On the Design of Optimum System of Viscoelastic Vibration Absorbers for a Given Viscoelastic Material

José João de Espíndola [1], Carlos Alberto Bavastri [2] and Eduardo Márcio de Oliveira Lopes [3]

[1] Federal University of Santa Catarina - UFSC, Florianópolis, SC, Brasil, espindol@mbox1.ufsc.br

[2] Federal Technological University of Paraná - UTFPR, Curitiba, PR, Brazil, bavastri@utfpr.edu.br

[3] Federal University of Paraná - UFPR, Curitiba, PR, Brazil, lopes@pisa.ufsc.br

Abstract: Vibration absorbers, also called vibration neutralizers, are mechanical devices to be attached to another mechanical system, or structure, called the primary system, with the purpose of reducing vibration and sound radiation. The simplest form of a vibration absorber is that of a single degree of freedom system where the "spring" is made of a viscoelastic material, perhaps with some metal inserts. This paper sets out to describe how to design a best possible system of viscoelastic vibration absorbers for an available viscoelastic material, known by its four fractional parameters model. A real example is presented and discussed.

Keywords: Vibration Absorber, Vibration Neutralizer, Viscoelastic Material, Vibration Abatement

INTRODUCTION

Dynamic Vibration Neutralizers, more often incorrectly (Crede, 1965, page 121) called Dynamic Vibration Absorbers (DVA) are mechanical devices to be attached to another mechanical system, or structure, called the primary system, with the purpose of reducing, or controlling vibrations and sound radiation from surfaces and structural panels.

Although conceptually incorrect, tradition has adopted the name Dynamic Vibration Absorber as standard. The phenomenon runs in parallel with the name *random variable*, also adopted by tradition, but which is not a variable at all: it is rather a function. Because only of this reason, in this paper the name absorber is used.

Since absorbers were first used to reduce rolling motions of ships (Den Hartog, 1956), many publications on the subject have steadily come to light, demonstrating their efficiency in mitigating vibrations and sound radiation in many structures and machines.

With modern technology of viscoelastic materials, which makes it possible to tailor a particular product to meet design specifications, vibration absorbers are easy to make and apply to almost any complex structure.

In recent times, a great deal of effort has been done to extend and generalize the theory of vibration absorbers, applied to more complex structures than the single degree of freedom undamped one, tackled by Ormondroyd & Den Hartog (1928).

Single degree of freedom vibration absorbers applied to particular positions of uniform beams, with particular boundary conditions, has been studied (Jacquot, 1978; Candir & Ozguven, 1986). Also mass distributed absorbers have been analyzed (Manikahally & Crocker, 1991; Esmailzadeh & Jalili, 1998). Simply supported uniform thin plates have also been considered as a primary system (Broch, 1946; Snowdon, 1975; Korenev & Reznikov, 1993).

In the work of Espíndola and Silva (1992), a general theory for the optimum design of absorber systems, when applied to a most generic structure of any shape, with any amount and distribution of damping, was derived. That theory has been applied to viscoelastic absorbers of various types (Espíndola & Silva, 1992; Freitas & Espíndola, 1993). The theory is based on the concept of equivalent generalized quantities for the absorbers, introduced by the first author.

With this concept, it is possible to write down the equations for the movement of the composite system (primary plus absorbers) in terms of the generalized coordinates (degrees of freedom), previously chosen to describe the configuration space of the primary system alone, in spite of the fact that the composite system has additional degrees of freedom introduced by the attached neutralizers. This fact was crucial in the development of the theory. It permits a coordinate transformation using the modal matrix of the primary system, which is invariant during the optimization process. With this transformation it is possible to obtain the modal space for the composite system without having to solve a complex eigenvalue problem for the whole composite system at each step of the iterative process, which could made it computationally out of question.

In the modal space of the composite structure, it is possible to retain only few modal equations, encompassing the band of frequencies of interest. If coupling is not considered between these equations, then the absorber system can be designed to be optimum for a particular mode, in parallel with Den Hartog's simple optimization method. If a set of coupled modal equations is retained, covering a particular frequency band, then a nonlinear optimization (or better, a

hybrid genetic algorithm/non-linear) technique can be used to design the absorber system to be optimum (in a certain sense) over that frequency band (Espíndola & Bavastri, 1995; Bavastri et alii, 1998).

In recent years, the concept of fractional derivative has been applied to the construction of parametric models for viscoelastic materials (Bagley & Torvik, 1979; Bagley & Torvik, 1986; Torvik & Bagley, 1987; Pritz, 1996; Liebst & Torvik, 1996; Rossikhin & Shitikova, 1998; Espíndola et alii, 2004; Espíndola et alii, 2005b).

This paper adds an important step to the above review: the design process is carried out for a particular set of fractional parameters, modelling the available viscoelastic material. In the end the anti natural frequencies of the absorbers are given together with their masses. With these parameters at hand, it is a matter of conceiving a spatial physical construction for the neutralizers.

General Ideas and Definitions

In this work the expressions *primary system*, or *primary structure* stand for the system, or structure, prior to the attachment of the set of absorbers. The primary structure, or primary system, considered in this paper may be of any shape, no matter how irregular or complex it is. Also, it may be inherently damped, the damping being considered viscous.
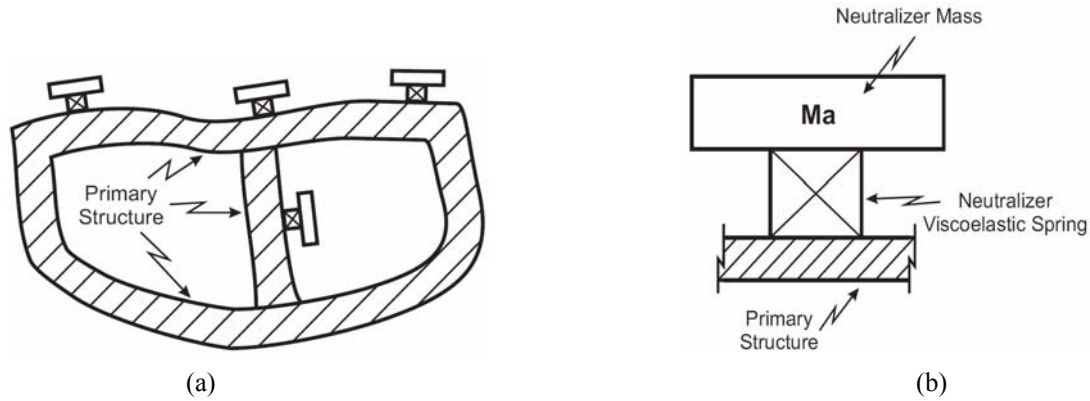

(a)   (b)

Figure 1: (a) Primary structure with absorbers attached to it. (b) A particular absorber.

The absorbers to be attached to the primary structure are single degree of freedom systems, the mass of each one of then being m_{aj}, j=1,p where p is the number of absorbers. The "springs" of the single degree of freedom absorbers are made with a viscoelastic material, perhaps with some metal inserts. They are denoted by $k_{aj}(\Omega)$, j=1,p , and are referred to a particular temperature. Each absorber is associated with a particular generalized coordinate of the configuration space of the primary system, where it is attached to. In this way, the jth absorber is attached at the point of the primary structure whose movement is described by the q_{k_j} generalized coordinate. The index j may be omitted when unnecessary. Figure 1 shows a structure of general shape with some absorbers attached to it.

The idea behind the attachment of a set of neutralizers on a primary structure is to reduce its responses to the action of input forces, or input displacements. How to design such a set of absorbers to achieve the best possible vibration abatement for a particular material, given in advance, is the subject of this paper.

Equivalent Generalized Quantities for an Absorber

For completeness, a brief review of the concept of generalized quantities for a simple vibration neutralizer, or absorber, is presented here (see Espíndola and Silva 1992).

The simple absorber (the one degree of freedom absorber) has a single lump of mass (m_a) connected to the primary structure through a resilient device (a "spring", see Fig. 2), assumed as having a viscoelastic nature, with *complex stiffness* $k_a(\Omega)$ equal to (Espíndola, 1995):

$$k_a(\Omega)=\vartheta G_c(\Omega)=\vartheta G(\Omega)\left[1+i\eta(\Omega)\right] \tag{1}$$

The base plate in figure 2 is assumed massless.

In the above expression, $G_c(\Omega)$ is the complex shear modulus of the viscoelastic material, $G(\Omega)$ is the dynamic shear modulus, $\eta(\Omega)$ is the loss factor of such material; Ω is the circular frequency and ϑ is a geometric factor, depending on the shape and metal inserts of the viscoelastic spring. According to the fractional derivative model with four parameters, the complex shear modulus is given in Eq. (2). There, G_0 and G_∞ are the low and high frequency asymptotes, b is the so called relaxation time of the viscoelastic material and α is the fractional (0<α<1) order of the derivative of the model.

$$G_c(\Omega) = \frac{G_0 + (b\Omega)^\alpha G_\infty}{1 + (b\Omega)^\alpha} \tag{2}$$

In figure 2, $Q(\Omega)$ and $F(\Omega)$ stand for the Fourier Transforms of the basis displacement $q(t)$ and the applied force $f(t)$, respectively. This applied force is a result of the interaction between the absorber and the point of the primary structure where it is attached.
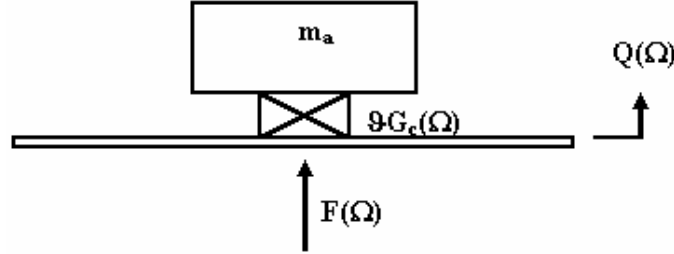

Figure 2: Scheme of a simple (single degree of freedom) absorber

It is a simple matter to verify that the interaction force $F(\Omega)$ at the attachment (massless) plate "feels" the neutralizer as a dynamic stiffness given by:

$$K_a(\Omega) = \frac{F(\Omega)}{Q(\Omega)} = \frac{m_a \Omega^2 \, \vartheta G(\Omega) \, (1 + i\eta(\Omega))}{m_a \Omega^2 - \vartheta G(\Omega)(1 + i\eta(\Omega))}. \tag{3}$$

The anti-resonant frequency of the simple absorber is defined as the one such that, in the absence of damping, makes the denominator of Eq. (3) equal to zero:

$$\Omega_a^2 = \vartheta G(\Omega_a)/m_a \tag{4}$$

In Eq. (4), Ω_a stands for the anti-resonant frequency of the absorber. In that equation, $\vartheta G(\Omega_a)$ is the stiffness of the viscoelastic spring at the anti-resonant frequency Ω_a. Note also that Eq. (4) is a transcendental equation for the anti-resonant frequency of the absorber. Since it is possible to write $G_c(\Omega) = G(\Omega_a) \, r_a(\Omega) \, (1 + i \, \eta(\Omega))$, Eq. (3) can be rewritten as:

$$K_a(\Omega) = -\Omega^2 m_a \frac{\left(\Omega_a^2 r_a(\Omega) - \Omega^2\right)\Omega_a^2 r_a(\Omega) + \left(\Omega_a^2 r_a(\Omega) \eta(\Omega)\right)^2}{D(\Omega)} + i\,\Omega\, m_a \frac{\Omega^3 \Omega_a^2 r_a(\Omega) \eta(\Omega)}{D(\Omega)} \tag{5}$$

where $r_a(\Omega) = G(\Omega)/G(\Omega_a)$ and $D(\Omega) = \left(\Omega_a^2 r_a(\Omega) - \Omega^2\right)^2 + \left(\eta(\Omega)\Omega_a^2 r_a(\Omega)\right)^2$.

Now imagine a single degree of freedom system in which a mass m is connected to a fixed reference ("earth") through a viscous dashpot of constant c. If a force $f(t)$ is applied to the mass, this mass will respond with movement $x(t)$.

The ratio between the input force and output movement, in the frequency domain, will be $k(\Omega) = F(\Omega)/X(\Omega) = -\Omega^2 m + i\,\Omega c$. If this equation is now compared with Eq. 5, one can see that the primary structure "sees" the absorber at the point of attachment as a mass $m_e(\Omega)$ connected to a viscous dashpot of constant $c_e(\Omega)$, the other end of this dashpot connected to the "earth". Figure 3 shows this interpretation. These two quantities are called here equivalent generalized mass and equivalent generalized viscous damping constant for the particular absorber. Dividing out both numerator and denominator of Eq. (5) by Ω_a^4, the equivalent quantities for the absorber can be written as:

$$c_e(\Omega) = m_a \Omega_a \frac{r_a(\Omega) \, \eta(\Omega)\varepsilon_a^3}{\left[\varepsilon_a^2 - r_a(\Omega)\right]^2 + \left[r_a(\Omega) \eta(\Omega)\right]^2} \tag{6}$$

and

$$m_e(\Omega) = m_a \frac{r_a(\Omega)\left\{r_a(\Omega)\left[1+\eta^2(\Omega)\right]-\varepsilon_a^2\right\}}{\left[\varepsilon_a^2-r_a(\Omega)\right]^2+\left[r_a(\Omega)\,\eta(\Omega)\right]^2}, \tag{7}$$
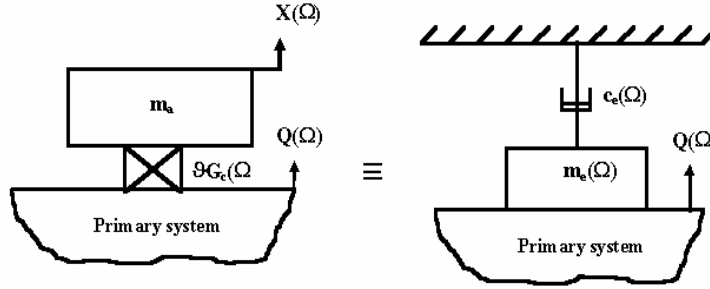
where $\varepsilon_a = \Omega/\Omega_a$.

Figure 3: Equivalent systems

It is a simple task to lift the hypothesis of massless base plate for the absorber and consider its mass in Eq. (7).

Now, it has been proved that both schemes shown in Fig. 3 are dynamically equivalent (Espíndola and Silva, 1992) in the sense that the stiffness "felt" by the primary system is the same in both cases. The primary system "feels" the absorber as a mass $m_e(\Omega)$, dependent on frequency, attached to it along a generalized coordinate $q(t)$ and a viscous dashpot (even if the damping is of viscoelastic nature) of constant $c_e(\Omega)$ (also dependent on frequency) linked to earth (a fixed reference). The dynamics of the resultant system (primary + absorbers) can then be formulated in terms of the original physical generalized coordinates alone (of which $Q(\Omega)$, in Fig. 3, is a representative coordinate), although the new system has now additional degrees of freedom (one for each absorber). This is a fundamental property of the concept of equivalent generalized quantities for the absorbers.

The Response of the Compound Structure

It can now be concluded from the previous discussion (and Fig. 3 helps this interpretation) that a linear structure modelled with many degree of freedom will have its damping and mass matrices modified (see below) by the attachment of the neutralizers, but not their size. If the primary system has been modelled as an n degree of freedom structure, both damping and mass matrices will still be of order $n \times n$ after the attachment of the absorbers, in spite of the fact that p (p absorbers) new degrees of freedom have been added to it. As for the stiffness matrix, it remains unchanged after the attachment of the absorbers.

Notice that Eq. (4) and (5) (or (6) and (7)) contain all the parameters of the fractional viscoelastic model. So, if p such absorbers, with equivalent generalized masses $m_{e1}(\Omega)$, $m_{e2}(\Omega)$, …, $m_{ep}(\Omega)$, and equivalent damping constants $c_{e1}(\Omega)$, $c_{e2}(\Omega)$, …, $c_{ep}(\Omega)$, are attached to the n degree of freedom primary system along the generalized coordinates $q_{k_1}, q_{k_2}, ..., q_{k_p}$, the equations of motion can be written, in the frequency domain, as:

$$\left[-\Omega^2\tilde{M}+i\Omega\tilde{C}+K\right]Q(\Omega)=F(\Omega) \tag{8}$$

where $\tilde{M}$ and $\tilde{C}$ are the modified mass and damping matrices and are given by:

$$\tilde{M}=M+\begin{bmatrix} 0 & & & & \\ & m_{e1}(\Omega) & & & \\ & & \ddots & & \\ & & & m_{ep}(\Omega) & \\ & & & & 0 \end{bmatrix}=M+M_A(\Omega)$$

$$\tilde{C}=C+\begin{bmatrix} 0 & & & & \\ & c_{e1}(\Omega) & & & \\ & & \ddots & & \\ & & & c_{ep}(\Omega) & \\ & & & & 0 \end{bmatrix}=C+C_A(\Omega) \tag{9}$$

and C and M are the ordinary viscous damping and mass matrices of the primary system, respectively.

Matrices $M_A(\Omega)$ and $C_A(\Omega)$ are diagonal and complex. Notice that the entry $\left(k_j, k_j\right)$ is $m_{ej}(\Omega)$ in $M_A(\Omega)$ and $c_{ej}(\Omega)$ in $C_A(\Omega)$, $j = 1, \, p$. Notice also that a particular generalized viscous damping coefficient is given by (see Eq. (6)):

$$c_{ej}(\Omega) = m_{aj}\,\Omega_{aj}\,\frac{r_{aj}(\Omega)\,\eta(\Omega)\,\varepsilon_{aj}^3}{\left[\varepsilon_{aj}^2 - r_{aj}(\Omega)\right]^2 + \left[r_{aj}(\Omega)\,\eta(\Omega)\right]^2}; \quad j = 1, \, p \tag{10}$$

where the index j stands for the jth neutralizer. A correspondent expression can be written for the jth generalized mass. Note also that $\varepsilon_{aj} = \Omega/\Omega_{aj}$ and $r_{aj}(\Omega) = G(\Omega)/G(\Omega_{aj})$, where Ω_{aj} is the anti resonant frequency of the jth absorber.

The anti resonant frequencies of the neutralizers will be given by the equation below:

$$\left(\Omega_{aj}\right)^2 = \frac{\vartheta_j\,G(\Omega_{aj})}{m_{aj}}; \quad j = 1, \, p \tag{11}$$

Now solve the following eigenvalue problem $K\varphi = \Omega^2\,M\varphi$, involving the ordinary mass and stiffness matrices of the primary system, and define the modal matrix $\Phi = \begin{bmatrix} \phi_{r_1} & \phi_{r_2} & \cdots & \phi_{r_m} \end{bmatrix} \in R^{n\times m}$ containing only m eigenvectors ϕ_{r_k}, k = 1,m . It is assumed that the corresponding band of frequencies $\left[\Omega_{r_1}, \Omega_{r_m}\right]$ covers the band of frequencies where the vibrations are to be abated and that m<<n.

Assume that all the eigenvectors are orthonormalized so that $\Phi^T M\Phi = I_m$ and $\Phi^T K\Phi = \Upsilon_m$, where $\Upsilon_m = \text{diag} \, (\Omega_{r_1}^2 \quad \Omega_{r_2}^2 \quad \cdots \quad \Omega_{r_m}^2)$.

Now, in equation (8), apply the following transformation:

$$Q(\Omega) = \Phi\,P(\Omega) \tag{12}$$

If Eq.(12) is taken into Eq.(8) and pre-multiplied by Φ^T, one gets, assuming proportional damping in the primary system:

$$\left\{-\Omega^2\,\tilde{M}_A(\Omega) + i\Omega\left[\Gamma_m + \tilde{C}_A(\Omega)\right] + \Upsilon_m\right\}P(\Omega) = \Phi^T F(\Omega) \tag{13}$$

where

$$\Gamma_m = \text{diag} \, (2\xi_{r_1}\Omega_{r_1} \quad 2\xi_{r_2}\Omega_{r_2} \quad \cdots \quad 2\xi_{r_m}\Omega_{r_m}) \tag{14}$$

$$\Upsilon_m = \text{diag}\left(\Omega_{r_1}^2 \quad \Omega_{r_2}^2 \quad \cdots \quad \Omega_{r_m}^2\right) \tag{15}$$

$$\tilde{M}_A(\Omega) = I_m + \Phi^T M_A(\Omega)\Phi \tag{16}$$

$$\tilde{C}_A(\Omega) = \Phi^T C_A(\Omega)\Phi \tag{17}$$

The Ω_{r_k}, k=1,m are undamped natural frequencies of the primary structure and ξ_{r_k}, k=1,m are the corresponding modal damping ratios. Eq. (13) represents a small system of m<<n equations and can be solved directly for any frequency with use of Eq. (6) and (7). But this may not be the best way to follow, since matrices $\tilde{M}_A(\Omega)$ and $\tilde{C}_A(\Omega)$ are not diagonal. Instead, a more robust approach will be offered. Eq. (13) can be written in the following way:

$$i\Omega\begin{bmatrix} \tilde{C}_A(\Omega) + \Gamma_m & \tilde{M}_A(\Omega) \\ \tilde{M}_A(\Omega) & 0 \end{bmatrix}\begin{Bmatrix} P(\Omega) \\ i\Omega P(\Omega) \end{Bmatrix} + \begin{bmatrix} \Upsilon_m & 0 \\ 0 & -\tilde{M}_A(\Omega) \end{bmatrix}\begin{Bmatrix} P(\Omega) \\ i\Omega P(\Omega) \end{Bmatrix} = \begin{Bmatrix} \Phi^T F(\Omega) \\ 0 \end{Bmatrix} \tag{18}$$

or

$$i\Omega\tilde{A}Y(\Omega)+\tilde{B}Y(\Omega)=\tilde{G}(\Omega) \tag{19}$$

where $\tilde{A} = \begin{bmatrix} \tilde{C}_A(\Omega)+\Gamma_m & \tilde{M}_A(\Omega) \\ \tilde{M}_A(\Omega) & 0 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} \Upsilon_m & 0 \\ 0 & -\tilde{M}_A(\Omega) \end{bmatrix}$ and $Y(\Omega) = \begin{Bmatrix} P(\Omega) \\ i\Omega P(\Omega) \end{Bmatrix}$.

The second set of equations in 18 is, in fact, an identity. Note that $\tilde{A}, \tilde{B} \in \mathbb{C}^{m\times m}$ and $Y(\Omega), \tilde{G}(\Omega) \in \mathbb{C}^{m\times 1}$. Note also that a time domain version of Eq. 19, say, $\tilde{A}\dot{y}(t)+\tilde{B}y(t)=g(t)$, where $y(t) = \mathscr{F}^{-1}(Y(\Omega))$ and $g(t) = \mathscr{F}^{-1}(\tilde{G}(\Omega))$, cannot be written simply because both matrices $\tilde{A}$ and $\tilde{B}$ are functions of frequency. This mixing of time and frequency domains would generate a set of non equations.

It is not difficult to show that matrix $\tilde{B}$ is positive definite. Consider the following eigenvalue problem:

$$\tilde{B}\theta = \lambda\tilde{A}\theta \tag{20}$$

and define the following modal matrix $\Theta = [\theta_1 \quad \theta_2 \quad \dots \quad \theta_{2m}]$ and the diagonal spectral matrix $\Lambda_{2m} = \mathrm{diag}(\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_{2m})$.

Assume that the eigenvectors are orthonormalized such that $\Theta^T\tilde{A}\Theta = I_{2m}$ and $\Theta^T\tilde{B}\Theta = \Lambda_{2m}$ and make the following transformation:

$$Y(\Omega) = \Theta Z(\Omega) \tag{21}$$

This transformation is possible because the columns of Θ are linearly independent, which makes this matrix non-singular. In fact, the inverse of Θ is $\Theta^{-1} = \Theta^T\tilde{A}$.

Substituting for $Y(\Omega)$ into Eq. (19) and pre-multiplying by Θ^T, one get:

$$(i\Omega I_{2m} + \Lambda_{2m})\,Z(\Omega) = \Theta^T G(\Omega) \tag{22}$$

Solving Eq. (22) for $Z(\Omega)$, substituting the result into Eq. (21) and remembering that $Y(\Omega) = [P(\Omega) \quad i\Omega P(\Omega)]^T$, one can get:

$$P(\Omega) = [\Theta_{11} \quad \Theta_{12}](i\Omega I_{2m} + \Lambda_{2m})^{-1}[\Theta_{11} \quad \Theta_{12}]^T \Phi^T F(\Omega) \tag{23}$$

Taking this result into expression 31, the following is obtained:

$$Q(\Omega) = \Psi(i\Omega I_{2m} + \Lambda_{2m})^{-1}\Psi^T F(\Omega) \tag{24}$$

where $\Psi = \Phi[\Theta_{11} \quad \Theta_{12}]$ and $[\Theta_{11} \quad \Theta_{12}]$ is the upper half of the matrix Θ. The matrix

$$A(\Omega) = \Psi(i\Omega I_{2m} + \Lambda_{2m})^{-1}\Psi^T \tag{25}$$

is the so called receptance matrix and is a model of the system in the frequency domain. Note that $A(\Omega) \in \mathbb{C}^{n\times n}$.

Having the receptance matrix for any frequency, the response at that frequency can be computed by:

$$Q(\Omega) = A(\Omega)F(\Omega) \tag{26}$$

The sth column of the receptance matrix $A(\Omega)$ is given by the expression (27):

$$\alpha_s(\Omega) = \sum_{j=1}^{2m}\psi_j\frac{\psi_{sj}}{i\Omega+\lambda_j} \tag{27}$$

So, assuming p absorbers attached to the primary structure, the theory described above tells how to compute the response of the compound system. But the problem at hand is the reverse: having a primary system strongly responding to input excitations, how to design a set of dynamic absorbers so as to mitigate the vibrations to acceptable levels.

First, it is assumed that a convenient viscoelastic material is available, its four fractional parameters are known, and that all the absorbers are to be constructed with that material. Second, since a modal model of the primary structure must also be know for the design process, it is assumed that the number and place of attachment of the absorbers have been decided beforehand.

The obvious places of attachment for the absorbers are the points of maximum displacement in each mode within the band of interest. An absorber placed at a nodal line of a mode will be completely inefficient in reducing vibration at that particular mode.

Specification of the Absorber's Masses

For primary systems with only one degree of freedom, the recommended ratio between the absorber mass (m_a) and primary structure mass (m_s) by Den Hartog (1956) is $\mu = m_a / m_s = 0.1$ to 0.25. The use of the modal mass ratio concept has been proposed by Espíndola and Silva (1992) for a system of multiple degrees of freedom as:

$$\mu_{s_j} = \frac{\left(\sum_{i=1}^{p} m_{ai} \, \phi_{k_i s_j}^2 \right)}{m_{s_j}}; \quad j = 1, d \tag{28}$$

where m_{ai} is the mass of the ith neutralizer, d is the number of modes taken inside the band of frequencies (d is, in general, smaller than m, the number of eigenvectors kept from problem $K\phi = \Omega^2 M\phi$. The symbol m_{s_j} stands for the s_j^{th} modal mass of the primary system, which in case of orthonormalization of eigenvectors is equal to one. The quantity $\phi_{k_i s_j}$ represents the element of Φ lying in the k_i^{th} line and s_j^{th} column. The numbers k_i, i = 1, p are of the coordinates q_{k_i}, where the p neutralizers are fixed to the primary structure. So, given μ_{s_j}, one for each of the modes of interest, a set of equations is established and m_{ai}, i = 1, p are computed by SVD decomposition of the system matrix associated with equations (28). The matrix of the system shown in Eq. 28 is of order $d \times p$. Note that the number of modes to be controlled (d) inside the band of eigenvectors in $\Phi \in \Re^{n \times m}$ may be smaller, equal to or greater than the number of neutralizers (p) attached to the primary system. This means that the system of Eq. (28) may be underdetermined, over determined or determined.

The arguments leading to Eq. 28 are too lengthy to be reproduced here.

Optimization for a Frequency Range

In what follows, it is assumed that a particular material is at hand, given by its four fractional parameters $\{\alpha, b, G_0, G_\infty\}$. In a different approach, the material (i.e., the four parameters) is searched for in the process of designing a optimum system of viscoelastic neutralizers (see Espíndola et alii, 2005a).

Assume that the input force vector $F(\Omega)$ is known or that the input forces can be simulated with reasonable accuracy. It is of interest to modify the anti resonance frequencies Ω_{aj}, j = 1, p in such a way that a norm of $P(\Omega)$ is minimum. In such manner the response given by Eq. (26) is also minimized. Define x as a vector of anti resonant frequencies:

$$x^T = \begin{bmatrix} \Omega_{a1} & \Omega_{a2} & \cdots & \Omega_{ap} \end{bmatrix} \tag{29}$$

In Eq. (23) call

$$V = \begin{bmatrix} \Theta_{11} & \Theta_{12} \end{bmatrix} \left(i\Omega I_{2m} + \Lambda_{2m} \right)^{-1} \begin{bmatrix} \Theta_{11} & \Theta_{12} \end{bmatrix}^T \tag{30}$$

Note that $V \in \mathbb{C}^{m \times m}$, that is, V is a small order square matrix.

Eq. (23) then becomes:

$$P(\Omega) = V\Phi^T F(\Omega) \tag{31}$$

Since the Frobenius norm of a matrix is a consistent one, the following expression is valid:

$\| P(\Omega) \|_2 = \| V\Phi^T F(\Omega) \|_F \leq \| V \Phi^T \|_F \| F(\Omega) \|_F \leq \| V \|_F \| \Phi^T \|_F \| F(\Omega) \|_2$. Since $\left\| \Phi^T \right\|_F$ is a positive

constant number and $\|F(\Omega)\|_2$ is fixed for every frequency, minimizing $\|P(\Omega)\|_2$ means minimizing $\|V\|_F$ in a certain way, for each and every frequency. So, take the following objective function:

$$f(x) = \max_{\Omega_{min} \leq \Omega \leq \Omega_{max}} \|V(\Omega, x)\|_F \tag{32}$$

and minimize it. Note that V (Ω, x) is precisely the matrix V in Eq. (30), with Ω and x in evidence. Remember also that x in Eq. (32) is the vector defined in Eq. (29).

As always, the better the information at hand the better the results will be. One should expect then that the results obtained using this definition for objective function (where no information about the input vector is used) are more conservative than those obtained using the previous one in Espíndola and Cruz (2005a). This is a price to be paid for our ignorance. The advantage of this present objective function is that it ignores the input excitation, which may be crucial in certain applications.

After a minimization procedure of f(x), the p anti-resonant frequencies $\Omega_{a1}, \Omega_{a2} \cdots \Omega_{ap}$ for the p respective neutralizers are known. Since m_{aj}, j = 1, p were given as input parameters, the ϑ_j, j = 1, p parameters of the viscoelastic element can be computed at each frequency Ω_{aj}, j = 1, p, from Eq. (11). This is only a geometric factor. It is now left to the designer to give shape and size to the absorbers so as to meet these anti-resonant frequencies and geometric factors. A simple example for heavy machinery illustrates this point.

For a uniform viscoelastic pad working in compression, it can be shown that

$$\vartheta = \frac{3(1+\beta S^2)A}{e} \tag{33}$$

where A is the one side load carrying area, e is the thickness, β is a factor equal to two for circular and square pads and approximately two for moderately rectangular pads. S is the so called shape factor and is defined as the ratio of the one side load carrying area to the free surface area.

In the design practice it may be convenient to make equal the resilient parts of all the absorbers. This calls for choosing the most significant (in a certain sense) of the form factors ϑ_j, j = 1, p (say ϑ_ℓ) and then recomputing the absorber's masses:

$$m_{aj} = \frac{\vartheta_\ell G(\Omega_{aj})}{(\Omega_{aj})^2} \quad ; j = 1,\, p \tag{34}$$

A possible criterion is to specify ϑ_ℓ as the *rms* value of all ϑ_j, j = 1, p.

Making the resilient parts equal for all the absorbers may signify an important saving in money (for instance in moulding and curing dies). Clearly this is an approximation often dictated by economy. The final result must then be checked. Simple as it is, this last approach may give excellent results as shown in Espíndola, & Bavastri, 1995, 1997.

Absorbers working in shear are in general very small in size (a few grams to few kilograms) and are normally designed to be applied to vibrating light surfaces (such as machine casings) to reduce vibration response and noise radiated from them. Those working in compression are normally bigger and heavier (a few hundredths kilograms) and are used to reduce vibration responses of heavy machinery.

In this paper an example of real application to heavy machinery is presented.

Example: Field application to a turbine-generator hydro group

The above theory was applied to a vertical hydroelectric group already assembled in the field and working for many years in the past. The entire shaft, generator rotor and turbine runner system is supported by a thrust bearing at the top of a six legged fabricated steel *crosspiece*. The thrust bearing rests on the crosspiece through ten Belleville springs. The steel crosspiece, on its turn, rests on the bulky and very stiff concrete structure. The total shaft length, from the top of thrust bearing to the bottom of the runner hub is 14 meters. The whole group runs at 300 [rpm].

I t was not until the replacement of the wear ring for a new one that strong axial vibration was felt mainly in the region near the thrust bearing. Measurements have shown a clear axial vibration pick at about 48 [Hz] while the group was running. No significant lateral vibration has been measured at any frequency.

Frequency responses, with the group at rest but with the turbine runner submerged, were measured with the use of a large piezoelectric hammer and many accelerometers placed along the shaft to pick up longitudinal acceleration signals. In most of measurements the hammer hits the top end of the shaft, just above the thrust bearing.

All frequency responses have shown a resonant frequency at around 48 [Hz], together with other picks above and below that frequency.

A long and thorough study of possible excitations inducing such a strong response led to the conclusion that only one cause was plausible: flow instability in the rotor-wear ring passage. This flow instability induces forces at the turbine runner of a narrow frequency band nature (Tomita, 1991), this band encompassing the 48 [Hz], thus producing a self excited vibration with a strong component at this frequency.

To solve this problem it was proposed to design an install six vibration absorbers, one in each of the six recesses in between the legs of the crosspiece. These vibration neutralizers have been installed as close as possible to the shaft.

To properly design a system of vibration absorbers, a modal model of the primary system must be available. In the present case this model was constructed via finite elements technique. In the finite element model of the whole system the generator rotor was considered a rigid body, but not the generator spider, which was considered flexible. Also, the turbine runner was considered to be a rigid body. Altogether 13314 elements were used with 7834 nodes.
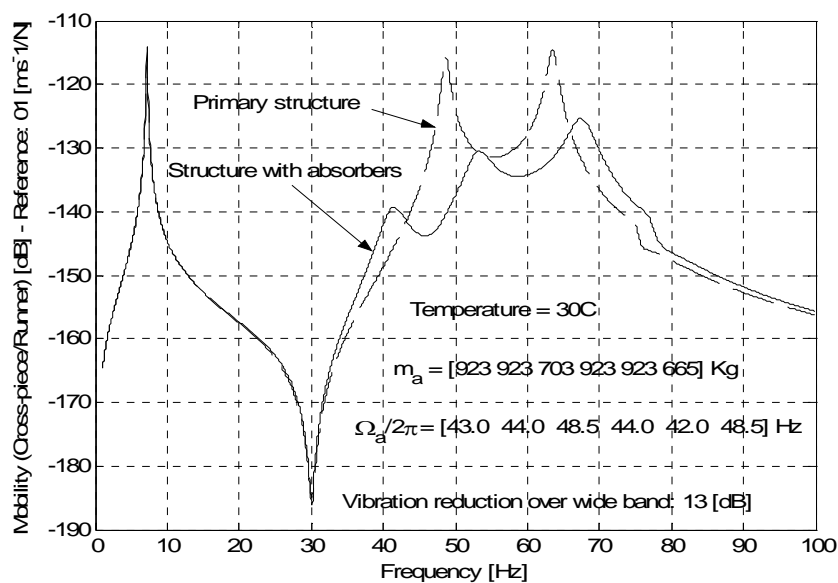

Figure 4: Frequency response of the primary system and of the compound system

The physical design of the six absorbers is precisely the same, except for the masses. It consist of three steel pillars supporting a cast steel shell filled with lead. The reason for using lead to fill the cast steel shells is to make it possible to house the absorbers in the available recesses of limited size. Also, this design solution allows for ease variation of the masses of the absorbers, simply by adding or taking off lead from inside the shells.

Each pillar is made of two metal parts connected together in tandem by a viscoelastic insert. These viscoelastic inserts consists of two cylindrical discs separated by a one millimetre thick steel disc. The viscoelastic inserts are vulcanized together with the two metal pillar parts. These viscoelastic inserts constitute the "springs" over which the absorber mass rests.

Figure 4 shows how a frequency response plot is modified by the use of absorbers. This is a particular response at the top of the crosspiece due to an excitation at the bottom of the turbine in the vertical direction.

To properly interpret Fig. 4, it must be realized that no excitation existed outside a narrow band around 48 Hz.

It is remarkable the effect of the absorbers over the frequency band of interest (40 to 70Hz). Since the excitation is in a very narrow indeed, around the 48 Hz, with no effect on other frequencies (for instance, in the low frequency range around 8,0 Hz) it seems that the absorbers, as designed, is a very good solution. In fact, field measurements in a wide band of frequencies has shown a 13 dB reduction in *rms* values of vibration velocity at 30 C. At 34 C (maximum temperature expected at the isolators) the abatement I reduced to 12 dB.

CONCLUSIONS

A general theory for the design of systems of viscoelastic vibration absorbers has been produced. It assumes that a particular viscoelastic material is available beforehand.

The theory takes in as raw material the modal model of the primary system, that is, the original system (structure) without neutralizers attached. This model may be developed through a finite elements technique or an experimental identification scheme.

So the theory is pretty general and by-passes any difficulties as related to the geometrical form of the primary system, assumed to be linear.

This theory has been applied to a hydroelectric group to reduce vertical vibrations due to fluid-structure interactions at the bottom of the turbine.

Six absorbers of the equal design, but allowing for different masses, have been constructed and installed.

The efficacy of the system of absorbers is shown in Fig. 3 of the text and field measurements have shown that 12 to 13 dB reduction in *rms* vibration speed was achieved, depending on the external temperature.

REFERENCES

Bagley, R.L. and Torvik, P.J. (1979). "A Generalized Derivative Model for an Elastomer Damper". The Shock and Vibration Bulletin, 49(2): 135-143.

Bagley, R.L.and Torvik,P.J. (1986). "On the Fractional Calculus Model of Viscoelastic Behaviour". Journal of Reology. Vol. 30: 133-155.

Bavastri, C.A., Espíndola, J.J. and Teixeira, P.H. (1998). "A hybrid algorithm to compute the optimal parameters of a system of viscoelastic vibration neutralisers in a frequency band". Proceedings of MOVIC'98, Zurich, Switzerland : 577-582.

Broch, J.E. (1946). "A Note on the Damped Vibration Absorber". Journal of Applied Mechanics, Trans. ASME. Vol. 68: A284.

Crede, E. C. (1965). "Shock and Vibration Concepts in Engineering Design". Prentice – Hall, Inc., Englewood Cliffs, N.J.: 121.

Candir, B. and Ozguven H.N., (1986). "Dynamic Vibration Absorbers for Reducing Resonance Amplitudes of Hysterically Damped Beam". Proc. of the Fourth International Modal Analysis Conference, Los Angeles, USA: 1628-1635.

Den Hartog, J.P. (1956). "Mechanical Vibrations". New York: McGraw-Hill.

Esmailzadeh, E. and Jalili, N. (1998). "Optimum Design of Vibration Absorbers for Structurally Damped Timoshenko Beams". ASME Journal of Vibration and Acoustics, Vol. 120: 833-841.

Espíndola, J.J. (1995). "Notes on viscoelastic damping (in Portuguese)". UFSC, Santa Catarina, Brasil.

Espíndola, J.J. and Bavastri, C.A. (2003). "Modal reduction of vibrations by dynamic neutralizers in a frequency band. In: VI Symposium on Dynamic Problems of Mechanics". V. 1: 214-217.

Espíndola, J.J. and Bavastri, C.A. (1997). "Viscoelastic neutralisers in vibration abatement: a non-linear optimisation approach". Journal of the Brazilian Society of Mechanical Sciences. XIX(2): 154-163.

Espíndola, J.J. and Silva, H.P. (1992). "Modal Reduction of Vibrations by Dynamic Neutralizers, Proc". Of the Tenth International Modal Analysis Conference, San Diego,USA: 1367-1373.

Espíndola, J.J., Silva Neto, J. M. and Lopes, E. M. (2004). "A New Approach to Viscoelastic Material Properties Identification Based on the Fractional Derivative Model". Proceedings of First IFAC Workshop on Fractional Differentiation and its Application (FDA' 04), Bordeaux, France: 19-21.

Espíndola, J.J., Silva Neto, J. M. and Lopes, E. M. (2005b). "A generalized fractional derivative approuch to viscoelastic material properties measurements". Apllied Mathematics and Computation,V. 164 (2) :493-506.

Espíndola, J.J., Cruz, G. A.M. (2005a). "On the design of optimum systems of viscoelastic vibration neutralizers. In: Modeling and control of autonomous decision support based systems". Hofer E. and Reithmeier (eds): 49-64.

Ferry, J.D. (1980). Viscoelastic Properties of Polymers. John Wiley & Sons, INC. New York.

Freitas, F.L. & Espíndola, J.J. (1993). "Noise and Vibration Reduction with Beam-Like Dynamic Neutralizers". 12th Brazilian Congress of Mechanical Engineering.

Jacquot, R.G. (1978). "Optimal Dynamic Vibration Absorbers for Timoshenko Beams". M. Sc. Thesis Mechanical Engineering, Sharif University of Technology Tehram, Iran.

Korenev, B.G. and Reznikov, L.M.(1993). "Dynamic Vibration Absorbers". New York: John Wiley & Sons.

Liebst, B.S. and Torvik, P.J. (1996). Asymptotic Approximations for Systems incorporating Fractional Derivative Damping. Journal of Dynamic Systems, Measurement, and Control. Vol. 118: 572-579.

Manikahally, D.N. & Crocker, M.J. (1991). Vibration Absorbers for Hysterically Damped Mass-Loaded Beams. ASME Journal of Vibration and Acoustics, Vol. 113: 116-122.

Nashif, A.D. et alii (1985). "Vibration Damping". John Wiley & Sons, New York.

Ormondroyd, J. and Den Hartog, J.P. (1928). The Theory of Dynamic Vibration Absorber. Journal of Applied Mechanics. Trans. ASME Vol. 49: A9-A22.

Pritz,T. (1996). "Analysis of Four- Parameter Fractional Derivative Model of Real Solid Materials". Journal of Sound and Vibration, Vol.195 (1) 103-115

Rossikhin, Y.A. and Shitikova, V. (1998). "Application of Fractional Calculus for Analysis of Nonlinear Damped Vibrations of Suspension Bridges". Journal of Engineering Mechanics. 124(9): 1029-1036.

Snowdon, J.C., (1975). "Vibration of Simply Supported Rectangular and Square Plats to which Lumped Masses and Dynamic Vibration Absorbers are Attached". Journal of Acoustical Society of America 57(6): 646-654.

Torvik, P.J. and Bagley, R.L. (1987). "Fractional Derivatives in the Description of Damping Materials and Phenomena, The Role of Damping in Vibration and Noise Control", ASME DE-5: 125-135.

Tomita H. (1991). "Modelling and Analysis. In: Vibration and Oscillation of Hydraulic Machinery". Hideo OHASHI (Ed.). Avery Technical, Brookfield, USA.

RESPONSIBILITY NOTICE