THERMALLY DEVELOPING LAMINAR FLOW OF POWER-LAW NON-NEWTONIAN FLUIDS INSIDE RECTANGULAR DUCTS

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Abstract. *The so-called generalized integral transform technique is employed in the hybrid analytical-numerical solution of laminar forced convection to power-law fluids inside rectangular ducts, allowing for the solution of this problem involving a non-separable eigenvalue problem. Reference results are established for quantities of practical interest within the thermal entry region, for a wide range of the axial variable, power law indices and for the typical situation of a square duct. The accuracy of previously reported results from older codes using the earlier mentioned method (for Newtonian fluids), as well as from direct numerical approaches are then critically examined, for both the developing and fully developed regions.*

Keywords: *Laminar rectangular duct flow, Power-law non-Newtonian fluids, Laminar forced convection, Integral transform solution.*

1. INTRODUCTION

Industrial applications in which processing of materials behaving as non-Newtonian fluids are those commonly encountered in the chemical, food processing, polymer and petrochemical industries which undergo thermal processing in heat exchange equipment, and in these applications the power-law model can describe adequately the rheology of such fluids. Therefore, heat transfer solutions for laminar forced flow inside ducts of various shapes is of great interest to the design of compact heat exchangers, solar collectors and several other low Reynolds number flow heat exchange devices (Shah & London, 1978). In this context, the establishment of benchmark results through numerical-analytical solutions is desirable for reference purposes, validation of direct numerical schemes and validation of old schemes using the hybrid numerical-analytical method (for Newtonian fluids), specially for thermally developing flows (Aparecido & Cotta, 1990), and a survey of the literature reveals a limited amount of works about heat and fluid flow of non-Newtonian fluids in rectangular and irregular ducts is available, and most contributions deal with purely numerical or approximate approaches (Chandrupatla & Sastri, 1977, Lawal & Mujumdar, 1985, Etemad & Mujumdar, 1994, Etemad, 1997 and 1998).

Particularly, the case of rectangular duct constitutes a typical example of the difficulties associated with solving multidimensional convection problems, requiring costly numerical solutions limited to regions away from the inlet (longer ducts). The exact solution of such a problem, through classical analytical methods (Mikhailov & Ösizik, 1984) is inhibited due to the non-separable nature of the related eigenvalue problem. The present work aims at applying the so-called Generalized Integral Transform Technique (GITT) (Cotta, 1993) in order to avoid the difficulties associated with the non-separable eigenvalue problem, consequently to give an accurate and reliable analysis to allow for the solution of this formally transformable but non-separable problem, providing an efficient algorithm for numerical computations.

The problem considered is that of a rectangular duct subjected to a constant wall temperature to illustrate the powerfulness of this hybrid approach. An analysis of convergence is made and a set of benchmark results established for quantities of practical interest, such as dimensionless average temperature, local Nusselt numbers, within a wide range of the dimensionless axial coordinate, power-law indices and for the typical situation of a square duct. Comparisons are then critically performed with previously reported results (Chandrupatla & Sastri, 1977, Shah & London, 1978, Etemad, 1997) from direct numerical approaches and from hybrid numerical-analytical approach (Cotta & Aparecido, 1990) for both, fully developed and thermally developing regions.

2. ANALYSIS

Laminar flow of a non-Newtonian power-law fluid inside a rectangular duct of sides 2a and 2b, according to Fig. 1, is considered. The velocity profile is taken as fully-developed and the duct walls are subjected to a constant temperature, so that the dimensionless energy equation for constant property flow, neglecting axial conduction and viscous dissipation, in thermally developing flow is written as:

$$\left(\frac{1+\alpha}{4\alpha}\right)^2 U(X,Y)\frac{\partial\theta(X,Y,Z)}{\partial Z} = \frac{\partial^2\theta(X,Y,Z)}{\partial X^2} + \frac{\partial^2\theta(X,Y,Z)}{\partial Y^2}, \text{ in } \quad Z>0,\ -1<X<1,\ -\alpha<Y<\alpha, \tag{1.a}$$

with inlet and boundary conditions given, respectively, as follows:

$$\theta(X,Y,0)=1, \quad -1\le X\le 1, \quad -\alpha\le Y\le\alpha \tag{1.b}$$

$$\theta(-1,Y,Z)=0; \quad \theta(1,Y,Z)=0, \quad Z>0 \tag{1.c,d}$$

$$\theta(X,-1,Z)=0; \quad \theta(X,1,Z)=0, \quad Z>0 \tag{1.e,f}$$

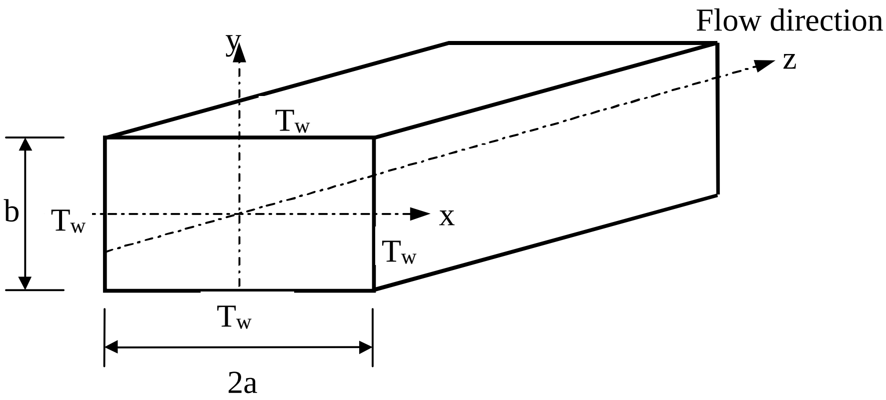

Figure 1 - Geometry and coordinates system for thermally developing rectangular duct flow.

where in Eqs. (1) above the following dimensionless groups were employed:

$$\theta(X,Y,Z)=\frac{T(x,y,z)-T_w}{T_i-T_w};\quad X=\frac{x}{a};\quad Y=\frac{y}{a};\quad Z=\frac{z}{D_h Pe};\quad \alpha=\frac{2b}{2a};$$
$$Pe=Re\,Pr=\frac{\rho cp}{k}u_m D_h;\quad U(X,Y)=\frac{u(x,y)}{u_m} \tag{2}$$

The main dimensionless groups in Eqs. (2) above are: Pe (Péclet number), α (aspect ratio), Re (Reynolds number) and Pr (Prandtl number). D_h is the hydraulic diameter defined as $D_h = 4b/(1+\alpha)$. The dimensionless velocity profile is given from the solution of momentum equations, for a non-Newtonian power-law fluid flowing within rectangular ducts, as an infinite series in the form (Lima et al., 2000):

$$U(X,Y)=4\alpha\frac{\sum_{k=1}^{\infty}\cos[\frac{(2k-1)\pi}{2}X]\,\tilde{u}_k(Y)}{\sum_{k=1}^{\infty}\tilde{h}_k\,\tilde{l}_k};\quad \tilde{h}_k=\frac{4(-1)^{k+1}}{(2k-1)\pi};\quad \tilde{l}_k=\int_{-\alpha}^{\alpha}\tilde{u}_k(Y)dY \tag{3.a-c}$$

In Eqs. (3), the quantities $\tilde{u}_k(Y)$ represent the transformed potentials for the velocity field, which were obtained numerically by the application of the GITT approach (Lima et al., 2000), so that the integral in Eq. (3.c) must also be obtained numerically through appropriate subroutines to evaluate integrals of a cubic spline such as the CSITG from the IMSL Library (1989).

Due to the non-separable nature of the velocity profile given in Eq. (3.a) and consequently, of the related eigenvalue problem needed to solve the energy equation through well-known analytical methods such as the classical integral transform technique (Mikhailov & Ösizik, 1984), an exact solution of problem (1) is not possible. On the other hand, with the advances on the so-called GITT approach for the hybrid analytical-numerical solution of this class of non-separable eigenvalue problem, it is possible to avoid these difficulties as now demonstrated (Cotta & Aparecido, 1990, Cotta, 1993). For this purpose, in order to alleviate the difficulties related to the eigenvalue problem and to permit the employment of the generalized integral transform technique, the following auxiliary eigenvalue problems are chosen:

$$\frac{d^2\psi_i(X)}{dX^2} + \mu_i^2\psi_i(X) = 0;\ -1 < X < 1 \tag{4.a}$$

$$\psi_i(-1) = 0; \qquad \psi_i(1) = 0 \tag{4.b,c}$$

and

$$\frac{d^2\phi_m(Y)}{dY^2} + \lambda_m^2\phi_m(Y) = 0;\ -\alpha < Y < \alpha \tag{5.a}$$

$$\phi_m(-\alpha) = 0; \qquad \phi_m(\alpha) = 0 \tag{5.b,c}$$

which are readily solved to yield eigenfunctions, eigenvalues, and normalization integrals as

$$\psi_i(X) = \cos(\mu_i X); \qquad \phi_m(Y) = \cos(\lambda_m Y) \tag{6.a,b}$$

$$\mu_i = \frac{(2i-1)\pi}{2}; \qquad \lambda_m = \frac{(2m-1)\pi}{2\alpha} \tag{6.c,d}$$

$$N_i = 1; \qquad M_m = \alpha, \qquad i, m = 1,2,.... \tag{6.e,f}$$

Eigenvalue problems (4) and (5) allow the development of the following integral transform pair:

$$\tilde{\tilde{\theta}}_{im}(Z) = \int_{-1}^{1}\int_{-\alpha}^{\alpha}\psi_i(X)\,\phi_m(Y)\theta(X,Y,Z)dXdY, \qquad \text{inversion} \tag{7.a}$$

$$\theta(X,Y,Z) = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\frac{\psi_i(X)\,\phi_m(Y)}{N_i M_m}\tilde{\tilde{\theta}}_{im}(Z), \qquad \text{transform} \tag{7.b}$$

Equation (1.a) is now operated on with $\int_{-1}^{1}\int_{-\alpha}^{\alpha}\psi_i(X)\,\phi_m(Y)dXdY$ to yield, after employing the inversion formula (7.b), the following system of coupled differential equations to compute the transformed potentials $\tilde{\tilde{\theta}}_{im}(Z)$:

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} D_{ijmn}\frac{d\tilde{\tilde{\theta}}_{jn}(Z)}{dZ} + \left(\frac{4\alpha}{1+\alpha}\right)^2(\mu_i^2 + \lambda_m^2)\tilde{\tilde{\theta}}_{im}(Z) = 0, \quad Z > 0 \tag{8.a}$$

where

$$D_{ijmn} = \int_{-1}^{1}\int_{-\alpha}^{\alpha}\psi_i(X)\psi_j(X)\,\phi_m(Y)\phi_n(Y)U(X,Y)dXdY \tag{8.b}$$

while the transformed inlet condition becomes

$$\tilde{\bar{\theta}}_{im}(0) = \tilde{\bar{g}}_{im} = 4\frac{(-1)^{i+m+2}}{\mu_i \lambda_m} \tag{8.c}$$

In Eq. (7.b) each summation is associated with the eigenfunction expansion in a corresponding spatial coordinate, for computational purposes, the series solution given by Eq. (7.b) is, in general, truncated to a finite number of terms for it summation, in order to compute the potential $\theta(X, Y, Z)$. The solution convergence is verified by comparing the values for the potential obtained with the truncated series for different numbers of retained terms. Such number of terms is commonly user-supplied and even taken as the same for each summation. This procedure certainly results in unnecessary computational effort due to the fact that each summation might be converged with a markedly different truncation order. Therefore, aiming at reducing computational costs to solve system (8), an ordering scheme is proposed as follows (Mikhailov & Cotta, 1996, Cotta & Mikhailov, 1997, Corrêa et al., 1997). Now, the criteria selected for this ordering procedure involves the summation of the eigenvalues in each direction, as:

$$\beta_p^2 = \mu_i^2 + \lambda_m^2 \tag{9}$$

Then, the indices i and m related to the temperature field are reorganized into the single index p, while the indices j and n are collapsed into the new index q. The associated double sums are then rewritten as:

$$\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} \rightarrow \sum_{p=1}^{\infty} \quad ; \quad \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \rightarrow \sum_{q=1}^{\infty} \tag{10.a,b}$$

The truncated version of system (8) is now written in terms of these new indices as:

$$\sum_{q=1}^{Nt} D_{pq} \frac{d\tilde{\bar{\theta}}_q(Z)}{dZ} + \left(\frac{4\alpha}{1+\alpha}\right)^2 \beta_p^2\, \tilde{\bar{\theta}}_p(Z) = 0, \quad Z > 0 \tag{11.a}$$

$$\tilde{\bar{\theta}}_p(0) = \tilde{\bar{g}}_p \tag{11.b}$$

The coupled system of ordinary differential equations (11) is solved by efficient numerical algorithms for initial value problems, such as in subroutine IVPAG from the IMSL package (1989), with high accuracy. Then, after the transformed potentials are obtained, quantities of practical interest are determined from the analytic inversion formula (7.b), such as the dimensionless average temperature

$$\theta_{av}(Z) = \frac{1}{A_c}\int_{A_c} U(X, Y)\theta(X, Y, Z)dA \tag{12.a}$$

or

$$\theta_{av}(Z)=\frac{1}{\alpha\sum_{k=1}^{Nv}\tilde{h}_k\,\tilde{l}_k}\sum_{p=1}^{Nt}\sum_{k=1}^{Nv}B_{ik}Q_{mk}\tilde{\bar{\theta}}_p(Z) \tag{12.b}$$

where

$$B_{ik}=\int_{-1}^{1}\cos[\mu_i X]\cos\left[\frac{(2k-1)\pi}{2}X\right]dX \quad ; \quad Q_{mk}=\int_{-\alpha}^{\alpha}\cos[\lambda_m Y]u_k(Y)dY \tag{13.a,b}$$

and the local Nusselt number can be calculated by making use of the temperature gradients at the wall integrated over the perimeter, or utilizing the axial gradient of the average temperature,

$$Nu(Z)=-\frac{\alpha}{(1+\alpha)^2}\frac{\left[\int_{-1}^{1}\left(\left.\frac{\partial\theta}{\partial Y}\right|_{Y=\alpha}-\left.\frac{\partial\theta}{\partial Y}\right|_{Y=-\alpha}\right)dX+\int_{-\alpha}^{\alpha}\left(\left.\frac{\partial\theta}{\partial X}\right|_{X=1}-\left.\frac{\partial\theta}{\partial X}\right|_{X=-1}\right)dY\right]}{\theta_{av}(Z)} \tag{14.a}$$

or

$$Nu(Z)=-\frac{1}{4\theta_{av}(Z)}\frac{d\theta_{av}(Z)}{dZ} \tag{14.b}$$

3. RESULTS AND DISCUSSION

Now, results are presented in terms of dimensionless average temperature and Nusselt numbers along the axial coordinate, within the range of $Z = 10^{-4} - 1$, for a square duct ($\alpha = 1$). System (11) was solved numerically with $Nt \leq 200$ and a relative user-prescribed tolerance of 10^{-4} in subroutine IVPAG from the IMSL package (1989).

To illustrate the convergence behavior of the present approach is showed in Table 1 the convergence of the Nusselt number in thermal entry region (i.e., $Z = 10^{-2}$, 10^{-1} and 1) for different power-law indices. It is observed in this table an excellent convergence ratio, with practically three digits converged for all positions studied. The comparison among the values of Nusselt numbers calculated in the present work and the values of Chandrupatla and Sastri (1977) is also showed, and it can be noticed that the values are in good agreement with each other, indicating that the numerical code developed here is well established.

In Fig. 2 the results of axial distribution of the dimensionless average temperature along the thermal entry region of a square duct are presented, for various power-law indices. It can be noticed a small influence of the power-law index in the dimensionless average temperature along the thermal entry region. But, in Fig. 3, it is observed a higher influence of the power-law index in Nusselt numbers in the thermal entry region. In the fully-developed region the effect of the power-law index in Nusselt numbers tends to disappear where the curves are practically coincident.

The effect of power-law index in the average temperature is small, as can be verified in Fig. 2 by a slight increase in the average temperature when $n > 1$. However, in Fig. 3, it is observed an opposite behavior for the Nusselt numbers. These aspects can be explained by the fact when n > 1 the viscous effects near the wall diminish and, consequently, the thermal exchange is less intensified resulting in lower values for the Nusselt numbers when compared with those values for n < 1.

Table 1. Convergence of the local Nusselt number for a square duct

Nu(Z)				
Z = 0.01				
Nt	**n = 0.50**	**n = 0.75**	**n = 1.00**	**n = 1.25**
50	4.6259	4.4479	4.3493	4.2887
100	4.6234	4.4456	4.3469	4.2868
150	4.6224	4.4446	4.3466	4.2859
200	4.6220	4.4445	4.3472	4.2858
a	4.604	4.446	4.357	NA
Z = 0.1				
50	3.2121	3.0660	2.9826	2.9288
100	3.2120	3.0660	2.9826	2.9288
150	3.2120	3.0660	2.9825	2.9288
200	3.2120	3.0660	2.9825	2.9288
a	3.189	3.055	2.976	NA
Z = 1				
50	3.2054	3.0630	2.9786	2.9241
100	3.2083	3.0615	2.9781	2.9237
150	3.1950	3.0618	2.9717	2.9238
200	3.2059	3.0620	2.9787	2.9232
a	3.184	3.050	2.975	NA

NA – Not available, a - Chandrupatla and Sastri (1977)

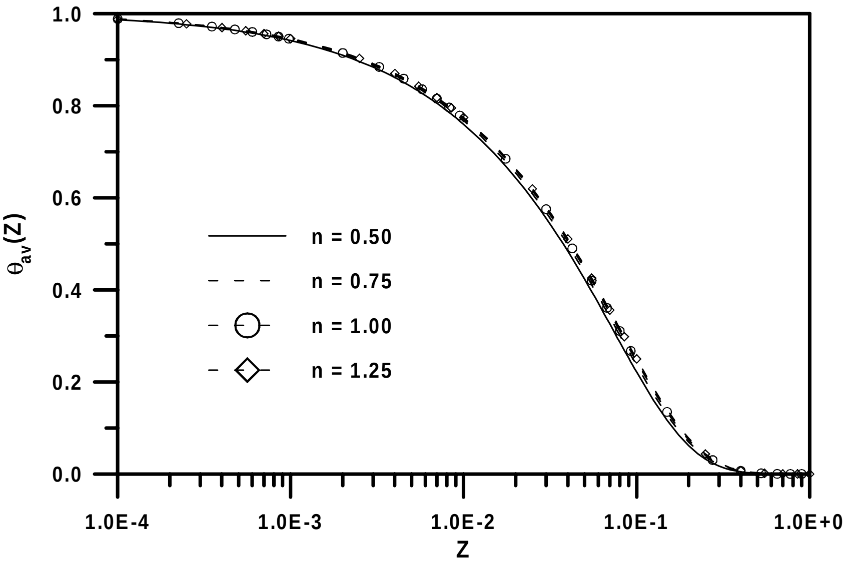

Figure 2. Dimensionless average temperature for different power-law indices.

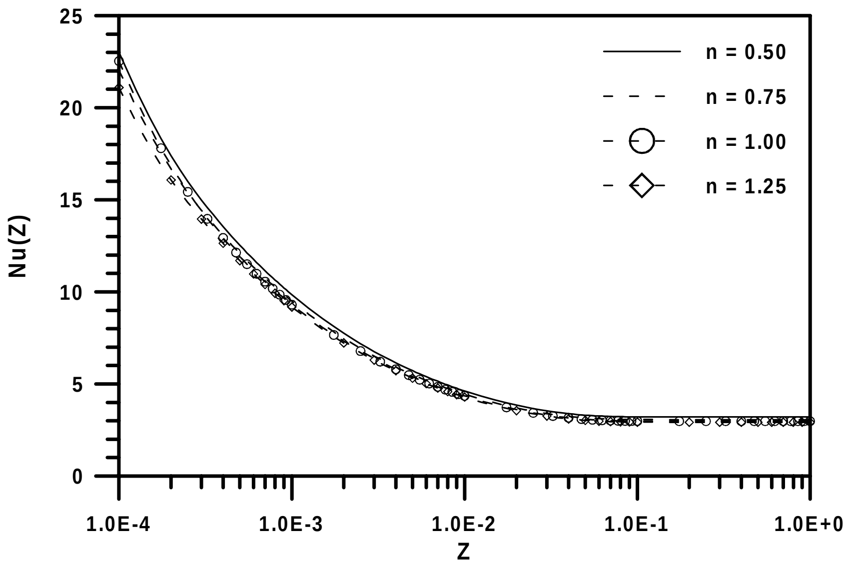

Figure 3. Nusselt numbers along the thermal entry region for different power-law indices.

4. CONCLUSIONS

The present approach demonstrated to be relatively cheap, within the range of Nt considered. Numerical results were tabulated and graphically presented providing sets of benchmark for the local Nuselt numbers and dimensionless average temperature. The next step in the application of the present methodology involving the flow of non-Newtonian fluids will be concerned to the case of irregularly shaped duct geometries as described in Cotta (1993).

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