NUMERICAL SIMULATION OF THREE DIMENSIONAL INCOMPRESSIBLE FLOWS USING THE FINITE ELEMENT METHOD

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Abstract. *A numerical algorithm to simulate 3-D incompressible flows of viscous fluids employing the finite element method is presented in this work. Space and time discretization of the complete set of differential equations were carried out using a semi-implicit two-step Taylor-Galerkin scheme and linear tetrahedral element. The code was written in FORTRAN language and was optimised in order to take advantages of vetorial processors existing in modern supercomputers. Examples including isothermal and non isothermal flows are presented to show the possibilities of the proposed algorithm as an important auxialiary tool for engineering design.*

Key words: *Computational fluid dynamic, finite element simulation, incompressible flows.*

1. INTRODUCTION

Applications of the Finite Element Method (FEM) to analyze viscous incompressible flows were introduced about thirty years ago by Oden and Welford (1972). Since then, different alternatives to solve this problem were presented by many other authors. Mixed methods (using different shape functions for velocity components and pressure in order to avoid pressure spurious modes) and the penalty function approach with selective integration (eliminating pressure as a primary variable) were used intensively during the three past decades. Malkus & Hughes (1978) and Engelman *et al.* (1982) showed the equivalence between both alternatives. The pseudo-compressibility approach, assuming a slightly compressible fluid, based in the ideas presented originally by Chorin (1967), were used by Ramshaw & Musseau (1990,1991) and by Ramshaw & Messina (1991) in the finite element context. Fractional or partitioned methods were introduced by Chorin (1967) and after were presented by several other authors such as Donea *et al.* (1982) and Kim & Moin (1985). These methods are initiated calculating the velocity field with the momentum equation, ommiting pressure gradients; the pressure is computed solving a Poisson equation, using the approximated velocity field obtained in the previous step; finally, the velocity field is

corrected using the pressure obtained by the Poisson equation. This way to analyze viscous incompressible flows is also known as velocity correction methods, and equal-order shape functions may be used for velocity components and pressure. Velocity correction methods were used by many authors, such as Gresho *et al.* (1995), Ren & Utnes (1993) and Kovacs & Kawahara (1991), among others.

In problems where convective terms are dominant with respect to diffusive or viscous terms, mesh refinement and upwinding techniques must be used in order to avoid spurious oscillations of the velocity components (which is an important shortcoming when the classical Bubnov-Galerkin weighted residual method is applied). An upwinding technique called SUPG (Streamline Upwinding Petrov-Galerkin) was introduced by Brooks & Hughes (1982) and used after, with some modifications, by several authors. This drawback may be also overcome using a Ballancing Diffusive Tensor (BDT) as presented by Gresho *et al*. (1984) or a Taylor-Galerkin scheme as presented by Donea (1984) and Zienkiewicz & Taylor (1991).

A partitioned velocity correction method to simulate numerically three-dimensional viscous incompressible flows using a semi-implicit two-step Taylor-Galerkin scheme is presented in this work. Advantages of vetorial processors of a supercomputer Cray T-94 were considered. Two examples are presented in order to show the good performance of the scheme described in this paper.

2. THE GOVERNING EQUATIONS

The conservative form of the momentum, continuity and energy equations are given, respectively by

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho\, u_i\, u_j\right)}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0 \quad (i,j=1,2,3) \text{ in } \Omega \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \quad (i=1,2,3) \text{ in } \Omega \tag{2}$$

$$\frac{\partial \left(\rho E\right)}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i\, \rho E\right) - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(u_i\, p\right) + \frac{\partial}{\partial x_i}\left(\tau_{ij}\, u_j\right) = 0 \quad (i,j=1,2,3) \text{ in } \Omega \tag{3}$$

where u_i are the velocity components, ρ is the specific mass, p is the pressure, τ_{ij} are the viscous stress tensor components, T is the temperature, E is the total specific energy, k the conductibility coefficient, x_i and t are the space and time coordinates and, finally, Ω is the domain to be analyzed. Source terms were ommited in Eq. (1) and Eq. (3). Initial and boundary conditions must be also prescribed for u_i, p and T. If necessary, convection, as well as a state equation relating the fluid viscosity, pressure, specific mass and temperature may be included.

Assuming constant entropy, the sound speed c is defined as

$$c^2 = \frac{\partial p}{\partial \rho} \tag{4}$$

and the continuity equation, Eq. (2), may be written as follows:

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \quad (i=1,2,3) \text{ in } \Omega \tag{5}$$

3. A FRACTIONAL TWO-STEP TAYLOR-GALERKIN ALGORITHM FOR NAVIER-STOKES EQUATIONS

Equation (1) may be written as

$$\frac{\partial U_i}{\partial t} = -\frac{\partial f_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} \quad (i,j=1,2,3) \text{ in } \Omega \tag{6}$$

where $U_i = \rho u_i$ and $f_{ij} = u_j(\rho u_i) = u_j U_i$. Integration of Eq. (6) in time is carried out in two steps. In the first step U_i is expanded in Taylor series considering the interval (t_n , $t_{n+1/2}$), obtaining

$$U_i^{n+1/2} = U_i^n + \frac{1}{2}\Delta t \frac{\partial U_i^n}{\partial t} \quad (i=1,2,3) \tag{7}$$

Taking into account that at t_n Eq. (6) may be expressed by

$$\frac{\partial U_i^n}{\partial t} = -\frac{\partial f_{ij}^n}{\partial x_j} + \frac{\partial \tau_{ij}^n}{\partial x_j} - \frac{\partial p^{n+\theta}}{\partial x_i} \quad (i,j=1,2,3) \text{ in } \Omega \tag{8}$$

where $p^{n+\theta} = p^n + \theta\ \Delta p$, $\Delta p = p^{n+1} - p^n$ with $0 \le \theta \le 1$ (in this work θ=1/2 is adopted), Eq. (7) is given by

$$U_i^{n+1/2} = \tilde{U}_i^{n+1/2} - \frac{\Delta t}{2}\theta \frac{\partial \Delta p}{\partial x_i} \quad (i=1,2,3) \tag{9}$$

In Eq. (9) $\tilde{U}_i^{n+1/2}$ is expressed by

$$\tilde{U}_i^{n+1/2} = U_i^n - \frac{1}{2}\Delta t\left(\frac{\partial f_{ij}^n}{\partial x_j} - \frac{\partial \tau_{ij}^n}{\partial x_j} + \frac{\partial p^n}{\partial x_i}\right) \quad (i,j=1,2,3) \tag{10}$$

Considering Eq. (9), the continuity equation, Eq. (5), is given by

$$\left(\frac{1}{c^2} - \Delta t^2 \theta \frac{\partial^2}{\partial x_i^2}\right)\Delta p = -\Delta t \frac{\partial \tilde{U}_i^{n+1/2}}{\partial x_i} \quad (i=1,2,3) \tag{11}$$

In the second step, U_i^{n+1} is computed as follows:

$$U_i^{n+1} = U_i^n + \Delta t \frac{\partial U_i^{n+1/2}}{\partial t} = U_i^n - \Delta t\left[\frac{\partial f_{ij}^{n+1/2}}{\partial x_j} - \frac{\partial \tau_{ij}^{n+1/2}}{\partial x_j} + \frac{\partial}{\partial x_i}\left(p^n + \frac{1}{2}\Delta p\right)\right] \quad (i,j=1,2,3) \tag{12}$$

Equations (10), (11), (9) and (12) may be computed using the finite element technique applying previously the classical Galerkin weighted residual method to these equations. In this work the linear tetrahedral element was adopted using the classical expansions for variables at time level $n\Delta t$ and $(n+1)\Delta t$, but employing a constant shape function for the variables at time level $(n+1/2)\Delta t$. Therefore, the variables are expanded as follows:

$$U_i^{n+1/2} = \mathbf{P}_E \overline{\mathbf{U}}_i^{n+1/2}; \quad \widetilde{U}_i^{n+1/2} = \mathbf{P}_E \widetilde{\overline{\mathbf{U}}}_i^{n+1/2}; \quad U_i^n = \mathbf{N}\overline{\mathbf{U}}_i^n; \quad p^n = \mathbf{N}\overline{\mathbf{p}}^n \tag{13}$$

where $\mathbf{N}$ is a vector containing the classical shape functions, $\overline{\mathbf{U}}_i^n$ and $\overline{\mathbf{p}}^n$ are vectors with nodal values of the unknowns and $\mathbf{P}_E$ is a shape function with a constant value for the element E and zero elsewhere. $\overline{\mathbf{U}}_i^{n+1/2}$ and $\widetilde{\overline{\mathbf{U}}}_i^{n+1/2}$ are constant over all the element domain. Tanking into account Eqs. (10), (11), (9), (12) and (13) and integrating by parts advective and viscous terms, the following system of equations are obtained at element level (assuming that $\mathbf{P}_E = 1$)

$$\widetilde{\overline{\mathbf{U}}}_i^{n+1/2} = \Omega_E^{-1}\left[\mathbf{C}\ \overline{\mathbf{U}}_i^n - \frac{\Delta t}{2}\left(\mathbf{T}\ \overline{\mathbf{U}}_i^n + \mathbf{L}_i\,\overline{\mathbf{p}}^n - \mathbf{f}_u\right)\right] \quad (i=1,2,3) \tag{14}$$

$$\left(\widetilde{\mathbf{M}} + \frac{\Delta t^2}{2}\theta\ \mathbf{H}\right)\Delta\overline{\mathbf{p}} = \Delta t\left(\mathbf{L}_i^T\,\widetilde{\overline{\mathbf{U}}}_i^{n+1/2} + \mathbf{f}_a\right) \quad (i=1,2,3) \tag{15}$$

$$\overline{\mathbf{U}}_i^{n+1} = \overline{\mathbf{U}}_i^n + \mathbf{M}^{-1}\Delta t\left[\mathbf{K}_u\,\overline{\mathbf{U}}_i^{n+1/2} + \mathbf{Q}_i\left(\overline{\mathbf{p}}^n + \frac{1}{2}\ddot{\mathrm{A}}\overline{\mathbf{p}}\right) - \mathbf{S} + \mathbf{f}_b\right] \quad (i=1,2,3) \tag{16}$$

where

$$\mathbf{C} = \int_\Omega \mathbf{N}\ d\Omega\ ; \quad \mathbf{L}_i = \int_\Omega \frac{\partial \mathbf{N}}{\partial x_i}\ d\Omega\ ; \quad \mathbf{T} = \int_\Omega \frac{\partial\left(u_i \mathbf{N}\right)}{\partial x_i}\ d\Omega$$

$$\mathbf{f}_u = \int_\Omega \frac{\partial \tau_{ij}^n}{\partial x_j}\ d\Omega\ ; \quad \Omega_E = \int_\Omega\ d\Omega\ ; \quad \widetilde{\mathbf{M}} = \int_\Omega \mathbf{N}^T\left(\frac{1}{c^2}\right)\mathbf{N}\ d\Omega$$

$$\mathbf{H} = \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}\frac{\partial \mathbf{N}}{\partial x_i}\ d\Omega\ ; \quad \mathbf{f}_a = \int_\Gamma \mathbf{N}^T.n_i\ \widetilde{\overline{\mathbf{U}}}_i^{n+1/2}\,d\Gamma; \quad \mathbf{M} = \int_\Omega \mathbf{N}^T\mathbf{N}\ d\Omega$$

$$\mathbf{K}_u = \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}u_i\ d\Omega\ ; \quad \mathbf{Q}_i = \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}\mathbf{N}\ d\Omega; \quad \mathbf{S} = \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_j}\tau_{ij}^{n+1/2}\ d\Omega$$

$$\mathbf{f}_b = -\int_\Gamma \mathbf{N}^T u_i.n_i\ \overline{\mathbf{U}}_i^{n+1/2}\,d\Gamma + \int_\Gamma \mathbf{N}^T\tau_{ij}^{n+1/2}.n_j\ d\Gamma - \int_\Gamma \mathbf{N}^T\mathbf{N}.n_i\left(\overline{\mathbf{p}}^n + \frac{1}{2}\ddot{\mathrm{A}}\overline{\mathbf{p}}\right)d\Gamma$$

The energy equation is solved by the same process, and corresponding matricial expressions are

$$\overline{\mathbf{E}}^{n+1/2} = \Omega_E^{-1}\left\{\mathbf{C}\,\overline{\mathbf{E}}^n - \frac{\Delta t}{2}\left[\mathbf{T}\ \left(\overline{\mathbf{E}}^n + \overline{\mathbf{p}}^n\right) + \mathbf{f}_e\right]\right\} \tag{17}$$

for the first step and

$$\overline{\mathbf{E}}^{n+1} = \overline{\mathbf{E}}^n + \mathbf{M}^{-1}\Delta t\left[\mathbf{K}_u\,\overline{\mathbf{E}}^{n+1/2} + \mathbf{V}\left(\overline{\mathbf{p}}^n + \frac{1}{2}\ddot{\mathrm{A}}\mathbf{p}\right) + \mathbf{R} + \mathbf{f}_c\right] \tag{18}$$

for the second step, where

$$\mathbf{f}_e = -\int_\Omega \frac{\partial \left(\tau_{ij} u_j\right)^n}{\partial x_i}\, d\Omega + \int_\Omega \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right)^n d\Omega$$

$$\mathbf{V} = \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}\left(u_i \mathbf{N}\right)\, d\Omega \;; \qquad \mathbf{R} = -\int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}\left(\tau_{ij} u_j\right)^{n+1/2}\, d\Omega - \int_\Omega \frac{\partial \mathbf{N}^T}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right)^{n+1/2} d\Omega$$

$$\mathbf{f}_c = -\int_\Gamma \mathbf{N}^T u_i E^{n+1/2} \cdot n_i \; d\Gamma - \int_\Gamma \mathbf{N}^T \left(u_i \mathbf{N}\right)\left(\overline{\mathbf{p}}^n + \frac{1}{2}\Ä\overline{\mathbf{p}}\right) n_i \; d\Gamma + \int_\Gamma \mathbf{N}^T \left(\tau_{ij} u_j\right)^{n+1/2} \cdot n_i \; d\Gamma$$

The sequence to calculate velocity components, pressure and specific total energy, after assemblying and applying boundary conditions to Eqs. (14) to (18) is

First step

- Calculate $\tilde{\overline{\mathbf{U}}}_i^{n+1/2}$ with Eq. (14).
- Calculate $\overline{\mathbf{E}}^{n+1/2}$ with Eq. (17).
- Calculate $\Ä\overline{\mathbf{p}}$ with Eq. (15).
- Calculate $\overline{\mathbf{U}}_i^{n+1/2} = \tilde{\overline{\mathbf{U}}}_i^{n+1/2} - \frac{\Delta t}{2\,\Omega_E}\theta\ \mathbf{L}_i \Delta\overline{\mathbf{p}}$

Second step

- Calculate $\overline{\mathbf{U}}_i^{n+1}$ with Eq. (16).
- Calculate $\overline{\mathbf{p}}^{n+1} = \overline{\mathbf{p}}^n + \Ä\overline{\mathbf{p}}$, where $\Ä\overline{\mathbf{p}}$ was obtained with Eq. (15).
- Calculate $\overline{\mathbf{E}}^{n+1}$ with Eq. (18).

The proposed scheme is conditionally stable, and the local stability condition for element *E* is given by

$$\Delta t_E \le \beta\, h_E / |\mathbf{u}| \tag{19}$$

where h_E is a characteristic dimension of element *E*, β is a safety factor and **u** is the fluid velocity. Equation (15) is solved using a conjugate gradient method with diagonal pre-conditioning. In this work, the mass matrix **M** in Eqs. (16) and (18) is substituted by the corresponding lumped mass matrix (with all its off-diagonal elements equal to zero), and the solution is carried out using a corrective iterative scheme.

4. NUMERICAL APPLICATIONS

4.1 **Analysis of viscous incompressible flow in a backward facing step**

The geometric characteristics and the boundary conditions of this problem are shown in Fig. 1. The Reynolds number Re=73 and H=1.0 were adopted, and although the flow is two-dimensional, a three-dimensional code was used. The finite element mesh, with 4782 nodes and 18120 tetrahedral elements is shown in Fig. 2. The mesh was built using hexahedral elements divided in six tetrahedral elements and in the normal direction to the flow (axis *z*) only two hexahedral elements were adopted.

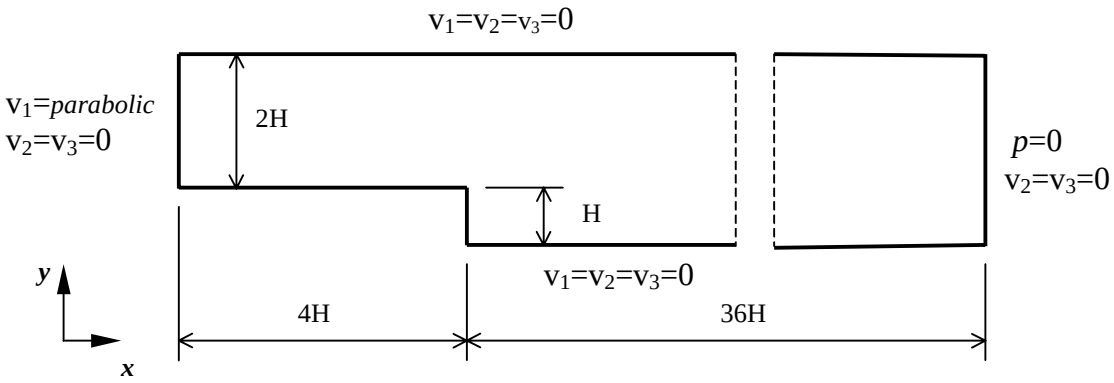

Figure 1 – A backward facing step: geometric characteristics and boundary conditions.

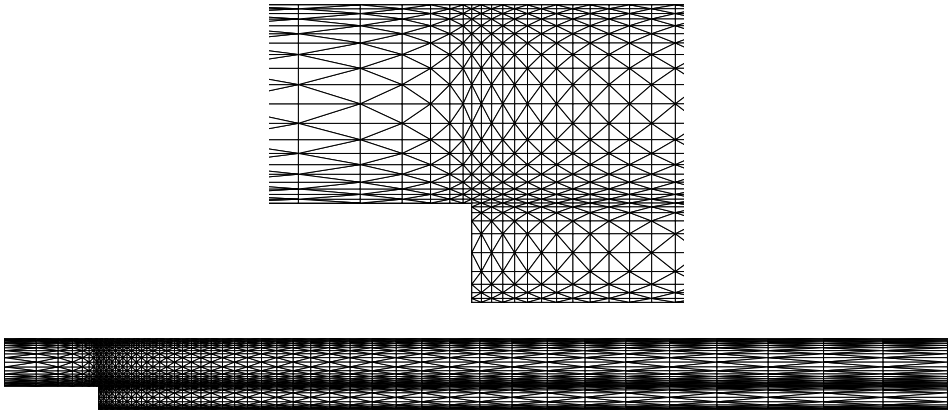

Figure 2 – The finite element mesh for the backward facing step.

At the inflow boundary the velocity components in the directions of axis y and z are $v_2=v_3=0$, whereas in the direction of the axis x, the velocity component v_1 was prescribed using a quadratic parabolic function, which is given by

$$u = \frac{3}{2}\left[1-\left(\frac{y-H/2}{H/2}\right)^2\right] \tag{20}$$

The pressure was prescribed at the outflow boundary, adopting $p=0$. The energy equation was not included, considering a isothermal process. A dimensionless time step $\Delta t=0.002$ was adopted, which satisfy the stability conditions. The steady-state was obtained after 18000 steps. In Fig. 3, Fig. 4 and Fig. 5, pressure distribution, details of the streamlines at the region where the step is located and profiles of the velocity component v_1 in the same region are shown.

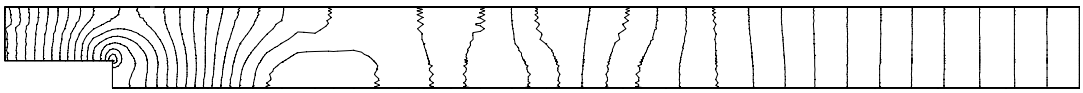

Figure 3 – Pressure distribution.

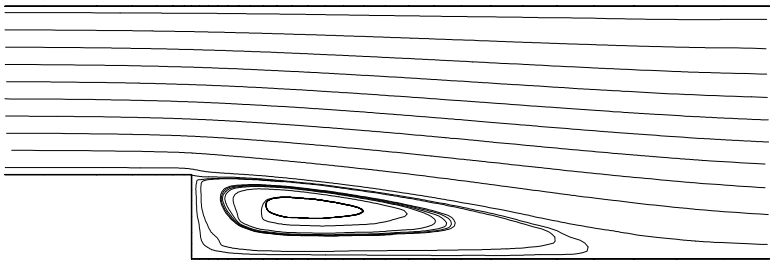

Figure 4 – Streamlines at the region where the step is located.

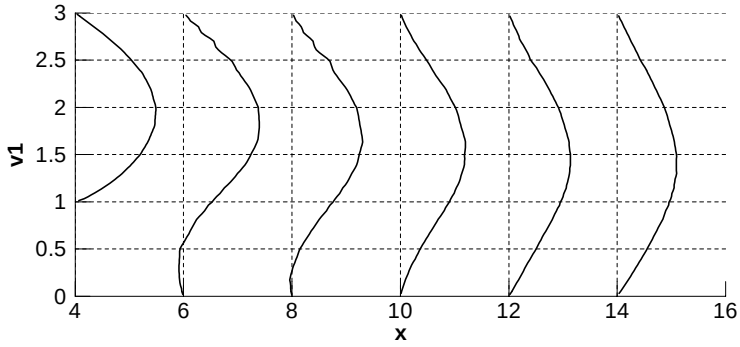

Figure 5 – Profiles of the velocity component v_1 at the region where the step is located.

In this work, a reattachment length equal to 4.95 was obtained. Zienkiewicz *et al*. (1996) and Taylor *et al*. (1981) obtained 4.8 and 5.3 , respectively.

4.2 **Analysis of viscous incompressible flow around a sphere**

This example consists in a 3-D flow around a sphere with a dimensionless diameter equal to 1.0 and the Reynolds number Re=100. A dimensionless diameter equal to 22.0 was adopted for an external sphere, which defines the region to be analyzed. A non structured mesh with 206722 tetrahedral elements and 37997 nodes was used. One quarter of the domain, with the planes of symmetry *yz* and *xz*, is shown in Fig. 6.

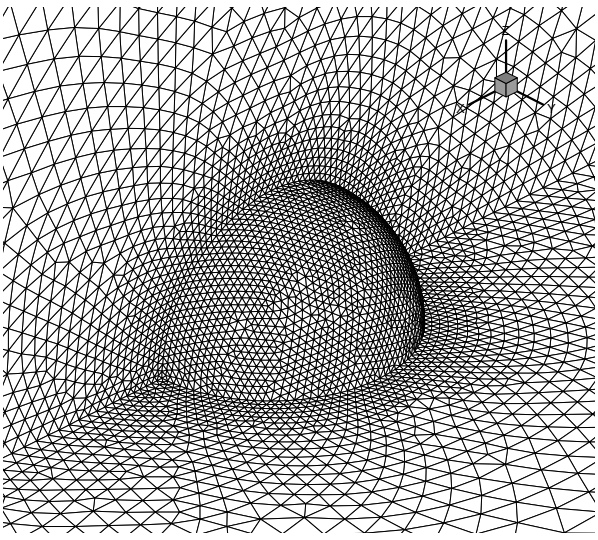

Figure 6 – Viscous incompressible flow around a sphere: finite element mesh.

A dimensionless time step Δt=0.0015 was adopted. In Fig. 7 and Fig. 8 the pressure distribution and streamlines are shown. The temperature contours are shown in Fig. 9.

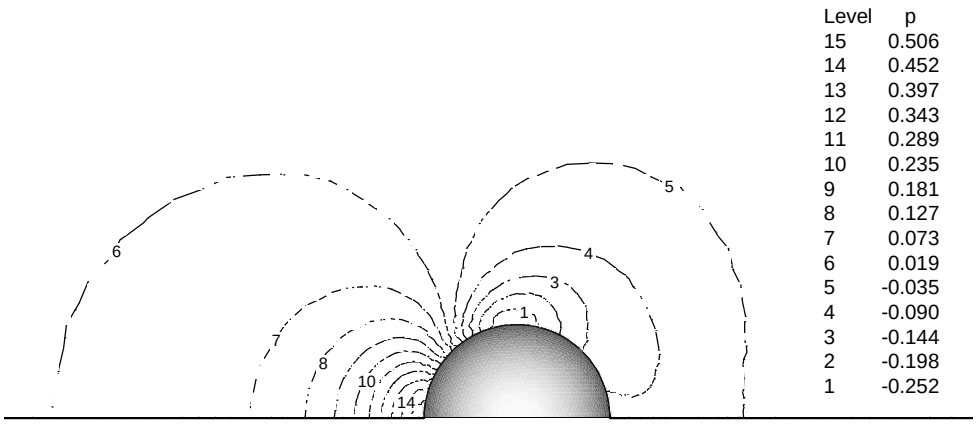

Figure 7 – Viscous incompressible flow around a sphere: pressure distribution.

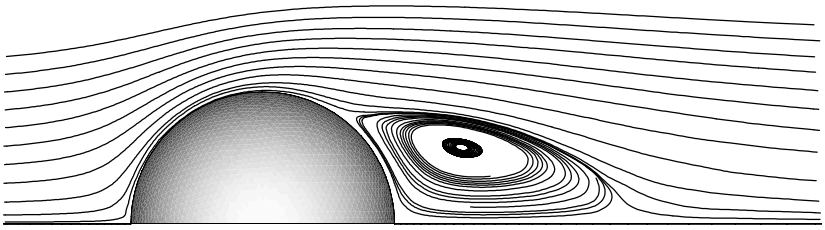

Figure 8 – Viscous incompressible flow around a sphere: streamlines.

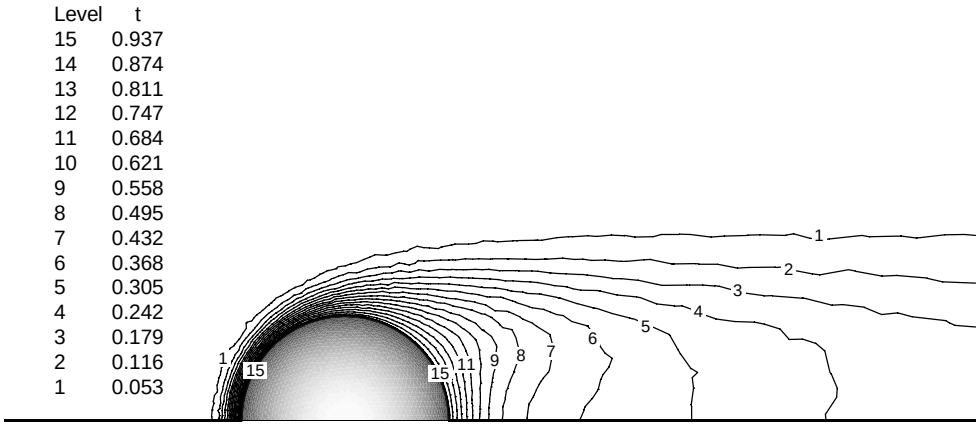

Figure 9 – Viscous incompressible flow around a sphere: temperature contours.

In Fig. 8 the separation points, characterized by a separation angle θ_s, and the length of the recirculation region, characterized by l_w/D (where D is the diameter of the sphere) can be obtained. In table 1, these values are compared with those obtained by Gülçan & Aslan (1997). In the same table, values of the total drag coefficient (C_D) and its relation with the coefficient due only to friction (C_{Df}) are also compared. Excellent agreement was obtained between the results of this work and those given by Gülçan & Aslan (1997).

Table 1 – Some parameters of the flow around a sphere

Reference	C_D	C_{Df}/C_D	l_w/D	$\theta_s(deg)$
Present work	1.07	0.53	0.94	55
Gülçat and Aslan	1.07	0.51	0.93	55

In Fig. 10 the pressure coefficients obtained here and by Gülçan & Aslan (1997) are also compared and, again, good results were observed.

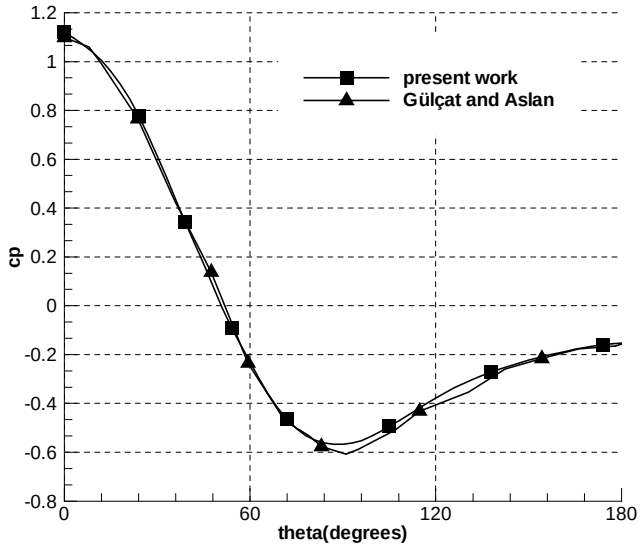

Figure 10 – Viscous incompressible flow around a sphere: pressure coefficients (*Cp*).

The steady-state was obtained in the dimensionless time t=12.0 and the performance was 515 Mflops in a Cray T-94 supercomputer. The average value of the number of iterations per time step in the conjugate gradient method with diagonal preconditioning was 54.

5. CONCLUSIONS

The efficiency and accuracy of a partitioned two-step Taylor-Galerkin scheme was shown in this work. Tetrahedral linear elements with the same interpolation functions for velocity components and pressure were used. The pressure increment was obtained solving a Poisson equation using the conjugate gradient with diagonal preconditioning. The average number of interations per time step was 20 for the first example and 54 for the second one. A good performance with respect to code vectorization was obtained (about 500 Mflops).

It is not very difficult to extend this code to solve compressible flows. It would be necessary to change only the continuity equation and to add the corresponding state equation.

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