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ON AN APPROXIMATE ANALYTICAL SOLUTION TO A VIBRATING SYSTEM WITH TWO DEGREES OF FREEDOM, EXCITED BY A NON-IDEAL MOTOR

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Abstract. *In this work the dynamic behavior of a non-ideal vibrating system with two degree of freedom is studied by taken analytical solution. It system is composed by two lumped masses and connected by springs and dampers. A non-ideal DC motor with limited power supply is connected to one of the masses in order to disturb the system. The interaction between the dynamic system and the limited power of the energy source make this a non-ideal system. Appropriate choices of the physical system parameters create resonance conditions to ts natural frequencies. Regular and irregular (chaotic) vibrations dependng of the choice of physical parameters and can be observed. An analytical solution for the system is obtained by using perturbation technique. Due to this solution one can observe typical non-ideal phenomenon like the amplitude motion dependency to the frequency of the excitation. Conditions for system stability and existence of bifurcations are also obtained.*

Keywords: *non-ideal system, resonance, perturbation technique, bifurcations, analytical solution.*

1. Introduction

The knowledge of the dynamic properties of current engineering systems is an important step in systems design and control. In the design of structures it is necessary to investigate the relevant dynamics in order to predict the structural response due to excitation. In the selection of rotating machines for applications in structures, often little thought is given to the effect that the structure has on the machine, i.e., the excitation is considered independent of the system response.

Mathematical models of real systems are usually idealized by prescribing the forcing term as a known function. In reality, for a great number of structures this is not the case, and such structures are called non-ideal. In general, it is possible that an energy source fixed to a structure may be affected by the structural response. Systems having dynamic coupling between structure and the energy source often exhibit peculiar behavior, especially systems with limited power supply. Non-ideal systems operating in the neighborhood of a resonant frequencies are often more expensive and perform poorly as compared with ideal counterparts.

Here we treated a problem with the passage of an unbalanced shaft through structural resonance, which can strongly magnified when the driving motor's power is limited and the same order of the driving motor's power consumption is required by vibrations at resonance. We know that this nonlinear phenomenon of interaction between shaft rotation and the system's vibration, which results in a reduction of shaft's angular acceleration, is known as the Sommerfeld effect (Sommerfeld, 1904).

One of the problems often faced by designers is how to drive a vibrating system with a limited power supply through resonance and avoid the Sommerfeld effect.

A number of authors studied this kind of vibrations. We mention some of them (Kononenko, 1969); (Evan-Iwanoski, 1976); (Nayfeh and Mook, 1979); (Dimentberg et al, 1994) and recently (Balthazar et al, 2001, 2003, 2004) present a complete and comprehensive review on this kind of problem called non-ideal system.

Note that the traditional cases where the motor has an unlimited power are called ideal systems. Then the non-ideal vibrating problem has a number of degree of freedom greater than their ideal counterparts, depending on the numbers of non-ideal motors exciting the structure. This fact can introduce the phenomenon of self- synchronization (Balthazar et al, 2004), which will not be treated here, because this paper concerns with only one non-ideal motor exciting the structure.

Here we will analyze a non-ideal problem with two degrees of freedom operating near a resonance, by searching an analytical solution for this kind of problem. The mathematical model of the problem was defined earlier in (Balthazar et al, 2001) where is studied a strategy of control based on Tikhonov regularization. Some numerical results in the particular cases of internal resonances 1:1 and 1:2 were done respectively in (Tsuchida et al, 2003, 2004). The first step towards an analytical solution for this kind of problem was done in (Guilherme, 2004).

We organized this paper as follows:

In the section 2 we define the mathematical model and exhibit the derivation of the governing equations. In the section 3 we obtain an approximate analytical solution to mathematical model defined in section 1. In the section 4 we analyze stability and bifurcation of the approximate analytical solution. In the section 5 we present the conclusions of the present work. In the section 6 some acknowledgements are presented and finally, in the section 7 we present the basic bibliographic references used here

2. Mathematical model and derivation of the governing equations

The vibrating problem that will be analyzed here is shown in Figure 1.

It consists of a block of mass m_1, a linear elastic spring with coefficient of elasticity k_1 and a linear damper with viscous damping coefficient c_1. On the body of mass m_1, a non-ideal motor is placed, with a driving rotor of moment of inertia J and an eccentric mass m_0 situated at a distance r from the axis of rotation. By means of a linear spring with coefficient of elasticity k_2 and a damper with coefficient of damping c_2, a body of mass m_2 has been attached to mass m_1.

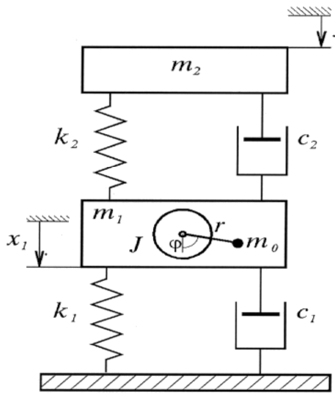

Figure 1. Non-ideal dynamic system with two degree of freedom. A non-ideal DC motor with limited power supply is connected to one of the masses in order to disturb the system.

The Lagrange equations of motion may be written as in (Balthazar et al, 2001):

$$
\begin{aligned}
&m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + F(x_1,x_2,\dot{x}_1,\dot{x}_2) + m_0r\omega^2\cos\varphi + m_0r\dot{\omega}sin\varphi;\\
&m_2\ddot{x}_2 = -F(x_1,x_2,\dot{x}_1,\dot{x}_2);\\
&J\dot{\omega} = L - H(\omega) + m_0\ddot{x}_1 sin\varphi;\\
&\dot{L} = -aL - b\omega + k_U U(\omega),\\
&F(x_1,x_2,\dot{x}_1,\dot{x}_2) = k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1)
\end{aligned}
$$

where

L is the torque generated by the DC motor of limited power supply and $H(\omega)$ is the resisting torque which will be ignored from now on. The parameters a and b are constants depending on the type and power of the DC motor, and $U(\omega)$ is the voltage of the motor. Note that above equations include only non-linear members resulting from the interaction between the vibrating system and the DC motor (non-ideal system).

Defining the new variables

$$\chi_1 = \tfrac{m_1}{m_0r}x_1 \qquad \chi_2 = \tfrac{m_1}{m_0r}x_2 \qquad \overline{\omega} = \sqrt{\tfrac{m_1}{k_1}}\omega \qquad \lambda = \tfrac{m_1}{Jk_1}L \qquad \tau = \sqrt{\tfrac{k_1}{m_1}}t;$$

We will obtain the dimensionless form of the equations:

$$\chi_1'' = -\chi_1 - \eta_1\chi_1' + \mu(\chi_2 - \chi_1) + \eta_2(\chi_2' - \chi_1') + \overline{\omega}^2 \cos\overline{\varphi} + \overline{\omega}' sen\overline{\varphi}$$
$$\chi_2'' = -\theta^2(\chi_2 - \chi_1) - \frac{\theta^2}{\mu}\eta_2(\chi_2' - \chi_1') \qquad (1)$$
$$\overline{\omega}' = \lambda + \rho\chi_1'' \operatorname{sen}\overline{\varphi}$$

where:

$\eta_1 = c_1 / \sqrt{k_1 m_1}$, $\eta_2 = c_2 / \sqrt{k_1 m_1}$, $\mu = k_2 / k_1$, $\rho = m_0^2 r / J m_1$ and $\theta^2 = \mu m_1 / m_2$.

Note that the functions $\overline{\omega}^2 \cos\overline{\varphi}$ and $\overline{\omega}' \operatorname{sen}\overline{\varphi}$ are inertia forces arisen from rotor action; The function $\rho\chi_1'' sen\overline{\varphi}$ represents the moment of these inertia forces, noting that it depends on the coordinates of motion of the structural system and of the non-ideal energy source, exhibiting the interaction between them.

We also mention that the inertia forces make that the unbalanced force m_0 deforms the spring's k1 e *K2* with an excitation frequency done by $\overline{\omega} = d\overline{\varphi}/dt$.

Note that the equation of the driving torque of the DC motor, i.e., the motive force associated with the source of energy, is giving by

$$\lambda' = -\alpha\lambda - \beta\overline{\omega} + u \qquad (2)$$

with adimensional constants, defined by:

$u = k_u U\, m_1 / J k_1 \sqrt{m_1/k_1}$; $\alpha = a\sqrt{m_1/k_1}$ and $\beta = b\, m_1 / J k_1$.

U represents the voltage, *a* and b *are* constants depending of the type and power f the used DC Motor of limited power supply.

The stability criterion depends essentially on the slope of the level curves of characteristic curves $\mathrm{N} = d\lambda/d\overline{\varphi}$ adjusting the values of the parameters an e *b*. Generally, in practice; they are obtained from experimental procedures. Numerical results obtained before in the region of internal resonance 1:1 showed that the model of torque is relevant (Tsuchida et al, 2003b), since if we taken linear model of torque (see equation 3) by choosing suitably values of the parameters one obtains only regular oscillations during passage through resonance. By other hand, if one takes nonlinear model for the applied torque F (equation 2) one obtains chaotic regimes.

In this paper, we only consider the case of linear torque in order to obtain an approximate analytical solution for the problem. Then the will consider the particular case of (2):

$$\lambda = -\overline{\alpha}\overline{\omega} + \overline{\beta}. \qquad (3)$$

2.1. Reduction to a normal generalized coordinates

The governing equations of motion (1) present linear coupling terms that must will be absent in order to use a perturbation technique as that called Krylov and Bogoliubov method (Nayfeh, 1973); (Nayfeh and Mook, 1979); (Evan-Iwanowski, 1976) and so on.

By introducing a linear coordinate transformation from the old to new coordinates, called principal coordinates and taking into account a suitably balancing order terms, we will obtain the governing equations of motion, in a matrix form:

$$M\ddot{u}(t) + Ku(t) = \varepsilon f(t) - \varepsilon C\dot{u}(t);$$
$$\bar{\varphi}'' = \varepsilon\,\lambda + \varepsilon\,\rho\,\chi_1'' \sin\bar{\varphi}, \qquad (4)$$

being $u = [\chi_1 \;\; \chi_2]^T$ the dimensionless generalized displacement of the vibrating system and

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \eta_1 + \eta 2 & -\eta_2 \\ -\frac{\theta^2}{\mu}\eta 2 & \frac{\theta^2}{\mu}\eta 2 \end{bmatrix}, \quad K = \begin{bmatrix} 1+\mu & -\mu \\ -\theta^2 & \theta^2 \end{bmatrix},$$

are the matrices of mass, damping and stiffness of the dimensionless system (1). Note also that the vectorial function f contains the non-linear coupling terms of (1). If we taken $\epsilon = 0$ in equations (4) we will obtain a no perturbed system. Rewritten the system (4) in the form:

$$\ddot{v}_i + \omega_i^2 v_i = \varepsilon f_i(\bar{\varphi}, v_1, v_2, \dot{v}_1, \dot{v}_2), \quad i = 1, 2, \tag{5}$$

Its solution is well known and has the form:

$$v_i = a_i \cos(\bar{\varphi} + \beta_i), \quad i = 1, 2 \tag{6}$$

and by introducing new variables v = $[v_1 \ \ v_2]^{T,}$ through a linear transformation u = Pv, where P is a quadratic matrix of order 2, which column pi = $[p_{i1} \ \ p_{i2}]^T$ are the right eigenvectors corresponding to the eigenvalues λ_1 e λ_2 (of e $Kp_i = \overline{\omega}_i^2 Mp_i$), we will obtain the normal system:

$$\begin{aligned} &I\ddot{v} + \Lambda v = \varepsilon\,[Q\,\dot{v} + f\,\cos\bar{\varphi} + h\,\sin\bar{\varphi}]\,; \\ &\ddot{\bar{\varphi}} = \varepsilon\,[\lambda + \rho\,(\ddot{v}_1 + \ddot{v}_2)\,\sin\bar{\varphi}]\,; \end{aligned} \tag{7}$$

with

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \quad \Lambda = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}; \quad f = \begin{bmatrix} \dot{\bar{\varphi}}^2 \\ 0 \end{bmatrix}; \quad h = \begin{bmatrix} \ddot{\bar{\varphi}} \\ 0 \end{bmatrix}; \quad Q = \begin{bmatrix} \eta_1 + \eta_2 + \eta_1\,p_{21} & \eta_1 + \eta_2 + \eta_1\,p_{22} \\ \frac{\theta^2}{\mu}\,\eta_2(1 + p_{21}) & \frac{\theta^2}{\mu}\eta_2(1 + p_{21}) \end{bmatrix}.$$

2.2 Method of variation of parameters

Next, we will transform (7) into a new where the new variables represent the essential parameters of the motion; the amplitudes of the oscillations, the angular phase displacements and frequency of the exciting force.

By introducing a variable transformation s to z, being $s = [v_1 \ \ \dot{v}_1 \ \ v_2 \ \ \dot{v}_2 \ \ \overline{\omega}]^T$ generalized coordinates of system (7) and taking the linear part of

$\dot{s} + \Lambda$ s = $\in$F (s, $\dot{s}$, t) and z = $[a_1 \ \ \beta_1 \ \ a_2 \ \ \beta_2 \ \ \overline{\omega}\,]$

we will obtain

$\dot{z} = \in f\,(z, t) + O\,(\in^2)$.

which coordinates (a_i, β_i , z) represent the amplitudes and phases of the oscillations of the coordinates v_i and $\overline{\omega}$ the frequency of the rotation of the DC motor.

Substituting them into (6) the equations (7) become:

$$\dot{v}_i = -\dot{a}_i\,\bar{\omega}_i\,\sin(\varphi + \beta_i), \quad i = 1, 2 \tag{8}$$

$$\dot{\bar{\varphi}} = \bar{\omega}\,. \tag{9}$$

Then, after some algebraic manipulation we will obtain the equations of motion in a suitably form

$$\begin{aligned} &a)\,\frac{da_i}{d\bar{\varphi}} = \frac{\varepsilon}{\bar{\omega}_i\bar{\omega}}\sin(\bar{\varphi} + \beta_i)[Tn_i]\,; \quad i = 1, 2\,, \\ &b)\,\frac{d\beta_i}{d\bar{\varphi}} = \frac{\varepsilon\alpha_i}{\bar{\omega}} + \frac{\varepsilon}{a_i\bar{\omega}_i\bar{\omega}}\cos(\bar{\varphi} + \beta_i)[Tn_i]\,; \quad i = 1, 2\,, \\ &c)\,\frac{d\bar{\omega}}{d\bar{\varphi}} = \frac{\varepsilon}{\bar{\omega}}\,[Hn]\,, \end{aligned} \tag{10}$$

with T_{ni} e H_n defined by:

$$Tn_1 = (\eta_1 + \eta_2 + \eta_2 p_{21})\, a_1 \bar{\omega}_1 \sin(\bar{\varphi} + \beta_1) + (\eta_1 + \eta_2 + \eta_2 p_{22})\, a_2 \bar{\omega}_2 \sin(\bar{\varphi} + \beta_2)\bar{\omega}^2 \cos(\bar{\varphi})$$

$$Tn_2 = \frac{\theta^2}{\mu} \eta_2 \left[(1 - p_{21})\, a_1 \bar{\omega}_1 \sin(\bar{\varphi} + \beta_1) + (1 - p_{22})\, a_2 \bar{\omega}_2 \sin(\bar{\varphi} + \beta_2)\right]$$

$$Hn = \lambda - \left[\rho a_1 \bar{\omega}_1 \bar{\omega} \cos(\bar{\varphi} + \beta_1) + \rho a_2 \bar{\omega}_2 \bar{\omega} \cos(\bar{\varphi} + \beta_2) \sin(\bar{\varphi})\right].$$

Next we will obtain an approximate analytical solution to (7) under internal 1: 1 resonance conditions.

3. Obtaining of an approximate analityical soltion by using an average method of perturbation theory

In order to obtain an analytical solution of first order relatively to the small parameter ε, under the conditions of internal resonance 1:1, one needs to introduce two detuning parameters σ_1 e σ_2 such that $\overline{\omega}_2 = \overline{\omega}_1 + \in \sigma_1$ and $\overline{\omega} = \overline{\omega}_2 + \in \sigma_2$. In order to simplify the algebraic manipulations we consider the following notation $\in \alpha_2 = - \in \sigma_2$.

The solutions of (10) will be taken into the form:

$$z = y_0 + \in U(y_0, \overline{\varphi}) \tag{11}$$

Where
y0T = [A_{10} ξ_{10} A_{20} ξ_{20} Ω_0] are constant values (representing stationary solutions), and $\in U^T(y_0, \overline{\varphi}) = [U_1(y_0, \overline{\varphi})\ U_2(y_0, \overline{\varphi}) \ldots U_5(y_0, \overline{\varphi})]$ are a periodic solutions.

By using (Nayfeh, 1973) we will obtain the Fourier series:

$$\dot{a}_i = \frac{\varepsilon}{\bar{\omega}_i \bar{\omega}} \sin(\bar{\varphi} + \beta_i) \left[T_{i0}(z) + \sum_{n=1}^{\infty} Tc_{in}(z) \cos(n\bar{\varphi} + n\beta_i) + \sum_{n=1}^{\infty} Ts_{in}(z) \sin(n\bar{\varphi} + n\beta_i)\right], \quad i = 1, 2; \tag{12}$$

$$a_i \dot{\beta}_i = (\bar{\omega}_i - \bar{\omega}) \frac{a_i}{\bar{\omega}} + \frac{\varepsilon}{\bar{\omega}_i \bar{\omega}} \cos(\bar{\varphi} + \beta_i) \left[T_{i0}(z) + \sum_{n=1}^{\infty} Tc_{in}(z) \cos(n\bar{\varphi} + n\beta_i) + \sum_{n=1}^{\infty} Ts_{in}(z) \sin(n\bar{\varphi} + n\beta_i)\right], \tag{13}$$
$i = 1, 2;$

$$\dot{\omega} = \varepsilon \left[\lambda + H_0(z) + \sum_{n=1}^{\infty} Hc_n(z) \cos(n\bar{\varphi}) + \sum_{n=1}^{\infty} Hs_n(z) \sin(n\bar{\varphi})\right]; \tag{14}$$

where

$$T_{i0}(z) = \frac{1}{2\pi} \int_0^{2\pi} T_i \, d\bar{\varphi}, \quad i = 1, 2 \qquad H_0(z) = \frac{1}{2\pi} \int_0^{2\pi} H \, d\bar{\varphi};$$

$$Tc_{in}(z) = \frac{1}{2\pi} \int_0^{2\pi} T_i \cos(n\bar{\varphi}) d\bar{\varphi}, \quad i = 1, 2 \qquad Hc_n(z) = \frac{1}{2\pi} \int_0^{2\pi} H \cos(n\bar{\varphi}) d\bar{\varphi};$$

$$Ts_{in}(z) = \frac{1}{2\pi} \int_0^{2\pi} T_i \sin(n\bar{\varphi}) d\bar{\varphi}, \quad i = 1, 2 \qquad Hs_n(z) = \frac{1}{2\pi} \int_0^{2\pi} H \sin(n\bar{\varphi}) d\bar{\varphi};$$

$$T_1 = (\eta_1 + \eta_2 + \eta_2 p_{21}) a_1 \bar{\omega}_1 \sin(\bar{\varphi} + \beta_1) + (\eta_1 + \eta_2 + \eta_2 p_{22}) a_2 \bar{\omega}_2 \sin(\bar{\varphi} + \beta_2) - \bar{\omega}^2 \cos(\bar{\varphi}) - \dot{\bar{\omega}} \sin(\bar{\varphi});$$

$$T_2 = \frac{\theta^2}{\mu} \eta_2 (1 - p_{21}) a_1 \bar{\omega}_1 \sin(\bar{\varphi} + \beta_1) + \frac{\theta^2}{\mu} \eta_2 (1 - p_{22}) a_2 \bar{\omega}_2 \sin(\bar{\varphi} + \beta_2);$$

$$H = -\frac{\rho\Omega}{2} a_1 \bar{\omega}_1 [\sin(2\bar{\varphi} + \beta_1) - \sin(\beta_1)] - \frac{\rho\Omega}{2} a_2 \bar{\omega}_2 [\sin(2\bar{\varphi} + \beta_2) - \sin(\beta_2)].$$

By using (7) and (11) and taking into account terms up to n=1 we will the averaging equations relatively to $\overline{\varphi}$:

$$\dot{A}_1 = \frac{\varepsilon}{4\bar{\omega}_1 \Omega} \left[(\eta_1 + \eta_2 + \eta_2 p_{21}) A_1 \bar{\omega}_1 \cos(\xi_1) + (\eta_1 + \eta_2 + \eta_2 p_{22}) A_2 \bar{\omega}_2 \cos(\xi_2)\right]; \tag{15}$$

$$\dot{\xi}_1 = \frac{\varepsilon}{\Omega} (\bar{\omega}_1 - \Omega) + \frac{\varepsilon}{4\bar{\omega}_1 \Omega A_1} \left[(\eta_1 + \eta_2 + \eta_2 p_{21}) A_1 \bar{\omega}_1 \sin(\xi_1) + (\eta_1 + \eta_2 + \eta_2 p_{22}) A_2 \bar{\omega}_2 \sin(\xi_2) - \Omega^2\right]; \tag{16}$$

$$\dot{A}_2 = \frac{\varepsilon}{4\bar{\omega}_2\Omega}\frac{\theta^2}{\mu}\eta_2\left[(1-p_{21})A_1\bar{\omega}_1\cos(\xi_1) + (1-p_{22})A_2\bar{\omega}_2\cos(\xi_2)\right] ; \tag{17}$$

$$\dot{\xi}_2 = \frac{\varepsilon}{\Omega}(\bar{\omega}_2 - \Omega) + \frac{\varepsilon}{4\bar{\omega}_2\Omega A_2}\frac{\theta^2}{\mu}\eta_2\left[(1-p_{21})A_1\bar{\omega}_1\sin(\xi_1) + (1-p_{22})A_2\bar{\omega}_2\sin(\xi_2)\right] ; \tag{18}$$

$$\dot{\Omega} = \frac{\varepsilon}{2\Omega}\left[2\lambda(\Omega) + \rho\Omega A_1\bar{\omega}_1\sin(\xi_1) + \rho\Omega A_2\bar{\omega}_2\sin(\xi_2)\right] . \tag{19}$$

In order to obtain the stationary solutions of (15)-(19), one considers them equal to zero.

By using (15) and (17) one has that $cos\,\xi_1 = 0$ and $cos\,\xi_2 = 0$ that are coordinates of equilibrium point of averaging equations. Then supposing that $\xi_1 = k\pi/2$ and $\xi_2 = n\pi/2$ it is possible to consider two cases:

1. *K* and n are even or k and n are odd, that is, ξ_1 and ξ_2 are in phase.

Then, one ha that

$$\sin\xi_1 = 1 \qquad \text{and} \qquad \sin\xi_2 = 1 \tag{20}$$

2. *K* and n are not even or odd simuostaneanaly, that is, ξ_1 e ξ_2 are not in phase.

Then $\sin\xi_1$ and $\sin\xi_2$ have opposite signals and one can considers $\sin\xi_1 = 1$ and $\sin\xi_2 = -1$.

Here, we only taking into account the when we consider ξ_1 and ξ_2 are in phase.

Then by considering the average equations (15)-(19) and by using the independent variable τ, and adopting

$l_1 = (\eta_1 + \eta_2 + \eta_2 p_{21})$ and $l_2 = (\eta_1 + \eta_2 + \eta_2 p_{22})$

we will obtain the following stationary states:

$$A_{10} = -\frac{\Omega^2}{\bar{\omega}_1}\left[\frac{4\mu}{\theta^2\eta_2}(\bar{\omega}_2 - \Omega) + (1-p_{21})\right]\left[\left\{\frac{4\mu}{\theta^2\eta_2}(\bar{\omega}_2 - \Omega) + (1-p_{21})\right\}\{4(\bar{\omega}_1 - \Omega) + l_1\}\frac{1}{(1-p_{21})} - l_2\right]^{-1} ; \tag{21}$$

$$A_{20} = \frac{\Omega^2}{\bar{\omega}_2}\left[\left\{\frac{4\mu}{\theta^2\eta_2}(\bar{\omega}_2 - \Omega) + (1-p_{21})\right\}\{4(\bar{\omega}_1 - \Omega) + l_1\} - l_2(1-p_{21})\right]^{-1} ; \tag{22}$$

$$\Omega_0 = \frac{-2\alpha}{\rho A_1\bar{\omega}_1 + \rho A_2\bar{\omega}_2 - \beta} . \tag{23}$$

Next, we will analyze the stability conditions of the obtaining solution

4. Stabilty analysis

The main purpose of the study of the stationary solution of the average equations is which concerns to the study of the stability of the considered system. According to Hartman-Grobman Classical Theorem if the considered solution is a hyperbolic point then the stability is equivalent to the stability of the original system.

Next, we analyze the Lyapunov stability conditions of the system (15)-(19) by using the classical de Routh-Hurwitz criterion and a condition to bifurcation of a particular solution which existence is based on Sotomayor Theorem (Sotomayor, 1986).

4.1. Routh-Hurwitz(R-H) algorthm

The necessary and sufficient conditions to stability analysis is given by the analysis of average system (15)-(19) are given by the analysis of the coefficients of the characteristic polynomial

$$\lambda^5 + B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0$$

given by the matrix

$$H_5 = \begin{bmatrix} B_1 & 1 & 0 & 0 & 0 \\ B_3 & B_2 & B_1 & 1 & 0 \\ B_5 & B_4 & B_3 & B_2 & B_1 \\ 0 & 0 & B_5 & B_4 & B_3 \\ 0 & 0 & 0 & 0 & B_5 \end{bmatrix}$$

being the coefficients B_i given by the following expressions:

$$B_1 = \quad -tr(A) = \beta - \frac{\rho}{2}(\bar{\omega}_1 A_{10} + \bar{\omega}_2 A_{20})\,; \tag{24}$$

$$B_2 = \frac{\theta^2\eta_2(1-p_{21})}{16\mu}\left[l_2 + \frac{l_2}{\bar{\omega}_1} - \frac{\theta^2\eta_2(1-p_{22})\bar{\omega}_1 A_{10}}{\mu\bar{\omega}_2 A_{20}}\right] + \frac{l_1(\Omega_0^2 - l_2\bar{\omega}_2 A_{20})}{16\bar{\omega}_1 A_{10}}\,; \tag{25}$$

$$B_3 = -\frac{\rho}{2}(\bar{\omega}_1 A_{10} + \bar{\omega}_2 A_{20})\left[\frac{\theta^2\eta_2(1-p_{21})}{16\mu}\left(l_2 + \frac{l_2}{\bar{\omega}_1} - \frac{\theta^2\eta_2(1-p_{22})}{\mu\bar{\omega}_2 A_{20}}\right) - \frac{l_1(\Omega_0^2 - l_2\bar{\omega}_2 A_{20})}{16\bar{\omega}_1 A_{10}}\right] + \tag{26}$$

$$+ \frac{\rho\,\Omega_0\,\bar{\omega}_1 A_{10}}{8\mu}\left[l_1\mu + \theta^2\eta_2(1-p_{21})\right]\left[1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right]$$

$$B_4 = \frac{\theta^4\,\eta_2^2\,(1-p_{21})}{(16\mu)^2}\left[l_2(1-p_{21}) - l_1(1-p_{22})\right]\left[\frac{\Omega_0^2 - l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_2 A_{20}} + \frac{l_2}{\bar{\omega}_1}\right]\;; \tag{27}$$

$$B_5 = -\frac{\theta^2\,\eta_2\,(1-p_{21})A_{10}A_{20}}{16\mu}\left[l_2(1-p_{21}) - l_1(1-p_{22})\right]\left[\frac{\theta^2\,\eta_2\,(1-p_{21})}{16\mu A_{10}A_{20}}\left(\frac{\rho}{2}\bar{\omega}_1 A_{10} + \frac{\rho}{2}\bar{\omega}_2 A_{20} - \beta\right)\times\right. \tag{28}$$

$$\left.\times\left[\frac{\Omega_0^2 - l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_2 A_{20}} - \frac{l_2}{\bar{\omega}_1}\right] + \frac{\rho\theta^2\,\eta_2\,(1-p_{21})\Omega_0}{8\mu A_{20}}\left(1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right)\left(1 + \frac{\bar{\omega}_1^2 A_{10}}{\bar{\omega}_2 A_{20}}\right) + \frac{\rho\,\Omega_0\,\bar{\omega}_2}{8A_{10}}\left(l_2 - \frac{\Omega_0^2 - l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_1 A_{10}}\right)\right]\,.$$

According to R-H criterion the real part of all eigenvalues are less than zero and if the following conditions are satisfied:

$$\Delta_1 = \; tr(A) < 0\,; \qquad \Delta_2 = B_1 B_2 - B_3 > 0\,; \qquad \Delta_3 = \; \Delta_2 B_3 + B_1(B_5 - B_1 B_4) > 0 \;\;;$$

$$\Delta_4 = \; \Delta_3 B_4 - B_5 B_2 \Delta_2 - B_5(B_5 - B_1 B_4) > 0 \;\;; \qquad \Delta_5 \; = \; B_5 > 0\,.$$

And according to these inequalities one will obtain the stability conditions:

$$\beta > \frac{\rho}{2}(\bar{\omega}_1 A_{10} + \bar{\omega}_2 A_{20})\,; \tag{29}$$

$$\frac{\rho\,\Omega_0\,\bar{\omega}_1 A_{10}}{8\mu}\left[l_1\mu + \theta^2\eta_2(1-p_{21})\right]\left[1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right] + \frac{\rho\,\Omega_0\,\bar{\omega}_2 A_{20}}{8\mu}\left[l_2\mu + \theta^2\eta_2(1-p_{21})\right] \;>\; 0\,; \tag{30}$$

$$\frac{\theta^2\,\eta_2\,A_{10}A_{20}}{16\mu}\left[l_2(1-p_{21}) - l_1(1-p_{22})\right]\left[-\frac{\rho\,\Omega_0\,\bar{\omega}_2}{8A_{10}}\left(l_2 - \frac{\Omega_0^2 l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_1 A_{10}}\right) - \right. \tag{31}$$

$$\left. - \left(1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right)\left(1 + \frac{\bar{\omega}_1^2 A_{10}}{\bar{\omega}_2 A_{20}}\right)\frac{\rho\,\theta^2\,\eta_2\,\Omega_0\,(1-p_{21})}{8\mu A_{20}}\right] \;>\; 0\,;$$

$$\left[l_2(1-p_{21}) - l_1(1-p_{22})\right]\left\{\left[\frac{\Omega_0^2 - l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_2 A_{20}} + \frac{l_2}{\bar{\omega}_1}\right]\left[\frac{\rho\,\Omega_0\bar{\omega}_1 A_{10}}{8}\left(\frac{\theta^2\,\eta_2\,(1-p_{22})}{\mu} + l_1\right)\left(1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right) + \right.\right. \tag{32}$$

$$\left. + \frac{\rho\,\Omega_0\bar{\omega}_2 A_{20}}{8}\left(\frac{\theta^2\,\eta_2\,(1-p_{21})}{\mu} + l_2\right)\right]\frac{\theta^4\,\eta_2^2\,(1-p_{21})}{(16\mu)^2} - \left[\frac{\theta^2\,\eta_2\,(1-p_{21})}{16\mu}\left(\frac{l_2}{\bar{\omega}_2} + l_2 - \frac{\theta^2\,\eta_2\,(1-p_{22})\bar{\omega}_1 A_{10}}{\mu\bar{\omega}_2 A_{20}}\right)\right]$$

$$\times \left[\left(1 + \frac{\Omega_0}{2\,\bar{\omega}_1 A_{10}}\right) \left(1 + \frac{\bar{\omega}_1^2 A_{10}}{\bar{\omega}_2 A_{20}}\right) \frac{\rho\,\theta^2\,\eta_2\,\Omega_0\,(1-p_{21})}{8\mu A_{20}} + \frac{\rho\,\Omega_0\,\bar{\omega}_2}{8A_{10}} \left(l_2 - \frac{\Omega_0^2 - l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_1 A_{10}}\right) + \right.$$

$$\left. \frac{l_1(\Omega_0^2 - l_2\bar{\omega}_2 A_{20})}{16\bar{\omega}_2 A_{20}} \right] \frac{\theta^2\,\eta_2\,A_{10}A_{20}}{16\mu} \Bigg\} > 0\,;$$

One can conclude that the stationary solution (20)-(23) of the dynamical system (15)-(19) is stable if the coefficients (24) to (28) of the characteristic polynomial are positive and if the conditions (29) to (32) are satisfied.

Next, we will analyze a bifurcation condition by using the same conditions of equilibrium that we considered here. We restrict our attention in the conditions to occurrence of saddle point bifurcation because it is the responsible to Sommefeld effect (Sommerfeld, 1904), that is, if we have an increase in the input power it causes the amplitude to decrease considerably and the frequency to increase considerably.

4.2. Saddle-node bifurcation

In order to apply Sotomayor Theorem (Sotomayor, 1986) in the vibrating system 15)-(19) it is necessary to have equilibrium points, i.e., the equations (20)-(23) are satisfied. Besides this condition it must have one eigenvalue of simple kind, too.

Then one need to have that the coefficient B_5 (28) of the characteristic polynomial vanishes. We need to remarked that this adopted condition is restrict to this eingenvalue and don't have another eigenvalues. Then, the terms

$$(1-p_{21}); \qquad [l_2(1-p_{21}) - l_1(1-p_{22})] \qquad \text{and} \qquad \left[\frac{\Omega_0^2 + l_2\bar{\omega}_2 A_{20}}{\bar{\omega}_2 A_{20}} - \frac{l_2}{\bar{\omega}_1}\right] \tag{33}$$

Cannot vanish. If them vanish the coefficient B_4 (27) would vanishes, too. This fact contradicts the fact that the characteristic polynomial has only one simple eigenvalue. Then, one can suppose that the eigenvector that corresponds to a simple overvalue of the Jacobian matrix A is $v^T = [v_1 \;\; v_2 \;\; v_3 \;\; v_4 \;\; v_5]$, and the corresponding eigenvector to the same eigenvalue to A^T is $w^T = [w_1 \;\; w_2 \;\; w_3 \;\; w_4 \;\; w_5]$, satisfying the second necessary condition. Note that the eigenvector is found through the non-trivial solution of Av=0. By using same procedure one can find the eingenvalue w of the A^T, making $A^Tv=0$.

Considering the necessary conditions, one can find easily the sufficient conditions.

If one considers *m* as the control parameter *m* and *f* the vector field of the equation that define the (15)-(19), the Sotomayor Theorem (Sotomayor, 1986) establishes the following conditions, as being sufficient to have a saddle-node bifurcation in the equilibrium point $(y_0\,,\,m_0)$ in the considered vibrating system:

$$x^T \frac{\partial f}{\partial m}(y_0, m_0) \neq 0 \quad and \quad x^T D_y^2 f(y_0, m_0).(v, v) \neq 0;$$

being:

$$x^T = \left[\begin{array}{ccccc} -\dfrac{\theta^2\eta_2(1-p_{21})\bar{\omega}_1}{\mu\bar{\omega}_2 l_1} & 1 & 1 & \dfrac{4\mu\bar{\omega}_2 A_{20}}{\theta^2\eta_2(1-p_{21})} & \dfrac{2}{\rho\bar{\omega}_2}\left(\dfrac{l_2\bar{\omega}_2}{4\bar{\omega}_1 A_{10}} - \dfrac{\bar{\omega}_1 A_{10}}{A_{20}}\right) \end{array} \right].$$

Then, according (Sotomayor, 1986) if we choose the control parameter to the system (15)-(19) the constant $\overline{\beta}$, parameter that depends on the power if the DC motor, one can obtain the necessary conditions to have saddle-node bifurcation in the averaging system (15)-(19):

$$\left(\frac{l_2\bar{\omega}_2}{4\bar{\omega}_1 A_{10}} - \frac{A_{10}\bar{\omega}_1}{A_{20}}\right) \neq 0; \qquad \Omega_0 \neq 0; \qquad \text{and} \tag{34}$$

$$x^T D_y^2 f(y_0, \beta_0).(v, v) \neq \frac{l_2\theta^2\,\eta_2\,(1-p_{21})}{4\mu l_1} v_4(v_1 + 2v_3) + \frac{l_2\bar{\omega}_2}{4\bar{\omega}_1 A_{10}^2} v_1(A_{20}v_1 - v_3) - \frac{\Omega_0}{2\bar{\omega}_1 A_{10}^2} v_1 \left(\frac{\Omega_0}{2} v_1 - v_5\right) - \tag{35}$$

$$-\frac{l_1}{4}v_2^2 - \frac{l_2\bar{\omega}_2}{4\bar{\omega}_1 A_{10}}(v_1 v_3 - A_{20}v_4^2) - 4\bar{\omega}_2 A_{20} v_4^2 - \frac{\theta^2\,\eta_2\,(1-p_{22})}{2\mu} v_4 v_3 - \frac{\bar{\omega}_2 + 1}{A_{20}} v_1 v_3 + \frac{2\bar{\omega}_1 A_{10}}{A_{20}^2} v_3^2 +$$

$$+\frac{v_5}{2\bar{\omega}_1 A_{10}}(\Omega_0 v_1 - v_5) - \left(\frac{l_2}{4\bar{\omega}_1 A_{10}} - \frac{\bar{\omega}_1 A_{10}}{\bar{\omega}_2 A_{20}}\right)(2\bar{\omega}_1 v_1 v_5 - \bar{\omega}_1 A_{10}\Omega_0 v_2^2 + 2\bar{\omega}_1 v_3 v_5 - \bar{\omega}_2 \Omega_0 v_4^2)\,.$$

5. Conclusions

We analyzed the non-ideal vibrating problem, with two degree of freedom, defined by Figure 1, by using linear torque model and considering the 1:1 resonance. This analysis was based on perturbation technique by using an averaging method based on Krylov – Bogoliubov. The results obtained here such as stability and bifurcation results obtaining conditions to the occurrence of saddle- node bifurcation is important because it is related to the Sommerfeld effect.

In summary, we results that we have obtained here, provide qualitative insight into the amplitude dependence of the external load on the source, and hence, can be used to explain why the coupling between source and response affect the power required to maintain steady-state conditions near resonance.

This work suggests that further investigation I the developed ways to avoid the Sommerfeld effect, that is, the energy sink by using active control systems that could yield way of achieving smooth passage through resonance.

We also notice that numerical results will be published separately.

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