# ANALYSIS OF A LOW SPEED WIND TUNNEL CONTRACTION BY GENERALIZED INTEGRAL TRANSFORM TECHNIQUE

**João Batista Aparecido** – jbaparec@dem.feis.unesp.br
**Edson Del Rio Vieira** – delrio@dem.feis.unesp.br
**João Batista Campos Silva** – jbcampos@dem.feis.unesp.br
Universidade Estadual Paulista – Unesp – Departamento de Engenharia Mecânica
Cx. P. 31 – 15385-000 – Ilha Solteira, SP, Brasil

**Abstract.** *The detailed knowledge of a flow field inside an incompressible subsonic wind or water tunnel contraction is of interest in designing this kind of equipment due to the needing to guarantee some flow characteristics such as: low turbulence level, small thickness of boundary layer and uniform velocity profile at inlet of the test section. Several approximate methods have been proposed in the technical literature to obtain two-dimensional flow fields inside axis-symmetric wind tunnel contraction. The purpose of this work is to apply the generalized integral transform technique (GITT) to analyze a low speed wind tunnel contraction with rectangular cross section by solving the streamline equation for the flow in a known axially non-symmetric contraction geometry.*

**Keywords:** C*ontraction, Generalized integral transform technique, Hydrodynamic tunnel.*

## 1. INTRODUCTION

Many experimental flow equipment to jet flow research and flow transducer calibration are fitted with a contracting nozzle. Contractions are also found in aerodynamics and hydrodynamics tunnels just before of the test section. The flow acceleration achieved in the contraction, basically, reduces the non-uniformity in the mean-flow and produces a uniform velocity profile at the inlet of the test section by flattening the boundary layer. A contraction also produces a strong reduction in the relative turbulence level of the flow field in the test section. In an aerodynamic tunnel a uniform free-stream flow with low turbulence is frequently required. In order to reduce the turbulence level, eliminate swirl and lateral mean velocity variations and undesirable pressure pulsation, originated from upstream blowers, many apparatus employed honeycombs and screens, usually upstream the test section. The use of a contraction permits the installation of screens and honeycombs in a very large dimension cross-section, usually named stagnation section. In this section, the velocity is very low, causing low pressure-loss and small dynamics loads in honeycombs and screens. Other information about contraction utilizing in aerodynamics tunnels can be obtained in (Bradshaw & Pankhurst, 1964) and (Gorecki, 1989).

Many analytical solutions, in a theoretical front, have been proposed, in the technical literature, for designing of contraction shapes in different cross-section geometry. Generally, they are solutions of the Stokes–Beltrami equation for stream function in a two-dimensional axis-symmetric geometry – only (Tulapurkara & Bhalla, 1988) relate fifteen different methods for axis-symmetric contraction shape design. These solutions produce an infinite set of stream surfaces and one of that, with tolerable pressure gradients, can be used as the contraction wall contour. However, many problems are related with this procedure, the main of this has relations with the infinite stream function that produce contraction of infinite length. In practice, actual contractions need a truncation process. Other related problem is related to a contraction cross-section where axis-symmetric calculus, depicted in literature, should be adapted to other geometry, remarkably rectangular and irregular octagonal cross-section. Evidently, it is an empirical process.

In this work, a method of design in two-dimensional rectangular coordinates using an ideal fluid flow has been proposed. This procedure using a known contraction contours shape permits to determine the flow field behavior. Physically, the problem consists of a flow that is incoming in the face $L_1$ and outgoing in the face $L_3$. The faces $L_2$ and p(x) are solid walls, p(x) is a prescribed shape function. Figure 1 shows the geometry for the problem.

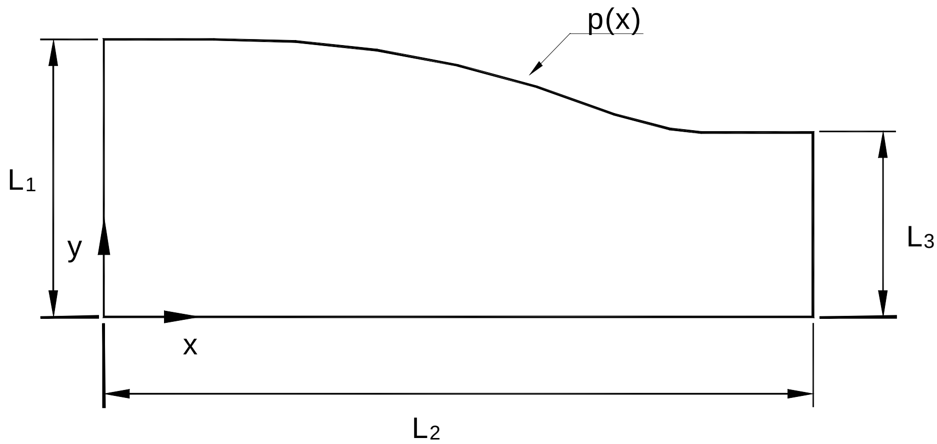



Figure 1 - Geometry of the contraction.

## 2. MATHEMATICAL MODEL

It is supposed in this flow that inertia forces prevails over the viscous ones. As is well-known this model fails nearby the solid walls but in this case we are interested just in the main stream, so this approximation is fairly good. For such a kind of two-dimensional flow it can be modeled as inviscid irrotational flow using the concept of streamline, so the governing formula for the flow is

$$\frac{\partial^2\Psi(x,y)}{\partial x^2}+\frac{\partial^2\Psi(x,y)}{\partial y^2}=0\,. \tag{1}$$

The boundary conditions are

$$\Psi(0,y)=u_o y\ ,\ \left.\frac{\partial\Psi(x,y)}{\partial x}\right|_{x=L_3}=0\,; \tag{2a,b}$$

$$\Psi(x,0)=0\ ,\ \Psi(x,p(x))=u_o L_1\,. \tag{2c,d}$$

The velocity components $u$ and $v$ can be calculated as

$$u(x,y) \equiv \frac{\partial \Psi(x,y)}{\partial y}, \quad \text{and} \quad v(x,y) \equiv -\frac{\partial \Psi(x,y)}{\partial x}. \tag{3a,b}$$

Defining the following dimensionless variables

$$X \equiv \frac{x}{L_1},\ R_s \equiv \frac{L_2}{L_1},\ Y \equiv \frac{y}{L_1},\ R_c \equiv \frac{L_3}{L_1},\ P(X) \equiv \frac{p(x)}{L_1} \quad \text{and} \quad \Psi^*(X,Y) \equiv \frac{\Psi(x,y)}{u_o L_1} \tag{4a-e}$$

equations (1, 2) becomes

$$\frac{\partial^2 \Psi^*(X,Y)}{\partial X^2} + \frac{\partial^2 \Psi^*(X,Y)}{\partial Y^2} = 0, \tag{5}$$

$$\Psi^*(0,Y) = Y \quad \text{and} \quad \left.\frac{\partial \Psi^*(X,Y)}{\partial X}\right|_{X=R_s} = 0, \tag{6a,b}$$

$$\Psi^*(X,0) = 0 \quad \text{and} \quad \Psi^*(X,P(X)) = 1. \tag{6c,d}$$

In equations above $R_c$ and $R_s$ are the contraction and the slenderness ratios. As a consequence of that dimensionless parameters we can also put the velocity components $u$ and $v$ in dimensionless form as follow

$$U(X,Y) \equiv \frac{u(x,y)}{u_o} = \frac{\partial \Psi^*(X,Y)}{\partial Y}, \quad \text{and} \quad V(X,Y) \equiv \frac{v(x,y)}{u_o} = -\frac{\partial \Psi^*(X,Y)}{\partial X}. \tag{7a,b}$$

The boundary conditions (6) are not homogeneous, so in order to homogenize them, the following change in the dependent variable $\Psi^*(X,Y)$ is done

$$\Psi^*(X,Y) = \Psi^+(X,Y) + \frac{Y}{P(X)} \tag{8}$$

then equations (5,6) become

$$\frac{\partial^2 \Psi^+(X,Y)}{\partial X^2} + \frac{\partial^2 \Psi^+(X,Y)}{\partial Y^2} = YH(X), \tag{9}$$

$$\Psi^+(X,0) = 0 \quad \text{and} \quad \Psi^+(X,P(X)) = 0, \tag{10a,b}$$

$$\Psi^+(0,Y) = 0 \quad \textit{and} \quad \left.\frac{\partial \Psi^+(X,Y)}{\partial X}\right|_{X=R_s} = \left.\frac{Y\dot{P}(X)}{P^2(X)}\right|_{X=R_s} = 0, \tag{10c,d}$$

where $H(X)$ the source term in equation (9) is given by

$$H(X) \equiv -\frac{d^2 P^{-1}(X)}{dX^2} = \frac{1}{P^2(X)}\left[\ddot{P}(X) - \frac{2\dot{P}^2(X)}{P(X)}\right]. \tag{10e}$$

## 2.1 ABOUT THE WALL FUNCTION *P(X)*

*P(X)* is a function that must have some special features. As one can see there are in the above formulation derivatives until the second order, so *P(X)* must be at least a function of class $C^2$ within the interval [0,$R_s$], meaning that it and its first and second derivatives must be continuous in that domain. Additionally *P(X)* must have a soft transition from the contraction to the straight part of the tunnel. To fit the contraction geometry, at least, the following geometric restrictions must be satisfied

$$P(0) = 1,\ \ P(R_s) = R_c,\ \ \dot{P}(X)\Big|_{X=0} = 0,\ \ \dot{P}(X)\Big|_{X=R_s} = 0 \tag{10f}$$

In this work the analysis about *P(X)* is quite general, but at the end it was used a third-order polynomial that obeys the above restrictions and is expressed by the formula

$$P(X) = 1 + (R_c - 1)\left[3 - 2\frac{X}{R_s}\right]\left[\frac{X}{R_s}\right]^2. \tag{10g}$$

Note that P(X) depends only on parameters $R_s$ and $R_c$.

## 3. GENERALIZED INTEGRAL TRANSFORM ANALYSIS

The Generalized Integral Transform Technique (GITT) has been used to solve several kind of problems in fluid flow and heat and mass transfer inside irregular domains. Some papers regarding irregular domains were published by Aparecido *et al.*, (1989) and Aparecido & Cotta (1990a,b, 1992).

### 3.1. Application of GITT for transforming the problem in the *Y*-direction

In order to transform the equations governing the problem, presented above, we choose in according with Aparecido (1997) the following eigenvalue problem

$$\frac{\partial^2 \psi(X,Y)}{\partial Y^2} + \lambda^2(X)\psi(X,Y) = 0, \quad 0 < Y < P(X) \tag{11a}$$

subject to the following boundary conditions

$$\psi(X,0) = 0 \quad \text{and} \quad \psi[X, P(X)] = 0. \tag{11b,c}$$

The solution for the problem (11) is

$$\psi_i(X,Y) = \sqrt{\frac{2}{P(X)}} sin[\lambda_i(X)Y]\ ;\ \lambda_i(X) = \frac{i\pi}{P(X)},\ \ i = 1,2,\ldots. \tag{12a,b}$$

Using such complete set of eigenfunctions $\psi_i(X)$ we can define an integral transform regarding the y-axis and its respective inverse as follow

$$\text{Transform: } \tilde{\Psi}_i^+(X) = \int_0^{P(X)} \psi_i(X,Y)\Psi^+(X,Y)dY, \ i = 1,2,3,... \tag{13a}$$

$$\text{Inverse: } \quad \Psi^+(X,Y) = \sum_{i=1}^{\infty} \psi_i(X,Y)\tilde{\Psi}_i^+(X). \tag{13b}$$

In order to transform the original partial differential equation (9), it is multiplied by the eigenfunctions $\psi_i(X)$; the auxiliary eigenvalue equation is multiplied by $\Psi^+(X,Y)$, the resulting equations are subtracted and then integrated over the domain [$0,P(X)$]. The final equation is

$$\frac{d^2\tilde{\Psi}_i^+(X)}{dX^2} + \sum_{j=1}^{\infty}\left[B_{ij}(X)\frac{d\tilde{\Psi}_j^+(X)}{dX} + C_{ij}(X)\tilde{\Psi}_j^+(X)\right] - \lambda_i^2(X)\tilde{\Psi}_i^+(X) = H(X)\tilde{g}_i(X), \ i = 1,2.... \tag{14}$$

where

$$B_{ij}(X) = 2\int_0^{P(X)} \psi_i(X,Y)\frac{\partial \psi_j(X,Y)}{\partial X}dY = \begin{cases} 0, & i = j, \\ \dfrac{4(-1)^{i+j}\, ij}{(i^2 - j^2)}\dfrac{\dot{P}(X)}{P(X)}, & i \neq j \end{cases} \tag{15a}$$

$$C_{ij}(X) = \int_0^{P(X)} \psi_i(X,Y)\frac{\partial^2 \psi_j(X,Y)}{\partial X^2}dY =$$

$$= \begin{cases} -\dfrac{3 + 4(i\pi)^2}{12}\left[\dfrac{\dot{P}(X)}{P(X)}\right]^2, & i = j \\ \dfrac{2\pi^2(-1)^{i+j+1}(ij)^2}{(i^2 - j^2)^2}\left[i^2[3\dfrac{\dot{P}^2(X)}{P^2(X)} - \dfrac{\ddot{P}(X)}{P(X)}] + j^2[\dfrac{\ddot{P}(X)}{P(X)} + \dfrac{\dot{P}^2(X)}{P^2(X)}]\right], & i \neq j \end{cases} \tag{15b}$$

The equation (14) is a system of second order ordinary differential equations having variable coefficients. To provide boundary conditions to such system is necessary to transform the primitive boundary conditions regarding the $X$-direction (10c,d). To accomplish that is necessary multiply that equations by $\psi_i(X)$ and integrate over the interval [$0,P(X)$], the resulting equations are

$$\int_0^{P(X)} \psi_i(X,Y)\Psi^+(0,Y)dY = 0 \ \Rightarrow \ \tilde{\Psi}_i^+(0) = 0 \ , \tag{16a}$$

$$\left.\int_0^{P(X)} \psi_i(X,Y)\frac{\partial \Psi^+(X,Y)}{\partial X}dY\right|_{X=R_s} = 0 \ \Rightarrow \ \left.\frac{d\tilde{\Psi}_i^+(X)}{dX}\right|_{X=R_s} = 0 \tag{16b}$$

### 3.2 Transforming the problem in the *X*-direction

Now to transform the system of ordinary differential equations (14) subjected to the boundary conditions (16a,b) we choose (Aparecido, 1997) an eigenvalue problem regarding the X-axis, stated as follow

$$\frac{d^2\phi(X)}{dX^2}+\mu^2\phi(X)=0, \quad 0<X<R_s \tag{17a}$$

subjected to the boundary conditions

$$\phi(0)=0 \quad \text{and} \quad \left.\frac{d\phi(X)}{dX}\right|_{X=R_s}=0. \tag{17b,c}$$

The solution for the problem (17) is

$$\phi_m(X)=\sqrt{\frac{2}{R_s}}sin(\mu_m X) \;\; ; \; \mu_m=\frac{(2m-1)\pi}{2R_s}, \;\; m=1,2,.... \tag{18a,b}$$

Using such complete set of eigenfunctions we can define an integral transform regarding the X-axis and its respective inverse as follow

$$\text{Transform: } \overline{\tilde{\Psi}}{}^{+}_{im}=\int_0^{R_s}\phi_m(X)\tilde{\Psi}^{+}_i(X)dX, \;\; m=1,2,3,... \tag{19a}$$

$$\text{Inverse: } \quad \tilde{\Psi}^{+}_i(X)=\sum_{m=1}^{\infty}\phi_m(X)\overline{\tilde{\Psi}}{}^{+}_{im}. \tag{19b}$$

In order to transform the system of ordinary differential equations (14), it is multiplied by the eigenfunctions $\phi_i(X)$; the auxiliary eigenvalue equation is multiplied by $\tilde{\Psi}^{+}_i(X)$, the resulting equations are subtracted and then integrated over the domain $[0,R_s]$. The final equation is

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(B_{ijmn}+C_{ijmn}-D_{iimn}\delta_{ij}-\mu_m^2\delta_{ij}\delta_{mn})\overline{\tilde{\Psi}}{}^{+}_{jn}=\tilde{\overline{g}}_{im}, \;\; i,m=1,2,3.... \tag{20}$$

where

$$\overline{\tilde{\Psi}}{}^{+}_{jn}=\int_0^{R_s}\phi_n(X)\tilde{\Psi}^{+}_j(X)dX \tag{21a}$$

$$B_{ijmn}=\int_0^{R_s}B_{ij}(X)\phi_m(X)\frac{d\phi_n(X)}{dX}dX \;\; , \;\; C_{ijmn}=\int_0^{R_s}C_{ij}(X)\phi_m(X)\phi_n(X)dX \tag{21b,c}$$

$$D_{iimn}=(i\pi)^2\int_0^{R_s}\frac{\phi_m(X)\phi_n(X)}{P^2(X)}dX \;\; , \quad \tilde{\overline{g}}_{im}=\int_0^{R_s}\phi_m(X)H(X)\tilde{g}_i(X)dX \tag{21d,e}$$

The equation (20) is a linear algebraic system and can be written in a matrix form as

$$\mathbf{A\Phi}=\mathbf{g}, \tag{22a}$$

where **A** is a square matrix, a proper representation of the coefficients $B_{ijmn}, C_{ijmn}, D_{iimn}\delta_{ij}, \mu_m^2\delta_{ij}\delta_{mn}$. The δ´s are the Kronecker delta and the vectors **Φ** and **g** are defined as follow

$$\mathbf{\Phi} \equiv [\overline{\tilde{\Psi}}_{11}, \overline{\tilde{\Psi}}_{12}, ..., \overline{\tilde{\Psi}}_{1N}, \overline{\tilde{\Psi}}_{21}, \overline{\tilde{\Psi}}_{22}, ..., \overline{\tilde{\Psi}}_{2N}, ..., \overline{\tilde{\Psi}}_{N1}, \overline{\tilde{\Psi}}_{N2}, ..., \overline{\tilde{\Psi}}_{NN}]^T, \tag{22b}$$

$$\mathbf{g} \equiv [g_{11}, g_{12}, ..., g_{1N}, g_{21}, g_{22}, ..., g_{2N}, ..., g_{N1}, g_{N2}, ..., g_{NN}]^T, \tag{22c}$$

where N is the truncation order in each summation in the infinite series.

The solution of the algebraic linear system (22a) is

$$\mathbf{\Phi} = \mathbf{A}^{-1}\mathbf{g}. \tag{22d}$$

After obtaining $\Phi$ in equation (22d) we have its elements $\overline{\tilde{\Psi}}_{im}^{+}$ and then we can invoke the two inversion formulae to get the streamline closed form profile given by

$$\Psi^*(X,Y) = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} \psi_i(X,Y)\phi_m(X)\overline{\tilde{\Psi}}_{im}^{+} + \frac{Y}{P(X)} \tag{23}$$

## 4. ENGINEERING COMPUTATIONS

Once we got the streamline profile (23) we are able to calculate some derived quantities of engineering interest such velocity components *u, v* and pressure for different values of the contraction and slenderness ratios. Using the velocity components definition (3a,b) and the streamline profile (23) we obtain its formulae

$$U(X,Y) = \frac{\partial \Psi^*(X,Y)}{\partial Y} = P^{-1}(X) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\partial \psi_i(X,Y)}{\partial Y}\phi_m(X)\overline{\tilde{\Psi}}_{im}^{+} \tag{24a}$$

$$V(X,Y) = -\frac{\partial \Psi^*(X,Y)}{\partial X} = Y\frac{\dot{P}(X)}{P^2(X)} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{\partial \psi_i(X,Y)}{\partial X}\phi_m(X) + \psi_i(X,Y)\frac{d\phi_m(X)}{dX}\right]\overline{\tilde{\Psi}}_{im}^{+} \tag{24b}$$

Another quantities of some interest are the velocities components $U_w(X)$, $V_w(X)$ and the pressure coefficient $Cp_w(X)$ from Bernoulli´s equation at the curved wall and $U_o(X)$ at the straight wall given by

$$U_w(X) \equiv U(X,P(X)) = P^{-1}(X) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left.\frac{\partial \psi_i(X,Y)}{\partial Y}\right|_{Y=P(X)} \phi_m(X)\overline{\tilde{\Psi}}_{im}^{+} \tag{24c}$$

$$V_w(X) \equiv V(X,P(X)) = \frac{\dot{P}(X)}{P(X)} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left.\frac{\partial \psi_i(X,Y)}{\partial X}\right|_{Y=P(X)} \phi_m(X)\overline{\tilde{\Psi}}_{im}^{+} \tag{24d}$$

$$U_o(X) \equiv U(X,0) = P^{-1}(X) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left.\frac{\partial \psi_i(X,Y)}{\partial Y}\right|_{Y=0} \phi_m(X)\overline{\tilde{\Psi}}_{im}^{+} \tag{24e}$$

$$Cp_w(X) \equiv \frac{2(p_w - p_o)}{\rho\, u_0^2} = 1 - U_w^2(X) - V_w^2(X) \tag{24f}$$

$p_0$ is the pressure at the inlet, $X=0$.

## 5. RESULTS

Some results of interest are presented in Figures 2 and 3 that show the velocity components $U$ and $V$ at the curved wall, respectively, for different contraction and slenderness ratios along the contraction axis X. The $U_w$ is strongly affected by the contraction ratio $R_c$ as the contraction decreases the velocity at the outlet increases. The slenderness ratio also have an important role as it decreases the region where the $U_w$ has a negative derivative grows, as in Figure 2. For short contractions with small contraction ratio the transversal velocity at the contour $V_w$ are in the worst situation as seen in Figure 3.

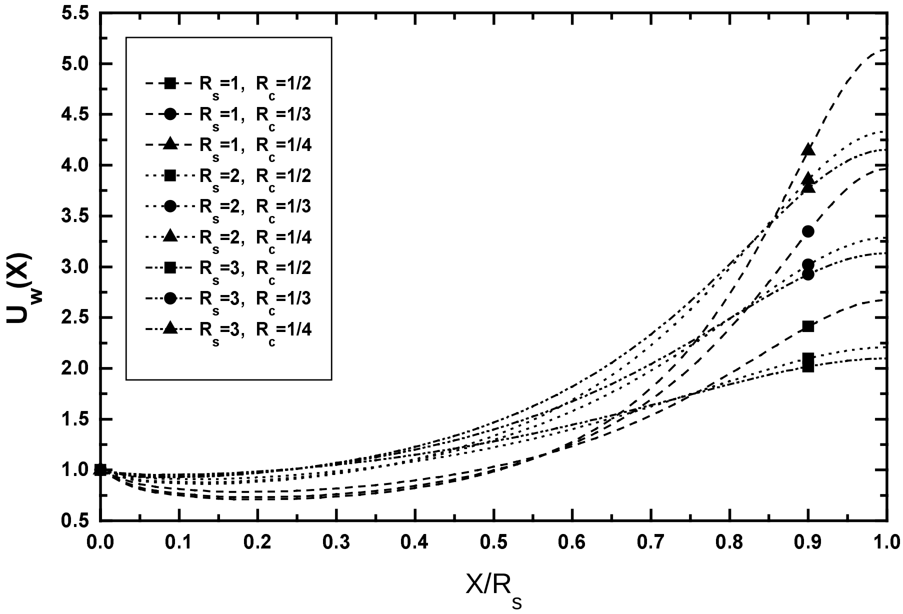



Figure 2 - $U$-velocity at the curved wall for different contraction and slenderness ratios.

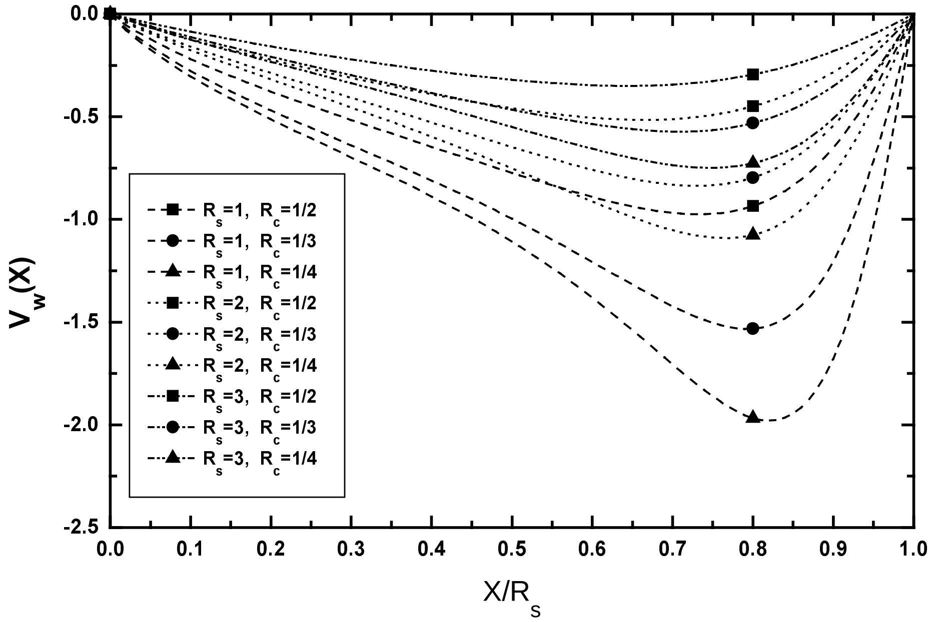



Figure 3 - $V$-velocity at the curved wall for different contraction and slenderness ratios.

Figure 4 presents the pressure coefficient at the curved wall. $C_{pw}$ decreases strongly as the contraction ratio decreases. Also it has an adverse pressure gradient near the inlet as the slenderness ratio decreases, so that region is a candidate to have flow detachment and such geometries must be avoided. Figure 5 presents the $U_o$ velocity at the straight wall that has a similar behavior as $U_w$ but without negative derivative along the axis X.

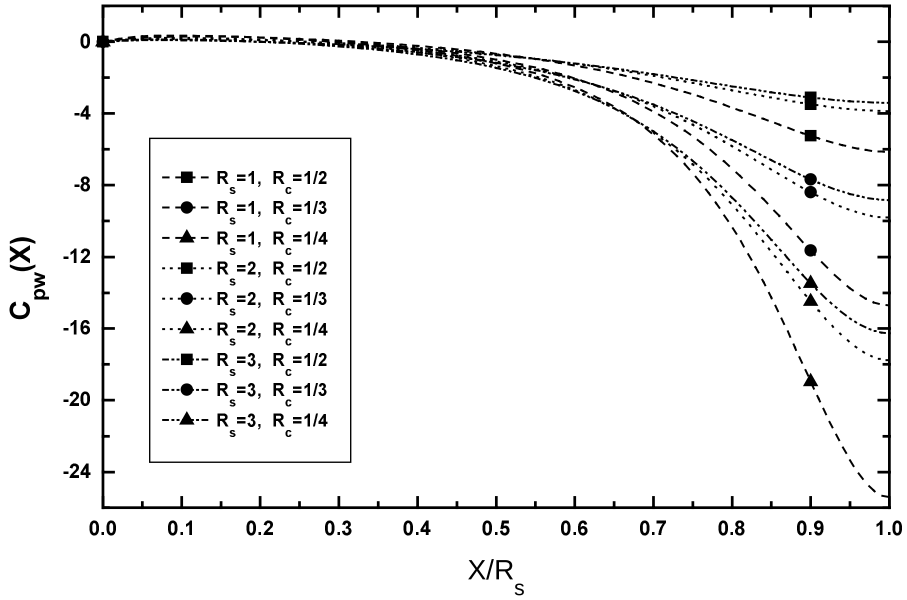



Figure 4 - Pressure coefficient $C_{pw}$ at the curved wall for different contraction and slenderness ratios.

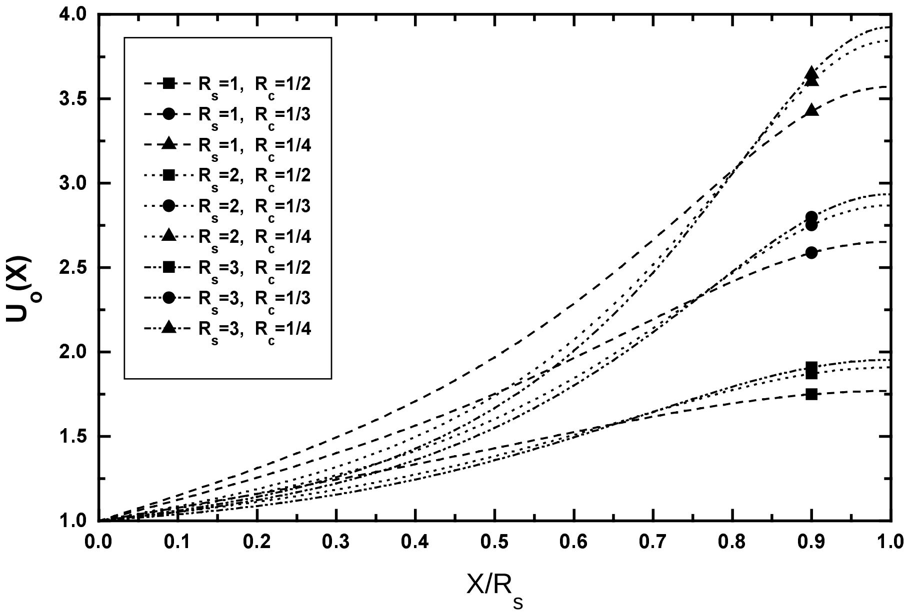
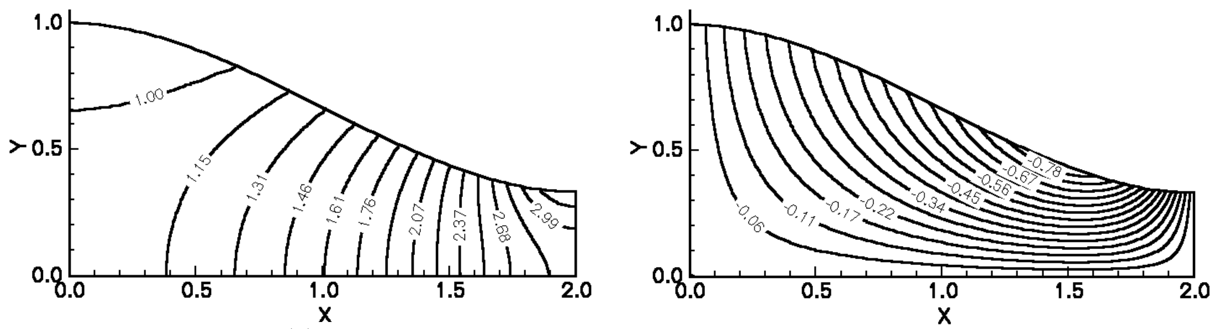



Figure 5 - $U$-velocity at the straight wall for different contraction and slenderness ratios.

Figure 6 is the isovalues for the velocities components $U$ and $V$ in the whole contraction for the length $R_s = 2$ and contraction ratio $R_c = 1/3$. A desirable feature for a contraction is that the axial velocity $U$ at the exhaust section be as uniform as possible and that the transversal velocity $V$ be as nearly to null as possible. The third-order polynomial contour geometry proposed achieved fairly good such goals as can be seen in Figure 6. When the slenderness and contraction ratios decreases those goals become worse, otherwise when the slenderness and contraction rations increases those features will be better in hydrodynamics view.




(a) (b)

Figure 6 - Isovalues for (a) $U$ and (b) $V$ inside the contraction for $R_s = 2$ and $R_c = 1/3$.

## 6. CONCLUSIONS

Contraction shapes of the existing tunnels vary widely and many of them have been projected basing only in designer's eye-judgments. A detailed knowledge of the flow velocity behavior become necessary in all these projects. In this work an analytical method for determination the flow characteristics for a given contraction geometry has been achieved. This method is based on Generalized Integral Transform Technique. An application of this proposed method of solution has been done successfully for a family of third-order polynomial contraction shape. This procedure permits the calculation of the flow field and so giving parameters for the effective analysis of contractions. In spite of the simple geometry used for the contraction the results revealed quite good.

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