Stability of Discrete Systems with Unilateral Contact and Coulomb Friction

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Abstract: The aim of this paper was to study the stability of equilibrium states in a mechanical system involving unilateral contact with Coulomb friction. Since the assumptions made in classical stability theorems are not satisfied with this class of systems, we return to the basic definitions of stability and study the time evolution of the distance between a given equilibrium and the solution of a Cauchy problem where the initial conditions are in a neighbourhood of the equilibrium. Such an analysis is possible because it was recently established that the dynamics is well-posed in the case of analytical data. In the present study, we focus in particular on the stability of the equilibrium states under a constant force, and deal essentially with a simple mass-spring system in R^2.

Keywords: Discrete dynamics, Unilateral contact, Shocks, Coulomb fricition, Stability

INTRODUCTION

This paper is exactly in line with a previous study of Basseville, Léger and Pratt, 2003, in which the authors explored the equilibrium states of a simple model involving unilateral contact and Coulomb friction. In the latter article, after exploring the set of equilibrium states, their stability was determined by performing a direct numerical study on the dynamics. The criteria used for this purpose were the reliability of the time discretization, which was of the "time stepping" type, and of the algorithm used in the "NonSmooth Contact Dynamics" software (NSCD) (see Jean, 1999). Since the convergence of this algorithm and its ability to approach the solution of the continuous problem were not yet established at that time, these studies have dealt mainly with numerical experiments. The first aim of the present study was to provide theoretical proof of the stability of the same simple model.

Existence of solutions to the dynamical problem with unilateral contact and Coulomb friction was proved in Monteiro-Marques, 1993, under the hypothesis that the external force is a bounded function of time. This result was generalized recently to the case where the external force is integrable. In the same way as for the result by Monteiro-Marques, this was obtained by the convergence of a sequence of approximate solutions associated with thinner and thinner time discretizations. But it was proved by counter-examples that, given initial data, the trajectory is not unique in general. In Ballard and Basseville, 2005, it was established that uniqueness holds only if the external force is an analytical function of time.

As a consequence of these results, it was proved that the sequence of approximations given by the algorithm NSCD converges uniformly towards the unique solution of the problem when the data are analytical, and in particular under a constant external force. This means in particular that the solution of the continuous problem can be studied starting with estimates based on the discretized problem. After this preliminary step, we can study the trajectory starting with any initial data, and thus analyze the stability of an equilibrium by studying the distance between the equilibrium and a trajectory starting from any point in a neighborhood of the equilibrium in a classical phase space.

The analysis is now relatively complete in R^2. This is the main part of the work presented here, which includes a detailled list of the stability properties of the equilibria of the simple mass-spring system. In the next section, we introduce a new stability notion, which seems specially suited to the dynamics with contact and friction. In the case of a larger size discrete system we discuss in the last part some remaining open problems and give some hints for future works. Although very useful for mechanical and engineering applications, a proper stability analysis in the case of a continuous medium submitted to the same unilateral constraints seems to be a long term work, as important mathematical difficulties remain.

FORMULATING THE DYNAMICS

This section recalls the basis of the study, namely the formulation of the dynamics and the main results that will be used later. This is done in the case of a very simple mass-spring system with a single mass moving in R^2, as represented on Figure (1), and we shall discuss after what can be naturally extended to R^n. The mass-spring system was first introduced by Klarbring, 1990, to study the set of solutions in the case of a quasi-static evolution. n and t denote respectively the normal and tangential components of the displacement U and of the reaction R of the mass m. We recall the nonregularized unilateral contact and Coulomb friction laws in which μ denotes the friction coefficient:

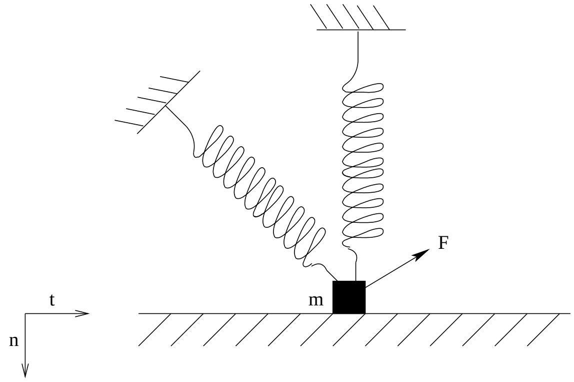

Figure 1 – The mass-spring model

$$\begin{cases} U_n \leq 0,\ R_n \leq 0,\ U_n.R_n = 0, \\ \\ |R_t| \leq \mu |R_n|, \\ |R_t| < \mu |R_n| \Longrightarrow \dot{U}_t = 0, \\ |R_t| = \mu |R_n| \Longrightarrow \dot{U}_t = -\lambda R_t,\ \lambda \geq 0, \end{cases} \tag{1}$$

The main qualitative feature of the dynamics is that, due to possible shocks, the velocity of particle m cannot be expected to be a continuous function, so that this velocity no longer admits a derivative in the classical sense, but admits derivatives only in the sense of distributions, and in fact in the sense of measures since the contact conditions (1) implies that R_n should satisfy a sign condition. We denote by $MMA([0,T];\mathrm{R}^2)$ (Motions with Measure Acceleration) the vector space of those integrable functions of $[0,T]$ into R^2 whose second derivative in the sense of distributions is a measure. It is nothing but the space of integrals of functions of bounded variation over $[0,T]$. $\mathcal{M}([0,T];\mathrm{R}^2)$ denotes in the same way the set of measures defined on $[0,T]$. Functions $U(t)$ in MMA are continuous and admit left and right derivatives (in the classical sense) $\dot{U}^-$ and $\dot{U}^+$ at any point, both being functions of bounded variation.

The dynamical problem

The evolution problem, formulated along the lines of Moreau, 1988, is the following:

Problem $(\mathcal{P})$. Find $U \in MMA([0,T];\mathrm{R}^2)$ and $R \in \mathcal{M}([0,T];\mathrm{R}^2)$ such that:

- $U(0) = U_0 \quad ; \quad \dot{U}^+(0) = V_0$ (initial data)
- $\ddot{U} + K \cdot U = F + R, \qquad in\ [0,T]$ (equation of motion)
- $U_n \leq 0, \quad R_n \leq 0, \quad U_n R_n = 0$ (unilateral contact) (2)
- $\displaystyle\int_{[0,T]} \left[R_t \cdot (V - \dot{U}_t^+) - \mu R_n (|V| - |\dot{U}_t^+|) \right] \geq 0, \quad \forall V \in C^0([0,T];\mathrm{R})$ (Coulomb friction)
- $U_n(t) = 0 \quad \Longrightarrow \quad \dot{U}_n^+(t) = -e\dot{U}_n^-(t), \ \ in\ ]0,T]$ (impact law).

F denotes the external force, $e \in [0,T]$ is a real constant (the so-called restitution coefficient) and (U_0, V_0) denotes some initial condition assumed to be compatible with the unilateral constraint, that is

$$U_{0n} \leq 0 \ \text{ and if } \ U_{0n} = 0 \text{ then } V_{0n} \leq 0. \tag{3}$$

In the following we consider completely inelastic shocks, i.e. $e = 0$. In this case, we can establish that the unilateral contact conditions (also called Signorini conditions) together with the impact law can be replaced in (2) by the so-called velocity Signorini conditions which read:

$$U_n \leq 0 \ \text{ with } \begin{cases} U_n < 0 \Longrightarrow R_n = 0, \\ \\ U_n = 0 \Longrightarrow \dot{U}_n^+ \leq 0, R_n \leq 0, \dot{U}_n^+ R_n = 0. \end{cases} \tag{4}$$

- Problem $(\mathcal{P})$ has a solution as long as function F is integrable (this result is obtained by the convergence of a time discretization using a similar method to that given in Monteiro-Marques, 1993, for $\mathcal{L}^\infty$ data).

- The uniqueness of the trajectory, given the data U_0 and V_0, is obtained only if F is an analytical function of time: two trajectories, solutions to problem ($\mathcal{P}$) with the same initial data, were explicitely built for a one-dimensional example with a force $F \in \mathcal{C}^\infty([0,\hat{T}];\mathrm{R})$.

- The problem obtained by discretizing the time interval into time steps of size h which will be used from now on, is written as follows, where V stands for the discrete values of $\dot{U}$:

Problem ($\mathcal{P}_d$)

- $h = \dfrac{\hat{T}}{K}$,
- $V^0 = V(0), \;\; u^0 = u(0)$,
- $V^{i+1} = V^i + \dfrac{h}{2m}\left((F^{i+1}+F^i) - k(u^{i+1}+u^i)\right) + \dfrac{h}{m}R^{i+1}$,
- $u^{i+1} = u^i + hV^{i+1}$,
- $$\left\{\begin{array}{l} |R_t^{i+1}| \;\leq\; \mu\,|R_n^{i+1}|, \\ \text{if } |R_t^{i+1}| \;<\; \mu\,|R_n^{i+1}|, \;\; \text{then} \;\; V_t^{i+1} = 0, \\ \text{if } |R_t^{i+1}| \;=\; \mu\,|R_n^{i+1}|, \;\; \text{then} \;\; \exists\,\lambda \geq 0 \;\text{ such that }\; R_t^{i+1} = -\lambda V_t^{i+1}, \end{array}\right. \tag{5}$$
- $$\left\{\begin{array}{l} \text{if} \quad U_n^{i+1} \geq 0 \Longrightarrow V_n^{i+1} \leq\, 0,\; R_n^{i+1} \leq\, 0,\; V_n^{i+1}R_n^{i+1} = 0, \\ \text{if} \quad U_n^{i+1} < 0 \Longrightarrow R_n^{i+1} = 0. \end{array}\right.$$

$*$ The “time stepping” type algorithm NSCD has been deduced from problem ($\mathcal{P}_d$) by Jean, 1999. We shall consider the iterates of this algorithm as a discrete dynamical systems, which will be useful to make estimates.

$*$ This algorithm, built from problem ($\mathcal{P}_d$), uses strongly the equivalence between classical Signorini conditions and velocity Signorini conditions just recalled above. Let us just note for the moment that this means that the convergence of this algorithm is consequently obtained only in the case $e = 0$. But the well-posedness of problem ($\mathcal{P}$), that is existence and uniqueness of the trajectory for sufficiently smooth data, holds for any $e \in [0, 1]$.

The set of equilibria

We first assume that the external force is constant. The equilibrium states of problem $(\mathcal{P})$ have been explored in Basseville, Léger and Pratt, 2003. They consist in the set of displacements U and reactions R satisfying, in addition to conditions (1), the following system, which is the restriction of the equation of motion of system (2) to the investigation of equilibrium solutions:

$$\left\{\begin{array}{l} K_t.U_t + W.U_n = F_t + R_t \\ W.U_t + K_n.U_n = F_n + R_n. \end{array}\right. \tag{6}$$

(F_t, F_n) denotes the external force, $K = \begin{pmatrix} K_t & W \\ W & K_n \end{pmatrix}$ is the stiffness matrix of the system of springs. We introduce a quantity A defined as $A \stackrel{def}{\equiv} K_t.F_n - W.F_t$. It is easily seen that system (6) has solutions without contact only if A is strictly negative. If A is greater or equal to zero, there are infinitely many equilibria, except if A is equal to zero and μ is small enough, in which case system (6) has a single solution which is the vertex of the Coulomb's cone. It is interesting to observe these sets of solutions in a table where the solutions are plotted in the $\{R_t, R_n\}$ plane for each possibility of the data μ and A. All the cases are represented on Figure (2). The two thin lines delimit the Coulomb's cone, and the thick line represents the equilibrium equation. The solutions in contact belong consequently to the intersection of the thick line with the cone.

SOME QUALITATIVE PHASES OF THE DYNAMICS

The stability results are obtained thanks to several technical lemmas, of which we give hereafter three significative examples.

Some technical tools

The first two lemmas are simple and intuitive. Considering an equilibrium state in contact perturbed by a normal or a tangential velocity, then, under some conditions, the trajectory will come into contact again. More precisely:

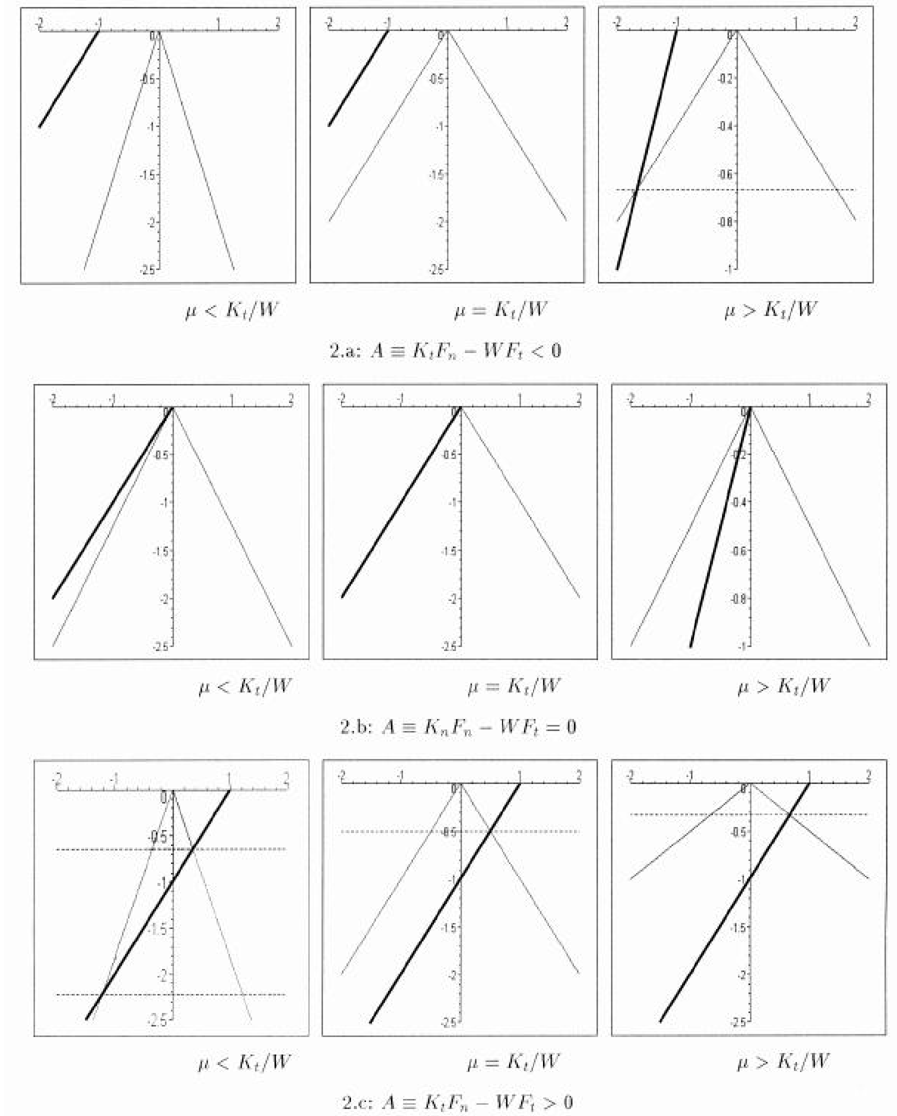

Figure 2 – The sets of equilibria in the $\{R_t, R_n\}$ plane

$*$ *Let (U^{eq}, R^{eq}) with R_n^{eq} strictly negative be an equilibrium solution, and let V_{0n} be a perturbation of this equilibrium at time t_0 by a normal velocity. Then there exists $\bar{t}$, $t_0 < \bar{t} < +\infty$, such that $U_n(\bar{t}) = 0$.*

$*$ *Let (U^{eq}, R^{eq}) be a grazing equilibrium state ($R_n^{eq} = 0$ and $U_n^{eq} = 0$) and let V_{0t} be a perturbation of this equilibrium at time t_0 by a tangential velocity. Then there exists $\tilde{t}$, $t_0 < \tilde{t} < +\infty$, such that $U_n(\tilde{t}) = 0$.*

These two preliminary results are proved by a direct analytical study, which is possible thanks to the fact that the solution to problem $\mathcal{P}$ is very smooth (made only of sine functions) when there is no contact, and theoretical regularity results (see Ballard and Bassevillle, 2005) allow the matching between successive parts of the trajectories at the impact times.

The last lemma is more technical, and is already very close to a stability result:

$*$ *Let (U^{eq}, R^{eq}) be an equilibrium state with R_n^{eq} strictly negative. Let the dynamics after any perturbation be such that there exists a time $t^* < +\infty$ with $V_t(t^*) = 0$ and $R_n(t^*) < 0$.*
If there exists an equilibrium state $(\hat{U}^{eq}, \hat{R}^{eq})$ such that $R_n^{eq}(t^*) = \hat{R}_n^{eq}$, then $V_t(t) = 0 \;\; \forall t > t^*$.

This lemma will be a key point for the stability analysis of the equilibria. The idea of the proof consists in performing estimates on the discrete dynamics which is the solution to problem ($\mathcal{P}_d$), by considering exactly successive time steps given by NSCD. Then we use the convergence of the algorithm as the time step tends to zero if the external force is analytical (it is constant in the present case). This is relatively long and not presented in this relatively short paper but this does not involve mathematical difficulties. Other technical results of the same kind are necessary, and could be obtained in the same way.

STABILITY OF THE EQUILIBRIA

We are now able to study the stability of all the equilibrium states presented in Figure (2). This amounts to studying the time evolution of the distance between a given equilibrium and the solution of a Cauchy problem in which the initial conditions belong to a neigbourhood of the equilibrium. If there exists a perturbation such that the trajectory diverges from the equilibrium in finite time or asympotically in time, then the equilibrium is unstable. On the other hand, if no perturbation leading to a divergence exists, then the equilibrium is Lyapounov or asymptotically stable.

We give below the list of the corresponding results, together with a few ideas of their proof in significative cases.

$*$ *Any equilibrium state without contact is stable.*

Without contact the particle is not subjected to any unilateral constraints. This stability result is therefore nothing but the classical result of the solutions of ordinary differential equations.

$*$ *The equilibrium in grazing contact characterized by $A = 0$ and $\mu < K_t/W$ is asymptotically stable.*

Although it is relatively long, the proof of this result can be completely obtained analytically. It consists of four steps of which we just give the matter below:

Preliminary: It is easily shown that proving the asymptotic stability reduces to establishing the following implication

$$\exists\, \eta > 0,\ V_{0t} \in]-\eta, 0] \Longrightarrow \begin{cases} \lim_{t\to+\infty} |U_n(t)| = 0, \\ \lim_{t\to+\infty} |U_t(t) - U_t^{eq}| = 0. \end{cases}$$

i) We establish that after a perturbation of the equilibrium in grazing contact by a negative tangential velocity V_{0t}, there exists a time t_1^{vertex} with $\dot{U}_t(t_1^{vertex}) > 0$, where t^{vertex} denotes a time where the particle reaches the vertex of the cone. It is easily shown that $U_t(t^{vertex}) = \dfrac{F_n}{W}$.

ii) The evolution of the particle includes a series of slip and motion without contact phases.

iii) Let t_i^{vertex}, t_{i+1}^{vertex} be two consecutive times such that $U_t(t_j^{vertex}) = \dfrac{F_n}{W}$, $j = i, i+1$. We establish that $\dot{U}_t(t_{i+1}^{vertex}) < \dot{U}_t(t_i^{vertex})$.

iv) Let t_{i+1}^{vertex} be a time such that $U_t(t_{i+1}^{vertex}) = \dfrac{F_n}{W}$, where i is the index of a cycle (slip, motion without contact). Then

$$\lim_{i\to+\infty} \dot{U}_t(t_{i+1}^{vertex}) = 0,$$

which is an easy consequence of

$$\dot{U}_t(t_{i+1}^{vertex}) < \gamma_i \dot{U}_t(t_i^{vertex}) \ \text{ with } 0 < \gamma_i < 1 \ \ \forall\, i.$$

$*$ *The equilibrium state in grazing contact ($U^{gr} = (0, F_n/W), R^{gr} = 0$) characterized by $A = 0$ and $\mu = K_t/W$ is Lyapunov stable.*

$*$ *All the equilibrium states in impending positive slip characterized by $A = 0$ and $\mu = K_t/W$ with a strictly negative reaction are unstable.*

$*$ *Let $A \geq 0$ and μ be such that $\mu > K_t/W$ if $A = 0$ or $\mu \in \mathrm{R}$ if $A > 0$.*
Then all the equilibrium states are Lyapunov stable.

We drop the sketch of the proof, and simply show on Figure (3) two examples of trajectories, observed in the $\{R_t, R_n\}$ plane, following a perturbation of a strictly sticked equilibrium by a small tangential velocity. The result follows from the analytical study of these trajectories with respect to the amplitude of the perturbation.

$*$ *The equilibrium state in impending positive slip characterized by $A < 0$ and $\mu > K_t/W$ is unstable.*

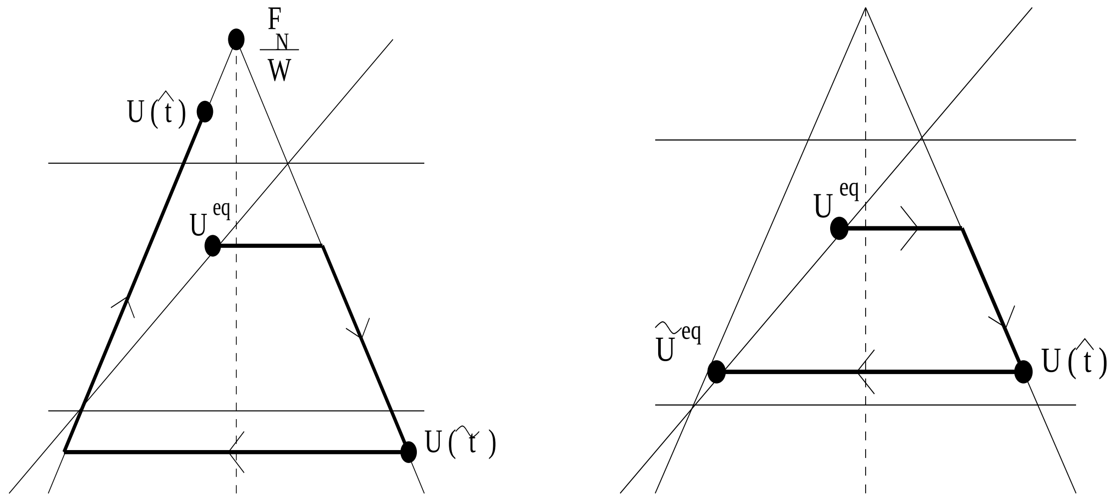

Figure 3 – Evolution after a perturbation by a tangential velocity of an equilibrium state in the case $A > 0$ **and** $\mu < K_T/W$

The latter result was announced in Klarbring, 1990, and proved in Martins et *al.*,1994. Martins et *al* obtained this result thanks to the convergence of a viscous problem.
We outline below the steps of a new proof based only on the integration of the dynamics of problem $(\mathcal{P})$.

We recall that for $A < 0$ and $\mu > K_t/W$, there exist two equilibrium solutions: one in impending positive slip, and the other without contact. The proof consists in establishing that the trajectory resulting from any perturbation of the impending slip solution leaves the neighbourhood of this solution and oscillates around the solution without contact. Basically, the tools that are required to prove this instability result are the same as those used above. In the first step we show that after an initial perturbation $V_{0t} > 0$ of the impending slip solution, there exists t_1^{vertex} such that $U_t(t_1^{vertex}) = \frac{F_n}{W}$ with $\dot{U}_t(t_1^{vertex}) > 0$. In the second step, we establish that the evolution of the particle includes a series of phases without contact and slip phases. Moreover, if t_j^{vertex}, $j = i, i+1$, are two consecutive times such that the reaction reaches the vertex of the cone, we get $\dot{U}_t(t_{i+1}^{vertex}) < \dot{U}_t(t_i^{vertex})$. In the third step, we establish that $\lim_{i \to +\infty} \dot{U}_t(t_{i+1}^{vertex}) = 0$.
After an infinitesimal perturbation $V_{0t} > 0$, the successive steps show that the particle passes through slip phases followed

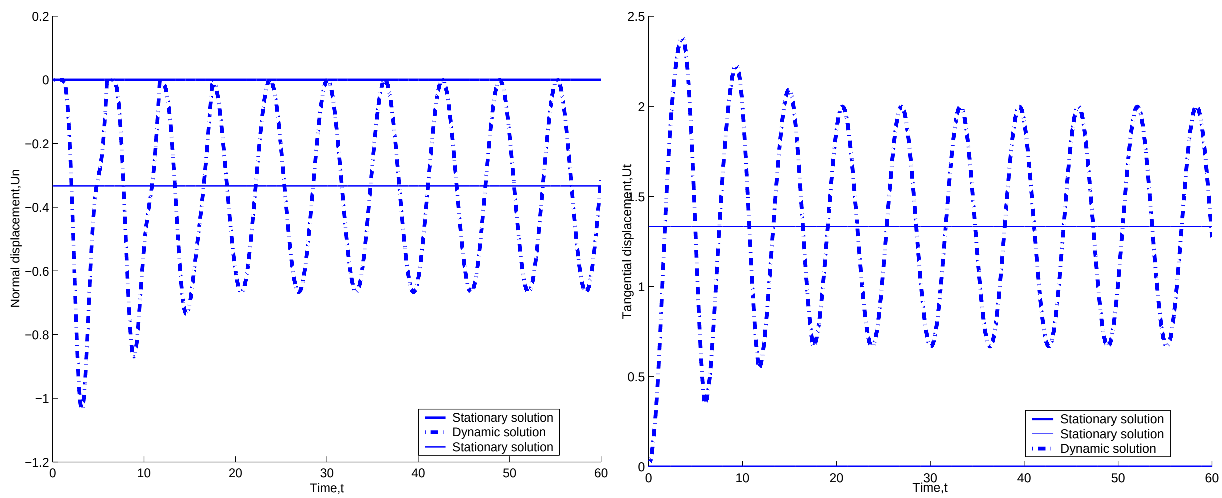

Figure 4 – Evolution of the normal and tangential velocity after a perturbation $V_{0t} > 0$ **in the case** $A < 0$ **and** $\mu > K_T/W$

by phases without contact, followed by further slip phases and so on. This motion is represented on Figure (4). As the result of the decrease of the initial tangential velocity at the beginning of each step and of the continuous dependance on the initial data in each phase without contact, the dynamics converges uniformly in any compact of $[0, \hat{T}]$ towards the solution of the oscillator without contact corresponding to a perturbation of the equilibrium without contact. This proves the instability of the equilibrium in impending slip.

ABOUT STABILITY NOTIONS FOR NONSMOOTH DYNAMICS

The idea of new notions of stability follows from the fact that relations (1) describing the unilateral contact and friction laws cannot be represented as functions but only as the graphs of multivocal applications, in the $\{U_n, R_n\}$ plane for the contact, and in the $\{\dot{U}_t, R_t\}$ plane for the friction (of course these laws would become simple functions if they were regularized, but the dynamics would be completely changed). This means that a given normal displacement (respectively a given tangential velocity) is associated by these laws with infinitely many normal (respectively tangential) reactions. The consequence of this observation is that a mechanical system may not come into motion even if it is submitted to changing (e.g. oscillating) external forces, which is the very strong difference between this dynamics and the one described by ordinary differential equations.

The two results given below prove a conjecture recently suggested to the Léger and Pratt by Michel Jean.

Let us choose the external force F under the form $F(t) = F_0 + \epsilon\xi(t)$, where $\xi(t)$ is an analytical function with values in R^2 and ϵ a positive parameter. $(\mathcal{P}_\mathbf{0})$ (resp. $(\mathcal{P}_\epsilon)$) denotes problem $(\mathcal{P})$ where $\epsilon = 0$ (resp. $\epsilon > 0$). In fact $(\mathcal{P}_\mathbf{0})$ represents the problem previously studied, i.e. with a constant external force and $(\mathcal{P}_\epsilon)$ a perturbation of $(\mathcal{P}_\mathbf{0})$.

A solution (U_ϵ, R_ϵ) of problem $(\mathcal{P}_\epsilon)$ where U_ϵ is constant will be referred to as a *space-equilibrium* solution.

The two following results, referred to as "stability under nonconstant forces", are given here without proof:

$*$ *The following statements i) and ii) are shown to be equivalent:*
i) (U_0, R_0) *is an equilibrium solution of problem* $(\mathcal{P}_\mathbf{0})$ *such that the reaction* (R_{0t}, R_{0n}) *is strictly inside the Coulomb cone.*
ii) $\exists\ \epsilon_0 > 0$ *such that* $\forall \epsilon < \epsilon_0$, *the solution* (U_ϵ, R_ϵ) *of problem* $(\mathcal{P}_\epsilon)$, *obtained with the equilibrium solution of* $(\mathcal{P}_\mathbf{0})$ *as initial data, is a space-equilibrium solution with* $U_\epsilon = U_0$.

$*$ *Assume* ϵ *is such that there exists a space-equilibrium solution to problem* $(\mathcal{P}_\epsilon)$. *Then any solution of* $(\mathcal{P}_\epsilon)$, *with a nongrazing equilibrium solution of problem* $(\mathcal{P}_\mathbf{0})$ *as initial data, leads in finite time to a space-equilibrium solution of* $(\mathcal{P}_\epsilon)$.

CONLUDING REMARKS

Here we studied the stability of the equilibrium states of a very simple mechanical system involving unilateral contact and Coulomb friction. Since the dynamical problem is well-posed for analytical external forces, the stability analysis used only basic definitions of stability, and not any stability theorem of classical mechanics which would no longer applied in presence of inequalities and dissipation. Then the study is based only upon a direct integration of the dynamics which follows initial data located in neighbourhoods of each equilibrium. The calculations are sometimes relatively long, but only quite simple tools are generally required. The following three qualitative aspects of the stability properties were already stressed in Basseville, Léger and Pratt, 2003, after performing numerical experiments, and are now completely proved:
$*$ the equilibrium states in impending slip involving a strictly negative reaction can be either stable or unstable;
$*$ the only asymptotically stable equilibrium state is the equilibrium state in grazing contact when it is the single equilibrium state; if the equilibrium in grazing contact coexists with other equilibrium states, then it is a Lyapounov stable state;
$*$ all the equilibria in strictly stuck contact are Lyapounov stable.

A part of the analysis uses estimates on the iterates of the discrete dynamics associated with the sofware NSCD. This means that the analysis has been done in the case of a restitution coefficient e equal to zero, that is within the so-called case of completely inelastic impacts. As we are dealing with stability properties, it is clear that any other choice of $e \in [0,1]$ might have changed the results. But an analysis with $e \neq 0$ would not be very different from the present one. The completely analytical part of the work could be performed exactly in the same way. Using only analytical tools would certainly become tedious, but remains completely possible anyway. This means that the present analysis of a very simple textbook case, moreover in a particular case of the restitution coefficient, is in a sense nothing more than an example of how to deal with stability for this kind of nonsmooth systems.

The new notions of stability, and the corresponding results of stability under nonconstant forces, seem very promising to tackle the mechanical behavior of systems involving unilateral contact and nonregularized Coulomb friction. Some works are presently in progress to extend these results to larger size discrete systems.

Extending the present results to discrete systems of any number of degrees of freedom is not an easy task. The analysis of the equation of motion shows that there is an important difference according to whether there is a coupling between the normal and tangential degrees of freedom by the mass matrix or not. The extension will be possible at short term if the mass matrix is diagonal. The case of a nondiagonal mass matrix is a typical feature in the dynamics of rigid bodies, and in this case, uniqueness cannot be recovered only by increasing the smoothness of the data. As a matter of fact it is known, since the famous Painlevé paradox about which lots of papers were published in the french Comptes Rendus de l'Académie des Sciences from 1895 to 1905, that the dynamical equations of a rigid body with Coulomb friction may exhibit several solutions, even under constant external forces. One can be easily convinced of this multiplicity by studying the motion of a rigid bar having one of its ends in contact with a rigid vertical wall and falling under its own weight. This multiplicity implies that the formulation of the dynamics should be reconsidered. This is presently in progress by Ballard, Léger and Pratt.

In order to understand the basic phenomenum involved by the dynamics with unilateral contact and Coulomb friction, it may be useful to study separately the dynamics with unilateral contact without friction (the frictionless case) and the dynamics with Coulomb friction but without allowing the motion to separate the particles from the obstacle (bilateral frictional contact). This is really enlightening as for instance the necessity of analyticity to recover uniqueness of the Cauchy problem was completely established for any discrete system in the frictionless case in Ballard, 2000. In the same way, recent works by Léger and Pratt, 2006, show the complexity of the trajectory in the bilateral frictional case and, in particular, it seems that the results of stability under nonconstant forces could be easily extended to systems of larger number of degrees of freedom in the latter case.

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