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ON THE DYNAMICS OF A CENTRIFUGAL VIBRATOR WITH BIG MOMENT OF INERTIA AND ITS IMPLICATION ON DETECTING OF SOMMERFELD EFFECT

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Abstract. *This paper deals with a mechanical system excited by a DC motor, with limited supply power, which base is leaned on a spring. Besides, the DC motor rotates a small mass m. Such machine is known as centrifugal vibrator. As it is usual, we assume that the damping, the stiffness of the spring, the difference between the resistive and driving torque are small. Moreover, the system is weakly coupled. Differently of the classical approach, in the mathematical model of this kind of problem, we assume that the moment of inertia of the rotating parts of the DC motor is big and we obtain several new results. By using the Averaging Method, in the region of resonance 1:1, we prove several results on stability of periodic orbits in the reduced phase space of this system.*
In our most important result, we find that there is no Jump Phenomenon occurring in this kind of vibrating problem, if we take into account big values of moment of inertia of the rotating parts of the DC motor

***Keywords**: Averaging Method, Non-Ideal Problem, Stable Periodic Orbit, Sommerfeld Effect*

1. Introduction

In the study of non-linear vibrating systems, it is almost always assumed that the energy source does not experience any influence from the vibrating system itself that is one say that the energy source is ideal. Here, we assume that motion of the system may have an influence on the action of the source. Because of this influence many of the properties of non-linear vibrating systems are found to depend on the properties of the energy source. In an ideal system we assume a motor operating on a structure requires a certain input (POWER) to produce a certain output (RPM), regardless of the motion of the structure. For a non-ideal system this may not be the case. So, we are interested in what happens to the motor, input and output, when it changes the response of the system.

Two important properties of non-ideal vibrating systems are mention, next.

1- Jump phenomenon is a non-linear effect that appears when a portion of the right branch of the frequency-response curve becomes unstable or when this curve becomes multivalued. As the driving frequency approaches the natural frequency, the vibrating system can suddenly jump from one side of resonance to other. That is, the system operating in a steady –state mode cannot realize certain frequencies near resonance. The Jump appears on the frequency response curve as a discontinuity, which indicates a region where steady-state conditions do not exist. Thus, we see that modeling systems by ideal model may be inadequate, if the driving frequency lies near a natural frequency of the system, as is often the case.

2- If one considers a typical frequency-response curve, we note that as the power supplied to the source increases, the RPM of the motor increases accordingly. However, this behavior does not continue indefinitely. That is, the closer the motor speed moves towards to the resonant frequency, the source requires more power in order to increase the motor speed. More formally a large change, in the power supplied to the motor, results in a small change in the frequency, but a large increase in the amplitude of the resultant vibrations. Thus, near resonance it appears that additional power supplied to the motor only increases the amplitude of the response while having little effect on the RPM of the motor

Note that the Jump phenomena and the increase in power required by a source operating near resonance are manifestation of a non-ideal energy source and are referred as the Somerfeld effect. We remarked that there is no suggestion of coupled systems (with its energy source) in which does not exist the Sommerfeld Effect in current literature, see Balthazar et all (2004), Balthazar et all (2003), Balthazar et all (2001), as well as in the classical book

Kononenko (1969). One of the problems often faced by designers is how to drive a vibrating system and avoid the Sommerfeld effect. Here, we give an example of this. Indeed, by using the Averaging Method, we prove that a centrifugal vibrator, with big inertia moment, has no Sommerfeld Effect. In the true for another non-Ideal problem, see Dantas and Balthazar, (2004), we have proved an analogous result.

We organize this paper as follows. In Section 2, we present the problem to be analyzed. In Section 3 we discuss the stability of the solutions and in Section 4 we present some discussions and concluding remarks of this paper. In Section 5 we do some acknowledgements and finally we mention the bibliographic references used.

2. A Centrifugal Vibrator with Big Moment of Inertia

Here, we consider a mechanical system excited by a DC motor, with limited supply power, which base is leaned on a spring. Besides, the DC motor rotates a small mass *m*, see Fig.1. This electro-mechanism has the main properties of a machine known as centrifugal vibrator.

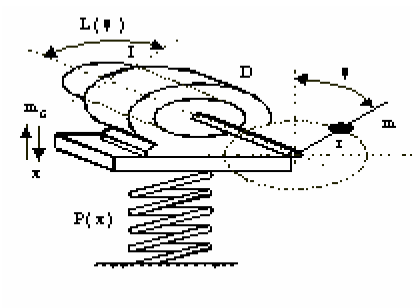

Figure 1: A Centrifugal Vibrator

The motion equations of this system were obtained before by Kononenko (1969), see page 38 . These equations are the following ones:

$$\begin{cases} m_1\,\ddot{x}+\beta\,\dot{x}+c\,x+d\,x^3 & = \quad m\,r\,\dot{\varphi}^2+m\,r\,\ddot{\varphi}\,\sin\varphi, \\ I\,\ddot{\varphi} & = \quad M(\dot{\varphi})+m\,r\,\ddot{x}\,\sin\varphi+mgr\,\sin\varphi. \end{cases} \tag{1}$$

where $m_1 = m_0 + m$ and m_0 denotes the mass of the DC motor. The resistance of the oscillatory motion is a linear force $\beta\,\dot{x}$. The constant c is the stiffness of the spring. And d the elasticity coefficient that describes how much the behavior of the spring moves away from the linear case.

For the remainder of this paper, we consider all constants that appear in Eq. (1) strictly positive. We will denote by r the distance between the mass m and the axis of rotation of the DC motor. J and $m\,r^2$ are the moment of inertia of the rotating parts of the motor and the moment of inertia of the rotating mass m, respectively. Therefore, the total moment of inertia of the system is given by $I = J + m\,r^2$. Furthermore, g denotes the acceleration of gravity. Note that the function $M(\cdot)$ is the difference between the driving torque of the source of energy (motor) and the resistive torque applied to the rotor. Such function $M(\cdot)$ is obtained from experiments.

As it is usual, we introduce a small parameter $\varepsilon > 0$ in Eq. (1) in following way: $m\,r \to \varepsilon\,m\,r, \beta \to \varepsilon\,\beta, d \to \varepsilon\,d, M(\cdot) \to \varepsilon\,M(\cdot)$. From this, we assume that $m\,g\,r \to \varepsilon\,m\,g\,r$. Now, we are interested in the situation which we have a "big motor." Such condition is included in the model by assuming that $I \to I/\varepsilon$. By using the earlier conditions in Eq.1 we get

$$\begin{cases} \quad m_1\,\ddot{x}+c\,x=\varepsilon\left(m\,r\,\dot{\varphi}^2+m\,r\,\ddot{\varphi}\,\sin\varphi\right), \\ I\,\ddot{\varphi}=\varepsilon^2\,\left(M(\dot{\varphi})+m\,r\,\ddot{x}\,\sin\varphi+mgr\,\sin\varphi\right). \end{cases} \tag{2}$$

By making $\omega^2 = c/m$, $a_2 = -\beta/m_1$, $a_3 = -d/m_1$, $a_4 = mr/m_1$, $a_5 = mr/I$, $a_6 = mgr/I$, $M_1(\dot{\varphi}) = M(\dot{\varphi})/I$ and from the following change of variables $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \varphi$, $x_4 = \dot{\varphi}$, we can write Eq. (2) as

$$\begin{cases} \dot{x}_1 = x_2\,, \\ \dot{x}_2 = -\omega^2\,x_1 + \varepsilon\left(a_4\,x_4^2\,\cos x_3 + a_4\,\dot{x}_4\,\sin x_3 + a_3\,x_3^2 + a_2\,x_2\right), \\ \dot{x}_3 = x_4\,, \\ \dot{x}_4 = \varepsilon^2\,\left(M_1(x_4) + a_5\,\dot{x}_2\,\sin x_3 + a_6\,\sin x_3\right). \end{cases} \tag{3}$$

Equation (3) can be written as

$$\left\{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -\varepsilon\, a_4 \sin x_3\\ 0 & 0 & 1 & 0\\ 0 & -\varepsilon^2 a_5 \sin x_3 & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}x_2\\ -\omega^2 x_1+\varepsilon\left(a_4 x_4^2 \cos x_3+a_4 \dot{x}_4 \sin x_3+a_3 x_3^2+a_2 x_2\right)\\ x_4\\ \varepsilon^2\left(M_1(x_4)+a_5 \dot{x}_2 \sin x_3+a_6 \sin x_3\right)\end{pmatrix}\right. . \tag{3}$$

By expanding Eq. (2) in the parameter ε we get

$$\left\{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\begin{pmatrix}x_2\\ -\omega^2 x_1+\left(a_4 x_4^2 \cos x_3+a_3 x_1^3+a_2 x_2\right)\varepsilon+O(\varepsilon^3)\\ x_4\\ \left(-a_5 (\sin x_3)\omega^2 x_1+M_1(x_4)+a_6 \sin x_3\right)\varepsilon+O(\varepsilon^3)\end{pmatrix}\right. . \tag{4}$$

Now, by making the following change of variables

$$\begin{cases}x_1=A\cos(x_3+B),\\ x_2=-\omega A\sin(x_3+B),\\ x_4=\omega+\varepsilon\Psi,\end{cases} \tag{5}$$

in Eq. (4), we obtain that

$$\left\{\begin{pmatrix}\dot{A}\\ \\ \dot{B}\\ \\ \dot{x}_3\\ \dot{\Psi}\end{pmatrix}=\begin{pmatrix}\dfrac{\varepsilon}{8\omega}\begin{pmatrix}-4a_4\omega^2 \sin B+4a_2\omega A-4a_4\omega^2 \sin(2x_3+B)-a_3A^3\sin(4x_3+4B)\\ -2a_3A^3\sin(2x_3+2B)-4a_2\omega A\cos(2x_3+2B)\end{pmatrix}+O(\varepsilon^3)\\ \dfrac{\varepsilon}{8\omega A}\begin{pmatrix}-8\omega\Psi A-4a_4\omega^2\cos B-3a_3A^3-4a_4\omega^2\cos(2x_3+B)\\ -4a_3A^3\cos(2x_3+2B)-a_3A^3\cos(4x_3+4B)+4a_2\omega A\sin(2x_3+2B)\end{pmatrix}+O(\varepsilon^3)\\ \omega+\varepsilon\Psi\\ \varepsilon\left(\dfrac{1}{2}a_5\omega^2 A\sin B+M_1(\omega)+a_6\sin x_3-\dfrac{1}{2}a_5\omega^2 A\sin(2x_3+B)\right)+O(\varepsilon^2)\end{pmatrix}\right. . \tag{6}$$

We would like to emphasize that Eq. $(5)_3$ takes into account the resonance condition.

We obtain, from Eq. (6), the following reduced system:

$$\begin{pmatrix}\dfrac{dA}{dx_3}\\ \dfrac{dB}{dx_3}\\ \dfrac{d\Psi}{dx_3}\end{pmatrix}=\varepsilon\begin{pmatrix}\dfrac{1}{8\omega^2}\begin{pmatrix}-4a_4\omega^2\sin B+4a_2\omega A-4a_4\omega^2\sin(2x_3+B)-a_3A^3\sin(4x_3+4B)\\ -2a_3A^3\sin(2x_3+2B)-4a_2\omega A\cos(2x_3+2B)\end{pmatrix}\\ \dfrac{1}{8\omega^2 A}\begin{pmatrix}-8\omega\Psi A-4a_4\omega^2\cos B-3a_3A^3-4a_4\omega^2\cos(2x_3+B)\\ -4a_3A^3\cos(2x_3+2B)-a_3A^3\cos(4x_3+4B)+4a_2\omega A\sin(2x_3+2B)\end{pmatrix}\\ \dfrac{1}{\omega}\left(\dfrac{1}{2}a_5\omega^2 A\sin B+M_1(\omega)+a_6\sin x_3-\dfrac{1}{2}a_5\omega^2 A\sin(2x_3+B)\right)\end{pmatrix}+O(\varepsilon^2). \tag{7}$$

By using the Averaging Method, see Guckenheimer and Holmes (1983), we get from Eq. (8) the following system :

$$\begin{pmatrix}\dfrac{d\bar{A}}{dx_3}\\ \dfrac{d\bar{B}}{dx_3}\\ \dfrac{d\bar{\Psi}}{dx_3}\end{pmatrix}=\varepsilon\begin{pmatrix}\dfrac{1}{2\omega}\left(-a_4\omega\sin\bar{B}+a_2\bar{A}\right)\\ \dfrac{1}{8\omega^2\bar{A}}\left(-8\omega\bar{\Psi}\bar{A}-4a_4\omega^2\cos\bar{B}-3a_3\bar{A}^3\right)\\ \dfrac{1}{\omega}\left(\dfrac{1}{2}a_5\omega^2\bar{A}\sin\bar{B}+M_1(\omega)\right)\end{pmatrix}. \tag{8}$$

Here, $\bar{A}$, $\bar{B}$, $\bar{\Psi}$ denote the averaged functions of A, B, Ψ respectively. As it is well known, see Guckenheimer and Holmes (1983), hyperbolic equilibrium points of Eq. (8) correspond to hyperbolic periodic orbits of Eq. (7).

3. A Stability Analysis

The equilibrium points $\left(\bar{A}_0, \bar{B}_0, \bar{\Psi}_0\right)$ and $\left(\bar{A}_0, \pi-\bar{B}_0, \bar{\Psi}_1\right)$ of Eq. (8) are given by:

$$
\begin{aligned}
&\bar{A}_0=\sqrt{\frac{2\,a_4\,M_1(\omega)}{\omega\,b_2\,a_5}}\ \text{where}\ b_2=-a_2>0,\\
&\sin\bar{B}_0=-\frac{1}{\omega}\sqrt{\frac{2\,b_2\,M_1(\omega)}{\omega\,a_4\,a_5}}\ \text{where}\ \cos\bar{B}_0>0,\\
&\bar{\Psi}_0=\frac{3\,b_3\,\bar{A}_0^3-4\,a_4\,\omega^2\cos\bar{B}_0}{8\,\omega\,\bar{A}_0}\ \text{where}\ b_3=-a_3>0,\\
&\bar{\Psi}_1=\frac{3\,b_3\,\bar{A}_0^3+4\,a_4\,\omega^2\cos\bar{B}_0}{8\,\omega\,\bar{A}_0}.
\end{aligned}
\tag{9}
$$

By making, the linearization of Eq. (8) in the equilibrium point $\left(\bar{A}_0, \bar{B}_0, \bar{\Psi}_0\right)$ we get the following linear system:

$$
\begin{pmatrix} \dfrac{d\bar{A}}{dx_3} \\ \dfrac{d\bar{B}}{dx_3} \\ \dfrac{d\bar{\Psi}}{dx_3} \end{pmatrix}
=\varepsilon\begin{pmatrix}
-\dfrac{b_2}{2\,\omega} & -\dfrac{a_4\cos\bar{B}_0}{2} & 0 \\
\dfrac{3\,b_3\,\bar{A}_0^3+2\,\omega^2\,a_4\cos\bar{B}_0}{4\,\omega^2\,\bar{A}_0^2} & \dfrac{a_4\sin\bar{B}_0}{2\,\bar{A}_0} & -\dfrac{1}{\omega} \\
\dfrac{\omega\,a_5\sin\bar{B}_0}{2} & \dfrac{\omega\,a_5\,\bar{A}_0\cos\bar{B}_0}{2} & 0
\end{pmatrix}
\begin{pmatrix} \bar{A} \\ \bar{B} \\ \bar{\Psi} \end{pmatrix}.
\tag{10}
$$

The characteristic polynomial of the matrix given in Eq. (10) is $P(\lambda)=\lambda^3+A_1\,\lambda^2+A_2\,\lambda+A_0$, where

$$
\begin{aligned}
&A_1=\varepsilon\,\frac{b_2\,\bar{A}_0-\omega\,a_4\sin\bar{B}_0}{2\,\omega\,\bar{A}_0},\\
&A_2=\varepsilon^2\,\frac{\left(4\,\omega^2\,a_5+3\,a_4\,b_3\right)\left(\cos\bar{B}_0\right)\bar{A}_0^3+2\,a_4\,\omega\left(\omega\,a_4\cos^2\bar{B}_0-b_2\left(\sin\bar{B}_0\right)\bar{A}_0\right)}{8\,\omega^2\,\bar{A}_0^2},\\
&A_3=\varepsilon^3\,\frac{a_5\left(\cos\bar{B}_0\right)\left(b_2\,\bar{A}_0-\omega\,a_4\sin\bar{B}_0\right)}{4\,\omega}.
\end{aligned}
\tag{11}
$$

From Eq. (11) a long but straightforward algebraic computation gives

$$
A_1\,A_2-A_3=\varepsilon^3\,\frac{a_4\left(b_2\,\bar{A}_0-\omega\,a_4\sin\bar{B}_0\right)\left(2\,\omega^2\,a_4\cos^2\bar{B}_0-2\,\omega\,b_2\left(\sin\bar{B}_0\right)\bar{A}_0+3\,b_3\left(\cos\bar{B}_0\right)\bar{A}_0^3\right)}{16\,\omega^3\,\bar{A}_0^3}.
\tag{12}
$$

Therefore, from Eq. (9), (11) (12) we conclude that $A_1>0$, $A_1\,A_2-A_3>0$ and $A_3>0$. By using the classical Hurwitz criterion, see Meirovitch (1970), we conclude that all roots of $P(\lambda)$ have negative real parts. Hence, it follows from Averaging Theorem, see Guckenheimer and Holmes (1983), that Eq. (7) has an asymptotically stable periodic orbit.

By using the same steps for the equilibrium point $(\bar{A}_0, \pi-\bar{B}_0, \bar{\Psi}_1)$, we obtain that $A_3<0$. Consequently, in the phase space of Eq. (7) there is an unstable hyperbolic periodic orbit.

4. Analysis of the Results and Concluding Remarks

As we are considering a DC motor, the parameter $M_1(\omega)$ can be interpreted as depending of the applied voltage. Moreover, from Eq. (9), we necessarily have that

$$0 < M_1(\omega) \le \frac{\omega^3 a_4 a_5}{2 b_2}. \tag{13}$$

Note that Eq. (13) is a condition for the existence of self-oscillations of Eq. (7). Let us denote the maximum value of $M_1(\omega)$ by k

From Eq. $(9)_2$, which holds for the both cases, we get $1 - cos^2 \vec{B}_0 = k M_1(\omega)$. In view of the earlier Section , we obtain the following diagram:

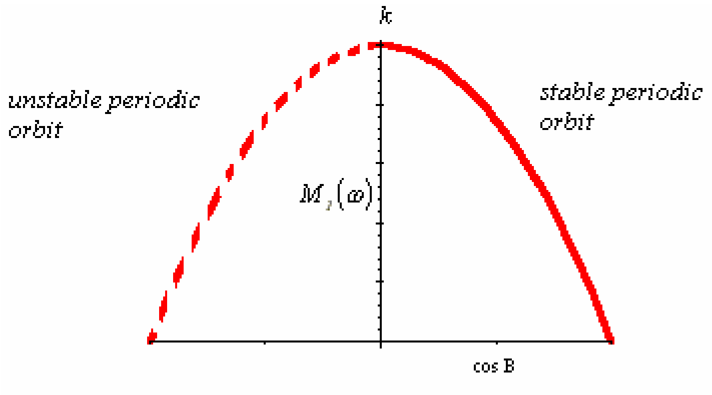

Figure 2 : Stability Diagram

By using Eq. $(9)_{3,4}$ we can plot the curve frequency x amplitude.

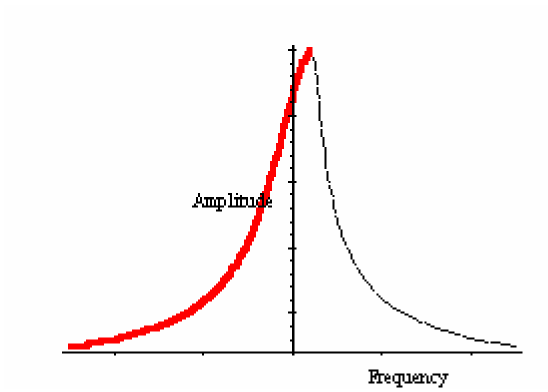

Figure 3: Frequency x Amplitude curve

In Fig. (3), the red line denotes the points on the graph where we have stability, in the other part we have unstability. Moreover, we have no jump in the amplitude, in other words, there is no Sommerfeld Effect.

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