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On Natural Frequencies of Conservative Multi-Degree-of-Freedom Mass-Spring Vibration Systems

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Abstract: Conservative linear chain-structured mass-spring vibration systems with N degrees of freedom are considered. Although this is a classical topic, cf. (Klotter, 1960), recently new problems have been discussed by Mikota (2001) und Braun (2003). These investigations stimulate to summarize existing analytical results and to develop new ones. Therefore, the paper deals with modelling of chain-structured vibration systems, with general aspects of analysing such systems, and with solving new eigenvalue/eigenvector problems. The paper shows the great variety of mathematical tools required for solving the problem depending on mass and stiffness distributions: Trigonometric functions, Laguerre polynomials, and binomial coefficients. A new subclass of vibration chains is defined for which further results are expected.

Keywords: Vibration chains, natural frequencies, eigenmodes, Mikota's conjecture

1 INTRODUCTION

Models of linear vibration chains

Conservative linear multi-degree-of freedom mass-spring vibration systems, as shown in Fig. 1, appear often in technical applications, cf. the classical book of Biezeno and Grammel (1953). Therefore, they have been investigated very much in detail, cf. (Klotter, 1960).

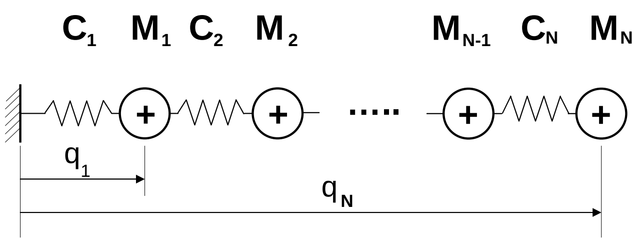

Figure 1 – Chain structured mass-spring vibration system

In an inertial coordinate system a conservative chain-structured mass-spring vibration system with N degrees of freedom is characterized by a diagonal mass matrix and a tridiagonal stiffness matrix depending on the boundary conditions (free or clamped on one or both ends of the chain). Defining the mass matrix $m\mathbf{M}_N$ by the elements $M_i = mm_i$, where $m_i, i = 1, \ldots N$, describes the mass distribution for the i-th mass, then

$$\mathbf{M}_N = \mathbf{diag}(m_i) \tag{1}$$

is obtained. Analoguously, the spring stiffnesses $\mathbf{C}_i = cc_i$ are summarized in a matrix of stiffnesses $c\mathbf{C}_N$:

$$\mathbf{C}_N = \mathbf{diag}(c_i). \tag{2}$$

The implications for the equations of motion,

$$m\mathbf{M}_N\ddot{\mathbf{q}}(t) + c\mathbf{K}_N\mathbf{q}(t) = \mathbf{0}, \tag{3}$$

depend on the boundary conditions, cf. (Klotter, 1960). Always a tridiagonal matrix $c\mathbf{K}_N$ appears where $\mathbf{K}_N$ shows the influence of the springs to the masses.

a) Free-free boundary conditions
If both ends of the vibration system are free, then

$$\mathbf{K}_N = \mathbf{K}_a = \mathbf{D}_a^T \mathbf{C}_{N-1} \mathbf{D}_a \quad \text{with} \quad \mathbf{D}_a = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & -1 & 1 \end{bmatrix} \tag{4}$$

is determined where $\mathbf{D}_a$ is a $(N-1) \times N$ matrix.

b) Free-clamped boundary conditions
In the case of Fig. 1 the result is

$$\mathbf{K}_N = \mathbf{K}_b = \mathbf{D}_b^T \mathbf{C}_N \mathbf{D}_b \quad \text{with} \quad \mathbf{D}_b = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}. \tag{5}$$

c) Clamped-clamped boundary conditions
For the case of clamped endpoints of the vibration chain, $N+1$ springs and N masses appear resulting in

$$\mathbf{K}_N = \mathbf{K}_c = \mathbf{D}_c^T \mathbf{C}_{N+1} \mathbf{D}_c \quad \text{with} \quad \mathbf{D}_c = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & 1 \end{bmatrix} \tag{6}$$

where $\mathbf{D}_c$ is a $(N+1) \times N$ matrix.

d) Closed vibration chain
If the vibration chain is closed, i. e. if the last mass is coupled by a spring to the first mass, then

$$\mathbf{K}_N = \mathbf{K}_d = \mathbf{D}_d^T \mathbf{C}_N \mathbf{D}_d \quad \text{with} \quad \mathbf{D}_d = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ -1 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 1 \end{bmatrix} \tag{7}$$

results.

To determine the natural frequencies of the vibration system (3) the related eigenvalue is considered. Defining $\Omega^2 = \frac{c}{m}\omega^2$, the (complex) natural modes $\mathbf{q}(t) = \tilde{\mathbf{q}} e^{i\Omega t}$ lead to the eigenvalue/eigenvector problem

$$(-\omega^2 \mathbf{M}_N + \mathbf{K}_N)\tilde{\mathbf{q}} = \mathbf{0}. \tag{8}$$

The solution of Eq.(8) will be discussed in the next sections for different assumptions on $\mathbf{M}_N, \mathbf{K}_N$.

2 GENERAL ASPECTS OF THE EIGENVALUE PROBLEM

The eigenvalue/eigenvector problem (8) cannot be solved analytically in general. For general mass and stiffness distributions a numerical solution has to be calculated. But it is obvious that N natural frequencies $\omega_i \neq 0$ exist for the cases b) and c) while for a) and d) one natural frequency vanishes and $N-1$ nonvanishing ω_i exist. Additionally the interlacing property holds: If a mass mm_{N+1} and a spring with stiffness coefficient cc_{N+1} are added to the chain, without changing the other masses and springs, the natural frequencies of the N-dof-system interlace the frequencies of the $(N+1)$-dof-system. In (Klotter, 1960) more results can be found on bounds of the natural frequencies and other properties of the vibration system.

Analytical results are available for homogeneous vibration chains where

$$m_i = 1 \quad , \quad c_i = 1, \tag{9}$$

cf. (Klotter, 1960). This is illustrated for case b) according to Fig. 1. Assuming $\tilde{\mathbf{q}} = [q_i]$ Eq.(8) leads to the scalar difference equations

$$\begin{aligned} (\omega^2 - 2)q_1 + q_2 &= 0\,, & (10.1)\\ q_{i-1} + (\omega^2 - 2)q_i + q_{i+1} &= 0\,,\; i = 2, \dots, N-1\,, & (10.2)\\ q_{N-1} + (\omega^2 - 1)q_N &= 0\,. & (10.3) \end{aligned}$$

These equations are solved by the approach

$$\omega^2 = 2(1 - \cos\varphi) \quad , \quad q_i = \sin(i\varphi). \tag{11}$$

Equations (10.1) and (10.2) are satisfied. The boundary condition (10.3) results in the requirement

$$\cos(N + \frac{1}{2})\varphi = 0. \tag{12}$$

This yields

$$\varphi_k = \frac{2k-1}{2N+1}\pi\,,\; k = 1, \dots, N. \tag{13}$$

According to Eq.(11) the natural frequencies and the eigenmodes are given by

$$\omega_k^2 = 2(1 - \cos\varphi_k)\,,\; \tilde{\mathbf{q}}_{\mathbf{k}} = [q_{ik}]\,,\; q_{ik} = \sin(i\varphi_k). \tag{14}$$

For nonhomogeneous vibration chains only a few special problems are known where Eq.(8) is solved analytically. Two recent examples are shown in the next two sections.

3 MULTIPLE PENDULUM

In the paper (Braun, 2003) some properties of a multiple pendulum has been discussed. As a special case the multiple pendulum is considered for N equal masses ($m_i = 1$), which are uniformly spaced at distances $a = L/N$ where L is the total length of the multiple pendulum. Using the horizontal displacements q_k of the kth mass point, then the linearized Eq.(8) is obtained with

$$\mathbf{M}_N = \mathbf{I}_N, \mathbf{K}_N = \begin{bmatrix} 2N-1 & -(N-1) & & & & \\ -(N-1) & 2N-3 & -(N-2) & & & \\ \ddots & \ddots & \ddots & & & \\ & -(N-j+1) & 2N-2j+1 & -(N-j) & & \\ & \ddots & \ddots & \ddots & & \\ & & -2(N-1) & 3(N-1) & -(N-1) & \\ & & & -N & N & \end{bmatrix} \tag{15}$$

The natural frequencies are given by $\Omega^2 = \frac{a}{g}\omega^2$.

Looking once again for the scalar relations of Eq.(8) it is obtained

$$\begin{aligned} [\omega^2 - (2N-1)]q_1 + (N-1)q_2 &= 0, & (16a)\\ (N-j+1)q_{j-1} + [\omega^2 - 2(N-j) - 1]q_j + (N-1)q_{j+1} &= 0\,,\; j = 2, \dots, N-1, & (16b)\\ q_{N-1} + [\omega^2 - 1]q_N &= 0. & (16c) \end{aligned}$$

Comparing Eq.(16) with the recurrence relation of Laguerre polynomials

$$L_i(x) = \sum_{j=0}^{i} \binom{i}{j} \frac{(-x)^j}{j!} \ , \ i = 1, 2, \dots, \tag{17}$$

cf. (Gantmacher, Krein, 1960, as well as Müller, Gürgöze, 2006), then

$$q_j = L_{N-j}(\omega^2) \ , \ j = 1, \dots, N, \tag{18}$$

and ω^2 is defined by the roots of $L_N(x)$:

$$\omega_k^2 : \quad L_N(\omega_k^2) = 0 \ , \ k = 1, \dots, N. \tag{19}$$

By these eigenfrequencies the eigenmodes are again obtained by $\tilde{\mathbf{q}}_k = [q_{ik}]$, $q_{ik} = L_{N-i}(\omega_k^2), i = 1, \dots, N$. Compared with the example of the previous section, here the solutions of Eq.(8) includes Laguerre polynomials instead of trigonometric functions as before.

4 MIKOTA'S PROBLEM

In (Mikota, 2001) a concept for the frequency tuning of multi-degree-of-freedom mass-spring oscillators have been presented which allows an exact placement of the natural frequencies at integer multiples $\Omega_i = i\Omega$, $i = 1, \dots, N$, of the base harmonic Ω_1. This has been achieved by the selection of masses and spring stiffnesses in the following manner:

$$m_i = \frac{1}{i} \ , \ c_i = N + 1 - i \ , \ i = 1, \dots, N. \tag{20}$$

Mikota conjectured the eigenfrequencies of the eigenproblem (8) at

$$\omega_i^2 = i^2 \ , \ i = 1, \dots, N, \tag{21}$$

but a proof has been not presented. Therefore, in (Müller, Gürgöze, 2006) some attempts have been made to verify this conjecture. The above approach of applying Laguerre polynomials failed and also a matrix square root approach (see below) did not succeed. Very recently, the problem has been solved independently by Müller, Hou (2006) and John (2006).

The problem is characterized by

$$\mathbf{M}_N = \mathbf{diag}(m_i) = \mathbf{diag}(\frac{1}{i}), \tag{22}$$

$$\mathbf{K}_N = \mathbf{D}_b^T \mathbf{C}_N \mathbf{D}_b \quad \text{with} \tag{23}$$

$$\mathbf{C}_N = \mathbf{diag}(c_i) = \mathbf{diag}(N + 1 - i). \tag{24}$$

The matrix $\mathbf{K}_N$ (23) agrees with the correspnding matrix of Eq.(15) but the mass matrix (22) is different to that of Eq.(15).

It is observed that

$$\mathbf{C}_N = \mathbf{P}\mathbf{M}_N^{-1}\mathbf{P} \ , \ \mathbf{P} = \mathbf{P}^T = \mathbf{P}^{-1} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} \tag{25}$$

holds. With respect to the symmetry $\mathbf{P}\mathbf{D}_b = \mathbf{D}_b^T\mathbf{P}$ and the definition of the matrix

$$\mathbf{A}_N = \mathbf{M}_N^{-1}\mathbf{P}_N\mathbf{D}_b = \begin{bmatrix} & & & -1 & 1 \\ & & \iddots & 2 & \\ & -(N-2) & \iddots & & \\ -(N-1) & (N-1) & & & \\ N & & & & \end{bmatrix} \tag{26}$$

the relation

$$\mathbf{A}_N^2 = \mathbf{M}_N^{-1}\mathbf{K}_N \tag{27}$$

is obtained. Therefore, an equivalent matrix square root eigenvalue/eigenvector problem

$$(-\lambda \mathbf{I}_N + \mathbf{A}_N)\tilde{\mathbf{q}} = \mathbf{0} \tag{28}$$

can be considered instead of Eq.(8). According to Mikota's conjecture (21) the problem (28) is conjectured to have eigenvalues $\lambda_i = \pm\omega_i, i = 1, \ldots, N$, where the sign has to be chosen correctly.

It should be mentioned that in (Müller, Hou, 2006) the problem (8) and in (John, 2006) the problem (28) have been solved. According to (Müller, Hou, 2006) it has been proven that

$$\mathbf{T}\mathbf{M}_N^{-1}\mathbf{K}_N\mathbf{T}^{-1} = \tilde{\mathbf{K}}_N = \begin{bmatrix} 1 & & & & & \\ -2(N-1) & 4 & & & & \\ & \ddots & \ddots & & & \\ & & -j(N-j+1) & j^2 & & \\ & & & \ddots & \ddots & \\ & & & & -N & N^2 \end{bmatrix} \tag{29}$$

holds where $\mathbf{T}$ is defined by binomial coefficients:

$$\mathbf{T} = [T_{ij}]\ ,\ T_{ij} = \begin{cases} 0 & , j < i \\ \binom{j-1}{i-1} & , j \geq i \end{cases} \tag{30}$$

$$\mathbf{T}^{-1} = [(T^{-1})_{ij}]\ ,\ (T^{-1})_{ij} = \begin{cases} 0 & , j < i \\ (-1)^{j-i} \quad \binom{j-1}{i-1} & , j \geq i \end{cases} \tag{31}$$

Mikota's conjecture (21) is verified by the bidiagonal matrix (29). But in addition the eigenmodes can be determined by a similarity transformation

$$\Omega^2 = \mathbf{diag}(i^2) = \tilde{\mathbf{P}}^{-1}\tilde{\mathbf{K}}_N\tilde{\mathbf{P}} \tag{32}$$

where

$$\tilde{\mathbf{P}} = [\tilde{\mathbf{p}}_1 \ldots \tilde{\mathbf{p}}_N]\ ,\ \tilde{\mathbf{p}}_i \equiv [p_{ji}] \tag{33}$$

$$p_{ji} = \begin{cases} 0 & , j < i \\ 1 & , j = i \\ \dfrac{(i+1)\ldots(i+k)\cdot(N-i-k+1)\ldots(N-i)}{[(i+1)^2-i^2][(i+2)^2-i^2]\ldots[(i+k)^2-i^2]} & , j = i+k, k = 1, \ldots N-i. \end{cases} \tag{34}$$

The eigenvectors of Eq.(8) are obtained by

$$\tilde{\mathbf{q}}_i = \mathbf{T}^{-1}\tilde{\mathbf{p}}_i \tag{35}$$

or component-wise by

$$q_{ki} = \sum_{j=k}^{N} (-1)^{j-k} \binom{j-1}{k-1} p_{ji}\ . \tag{36}$$

The matrix square root problem (28) is solved by

$$\lambda_i = (-1)^{i-1} i \quad \text{and} \quad \tilde{\mathbf{q}}_i\ ,\ i = 1, \ldots, N. \tag{37}$$

The intermediate similarity transformation (30) can be also applied to $\mathbf{A}_N$ and leads to the provisional result

$$\begin{aligned} \mathbf{T}\mathbf{A}_N\mathbf{T}^{-1} &= \tilde{\mathbf{A}} = [\tilde{A}_{ij}], \\ \tilde{A}_{ij} &= \begin{cases} 0 & , j > i \\ (-1)^{i-1} i & , j = i \\ (-1)^{j-1} i \quad \binom{N-j}{N-i} & , j < i \end{cases} \end{aligned} \tag{38}$$

Summarizing, the eigenvalue/eigenvector problems (8) and (28) have been completely solved for system matrices (22, 23). By that, Mikota's conjecture has been proved. The main point of the solution was to find the similarity transformation (30) including binomial coefficients.

5 CONCLUSION AND OUTLOOK

It has been demonstrated that the classical eigenvalue/eigenvector problem (8) can be solved analytically for special cases: Homogeneous vibration chains, multiple pendulum with equal and uniformly spaced masses, Mikota's problem. The discussion has shown how large the variety of solutions is depending on the distributions of masses and stiffnesses appearing in the vibration system, i.e. on the entries of $\mathbf{M}_N$ and $\mathbf{K}_N$. The required tools may be trigonometric functions, Laguerre polynomials or binomial coefficients to describe the solutions of the eigenvalue problem (8). Obviously, it will be not possible to find an analytic solution for arbitrary vibration chains with a diagonal mass matrix $\mathbf{M}_N$ and a tridiagoanl stiffness matrix $\mathbf{K}_N$.

But Mikota's problem illustrated that there may be a subclass of vibration chains where the eigenvalue/eigenvector problem is analytically solvable. Assuming

$$\mathbf{K}_N = \mathbf{D}_b^T \mathbf{P} \mathbf{M}_N^{-1} \mathbf{P} \mathbf{D}_b \; , \; \mathbf{D}_b^T \mathbf{P} = \mathbf{P} \mathbf{D}_b \tag{39}$$

the matrix square root approach can be applied. Instead of Eq.(8) the eigenvalue/eigenvectorproblem (28) can be considered equivalently. The related system matrix $\mathbf{A}_N$ is an anti-bidiagonal matrix:

$$\mathbf{A}_N = \begin{bmatrix} & & & -\frac{1}{m_1} & \frac{1}{m_1} \\ & & \iddots & & \\ -\frac{1}{m_{N-1}} & & & \iddots & \\ \frac{1}{m_N} & \iddots & & & \end{bmatrix} \tag{40}$$

for general diagonal mass matrices. In the literature not much is found about such matrices. Therefore, it would be very welcome to start a development of a related matrix theory. Applications in the field of machine dynamics, or more abstract in chain-structured vibration systems are waiting for that.

Some properties may be seen immediately.

(i) If the masses change uniformly by a factor μ, then the natural frequencies ω_i will change uniformly by μ^{-1}.

(ii) If the matrices $\mathbf{M}_N, \mathbf{P}\mathbf{D}_b$ commute ($\mathbf{M}_N$ not necessarily diagonal anymore but still with different eigenvalues), then the calculation of the natural frequencies can be partitioned in the calculation of the eigenvalues $\mu_i(\mathbf{M}_N)$ and $\pi_i(\mathbf{P}\mathbf{D}_b)$:

$$\omega_i^2 = \frac{\pi_i^2}{\mu_i^2} \; . \tag{41}$$

The related eigenvectors are the common eigenvectors of $\mathbf{M}_N$ and $\mathbf{P}\mathbf{D}_b$:

$$\tilde{\mathbf{q}}_i = \tilde{\mathbf{q}}_i(\mathbf{M}_N) = \tilde{\mathbf{q}}_i(\mathbf{P}\mathbf{D}_b) \; . \tag{42}$$

All over, there is a great demand for further investigations.

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