments of Mo it et al. (1991).

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