Proceedings of the XI DINAME, 28th February-4th March 2005, Ouro Preto - MG - Brazil
Edited by D.A. Rade and V. Steffen Jr.

A 7-DEGREE-OF-FREEDOM MATHEMATICAL MODEL OF A VEHICLE SUBJECTED TO RANDOM PAVEMENT IRREGULARITIES

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Abstract. *This paper is part of a project in which we aim to study some aspects of the dynamics of a class of road vehicles for light loads transportation. Specially, we research their response to ground excitation due to irregular pavement conditions. It is an important subject in Brazilian reality. The technology used in our automotive industry is mostly imported from the main offices of controlling multinational enterprises. Nevertheless, a special branch of this industry is very Brazilian: that of bodybuilding for vehicles for load and mass transportation. This branch is in need of larger investments in technological research. In this paper, the vehicle is modeled as a seven-degree-of-freedom lumped parameter system: the two independent suspension front wheels, the two degrees of freedom of the rigid rear axis and three degrees of freedom of the suspended mass. The linear Equations of Motion (due to the linearization hypothesis of small displacements) are deduced via Lagrange's Equations. The support excitation due to pavement irregularities is included in this model.*

Keywords*: random vibrations, vehicle dynamics, pavement irregularities*

1. Introduction

In its 50 years history, Brazilian automotive industry has imported most of its technology from the headquarters of the multinational concerns that control the industry.

Nevertheless, a sector of the industry is very Brazilian, that of bodybuilding of trucks and busses. Here larger investments in research and local technology are needed. One of the most interesting branches of these researches is vehicle dynamics.

Vehicle dynamics is concerned with the motions to which a certain vehicle and its parts are submitted due to the several applied actions. One of the several concerns of this discipline is the support excitation due to the random road surface irregularities.

It is the authors' view that here we have a common point of interest between mechanical engineering, particularly dynamics and control, with some aspects related to the environment. In fact, that is an excitation originated in the environment in which the vehicle runs and, as most of this kind of actions, it is random in nature and can only be properly taken in account in terms of statistical concepts such as average values, variances, probability distributions, spectral densities, etc.

Here, we prefer to analyze a model with a small number of degrees of freedom to better understand the underlying concepts, leaving to future work the use of the Finite Elements Method. A more complete description of the model is to be found in the Master Dissertation of the second author, Colombo (2001).

Due to the random nature of the excitations, this research is based on the theory of random vibrations. Good introductions to the field are to be found in Brasil (1993) and in the classic textbook by Newland (1993).

The most intricate phase of the work is the mathematical modeling of irregularities of usual road surfaces. We adopt a power spectrum of these irregularities, considered to be a stationary, ergodic and gaussian random process. To that purpose, we borrowed heavily from Wong (1978) and Costa Neto (2000).

Our proposition is to perform a Monte Carlo type simulation of the support excitation based on the chosen power spectrum. A large number of loading time histories is generated by superposition of harmonic functions whose amplitudes are taken from the spectrum for each of the several frequency ranges it is divided. The phases are randomly set. The idea is inspired by the so-called "synthetic Wind", an analogous procedure to analyze the wind effect on tall civil structures proposed by Franco (1993).

2. The Mathematical Model

2.1 Model description

Our structure is a road vehicle for lightweight transport.

We consider a X , Y , Z reference frame. Our model comprises a suspended mass m_7 with moments of inertia I_5 and I_6, with respect to axis X , Y, supported by suspension setups (spring and damper) and tires.

The two forward suspensions are of the independent type and the rear suspensions are united by a rigid axis.

The forward wheels 1 and 2 are modeled as lumped masses, m_1 and m_2 respectively and the rear wheels are part of the rear rigid axis with mass m_3 and moment of inertia about X axis, I_4.

Figures 1 and 2 present a schematic representation of the vehicle.

The spring are supposed to have linear behavior with stiffness coefficients

K_{sf} = forward suspension stiffness

K_{st} = rear suspension stiffness

The dampers are supposed to have viscous linear behavior with damping coefficients

C_{sf} = forward suspension damping coefficient

C_{st} = rear suspension damping coefficient

Tires are considered to have stiffness given by

K_{pf} = forward tire stiffness

K_{pt} = rear tire stiffness

The damping coefficients of the tires are:

C_{pf} = forward tire damping coefficient

C_{pt} = rear tire damping coefficient

The ground irregularities are represented by the following variables

w_i = support excitation of a certain wheel:

w_1 : Vertical displacement under left forward wheel

w_2 : Vertical displacement under right forward wheel

w_3 : Vertical displacement under left rear wheel

w_4 : Vertical displacement under right rear wheel

As presented in the figure, the model comprises 4 masses and 7 degrees of freedom.

The chosen degrees of freedom are as follows:

Main Mass (m_7):

- vertical motion q_7 (Heave)
- rotation q_5 about X axis (Roll)
- rotation q_6 about Y axis (Pitch)

Rear Mass (m_3)

- vertical motion q_3
- rotation q_4, rear axis rotation about X axis

Forward Masses (m_1 and m_2) :

- vertical motion q_1 and q_2

Important dimensions :

a_f - Half distance along axis Y of forward wheel and damper/spring set with respect to the vehicle CG .

a_t - Half distance along axis Y of the rear wheel damper/spring set with respect to the vehicle CG .

b_f - Distance along axis X of the forward wheel with respect to the vehicle CG .

b_t - Distance along axis X of the rear wheel with respect to the vehicle CG .

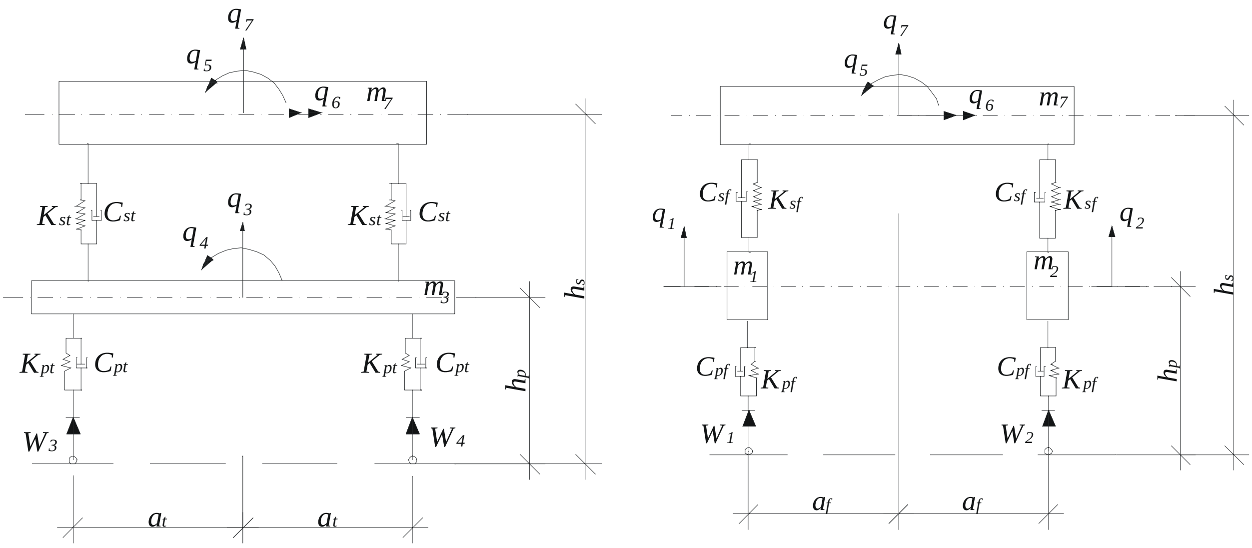

Figure 1: rear and frontal view of the model

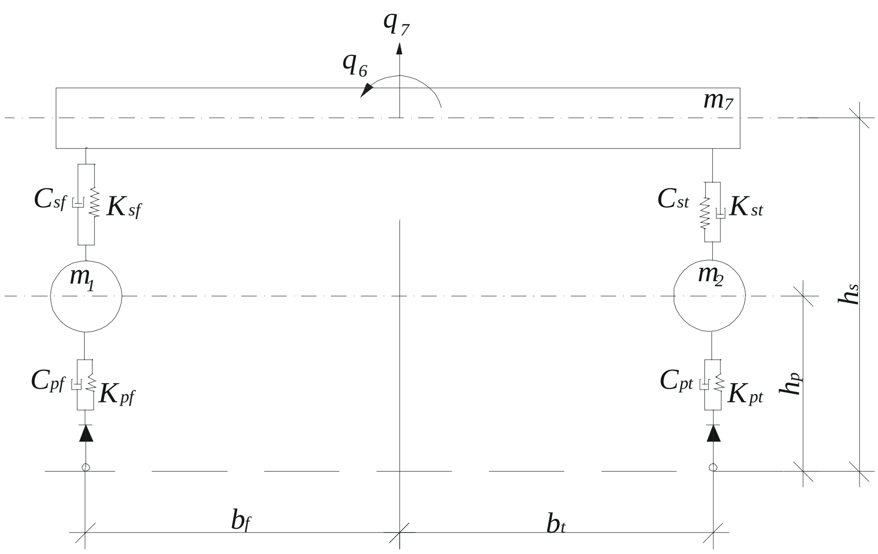

Figure 2: lateral view of the model

2.2 Derivation of Equations of motion

A convenient way to derive the equations of motion are the Lagrange Equations.
We need N independent coordinates to describe the motion of a N degrees of freedom system.
These are the so called generalized coordinates here represented by q_i.
The Lagrange Equations are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i + p_{ci} \tag{1}$$

Q_i = non-conservative generalized forces
p_{ci} = conservative generalized forces
$i = 1,2,\ldots,N$

where N is the number of degrees of freedom
Equation (1) allows analyses of non-conservative systems as our damped model.

As in our case, the kinetic energy *T* depends only on the velocities $\frac{\partial T}{\partial q_i} = 0$

Thus, Lagrange equations reduce to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = Q_i + p_{ci} \tag{2}$$

2.2.Kinetic Energy

For our model, the kinetic energy is

$$T = \frac{1}{2}\left[m_1\dot{q}_1^2 + m_2\dot{q}_2^2 + m_3\dot{q}_3^2 + I_4\dot{q}_4^2 + I_5\dot{q}_5^2 + I_6\dot{q}_6^2 + m_7\dot{q}_7^2\right] \tag{3}$$

where m_i represent the masses, I_i the moments of inertia and $\dot{q}_i$ the velocities

2.2.2 Contributions to the Lagrange equations

The contributions to Lagrange equations given by the kinetic energy, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$ for each degree of freedom, lead to the Mass Matrix:

$$M = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & m_3 & & & & \\ & & & I_4 & & & \\ & & & & I_5 & & \\ & & & & & I_6 & \\ & & & & & & m_7 \end{bmatrix} \tag{4}$$

2.2.3 Strain Energy

We consider small rotations (linearization) leading to Strain Energy given by

$$U = \frac{1}{2}Kx^2 \tag{4}$$

where K is the stiffness of the springs of our model.

$$U = \frac{1}{2}\left\{\begin{aligned} & K_{pf}\left[(q_1 - w_1)^2 + (q_2 - w_2)^2\right] + \\ & K_{pt}\left[(q_3 - a_t q_4 - w_3)^2 + (q_3 + a_t q_4 - w_4)^2\right] + \\ & K_{sf}\left[(q_1 + a_f q_5 + b_f q_6 - q_7)^2 + (q_2 - a_f q_5 + b_f q_6 - q_7)^2\right] + \\ & K_{st}\left[(q_3 - a_f q_4 + a_f q_5 - b_f q_6 - q_7)^2 + (q_3 + a_f q_4 - a_f q_5 - b_f q_6 - q_7)^2\right] \end{aligned}\right\} \tag{5}$$

2.2.4 Work of conservative forces

These forces are due to gravity (self weight), that is, in the negative vertical direction. As the coordinates increase in the opposite sense, the work is always negative:

$$W_c = Uq_i \tag{6}$$

$$W_c = -g\left(m_1 q_1 + m_2 q_2 + m_3 q_3 + m_7 q_7\right) \tag{7}$$

2.2.5 Total Potential Energy

$$V = U - W_c \tag{8}$$

2.2.6 Contributions to the Lagrange equations

The contributions to Lagrange equations given by the potential energy, $\frac{\partial V}{\partial q_i}$ for each degree of freedom, lead to the Stiffness Matrix:

$$\mathbf{K} = \begin{bmatrix} K_{pf} + K_{sf} & 0 & 0 & 0 & a_f K_{sf} & b_f K_{sf} & -K_{sf} \\ 0 & K_{pt} + K_{sf} & 0 & 0 & -a_f K_{sf} & b_f K_{sf} & -K_{sf} \\ 0 & 0 & 2\left(K_{pt} + K_{st}\right) & 0 & 0 & -b_t K_{st} & -2K_{st} \\ 0 & 0 & 0 & 2\left(a_t^2 K_{pt} + a_f^2 K_{st}\right) & -a_f^2 K_{st} & 0 & 0 \\ a_f K_{sf} & -a_f K_{sf} & 0 & -2a_t^2 K_{st} & a_f^2\left(K_{sf} + K_{st}\right) & 0 & 0 \\ b_f K_{sf} & b_f K_{sf} & -2b_t K_{st} & 0 & 0 & 2\left(b_f^2 K_{sf} + b_t^2 K_{st}\right) & 2\left(b_t K_{st} - b_f K_{sf}\right) \\ -K_{sf} & -K_{sf} & -2K_{st} & 0 & 0 & 2\left(b_t K_{st} - b_f K_{sf}\right) & 2\left(K_{sf} - K_{st}\right) \end{bmatrix} \tag{9}$$

2.2.7 Generalized Non-Conservative forces

Using the Virtual Work Principle in terms of generalized coordinates we get the non-conservative generalized forces as a sum of dot products:

$$Q_i = \sum_j \vec{F}_j \cdot \frac{\partial \vec{R}_j}{\partial q_i} \tag{10}$$

where $j = 1, 2, \ldots, M$

M = number of applied forces

$\vec{F}_j$ = applied forces

$\vec{R}_j$ = position vector of the applied forces

As we consider a linearized system, due to small rotations, the dumping forces may be considered applied in only the vertical direction (z axis) and their position vector will have only vertical components.

Thus:

$$Q_i = \sum_j Z_j \cdot \frac{\partial z_j}{\partial q_i} \tag{11}$$

Z_j are the vertical components of forces

z_j are the z components of position vectors of the force j

Figure 3 displays those components.

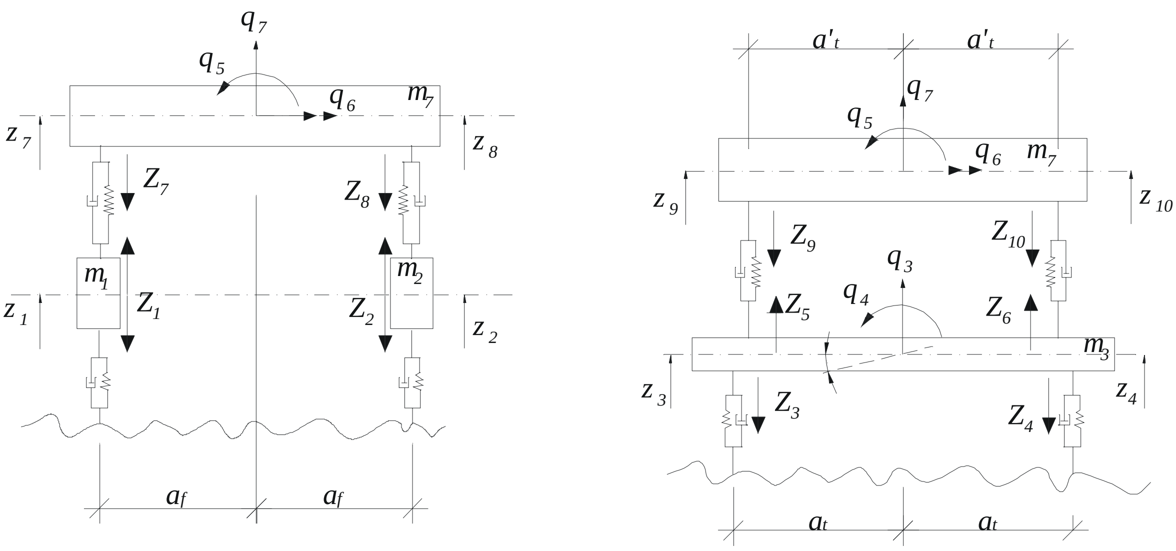

Figure 3: non-conservative forces and their virtual displacements, frontal and rear view of the model

The Generalized non-conservative forces are:

$$
\begin{aligned}
Q_1 &= -C_{pf}\left(\dot{q}_1-\dot{w}_1\right)+C_{sf}\left(\dot{q}_7-\dot{q}_1-a_f\dot{q}_5-b_f\dot{q}_6\right)\\
Q_2 &= -C_{pf}\left(\dot{q}_2-\dot{w}_2\right)+C_{sf}\left(\dot{q}_7-\dot{q}_2-a_f\dot{q}_5-b_f\dot{q}_6\right)\\
Q_3 &= -C_{pt}\left[\left(\dot{q}_3-a_t\dot{q}_4-\dot{w}_3\right)+\left(\dot{q}_3+a_t\dot{q}_4-\dot{w}_4\right)\right]+\\
&\qquad C_{st}\left[\left(\dot{q}_7-\dot{q}_3+a_f\dot{q}_4-a_f\dot{q}_5+b_t\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_3-a_f\dot{q}_4+a_f\dot{q}_5+b_t\dot{q}_6\right)\right]\\
Q_4 &= a_tC_{pt}\left[\left(\dot{q}_3-a_t\dot{q}_4-\dot{w}_3\right)-\left(\dot{q}_3+a_t\dot{q}_4-\dot{w}_4\right)\right]+\\
&\qquad a_fC_{st}\left[-\left(\dot{q}_7-\dot{q}_3+a_f\dot{q}_4-a_f\dot{q}_5+b_t\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_3-a_f\dot{q}_4+a_f\dot{q}_5+b_t\dot{q}_6\right)\right]\\
Q_5 &= a_f\left\{\begin{matrix}C_{sf}\left[\left(\dot{q}_7-\dot{q}_1-a_f\dot{q}_5-b_f\dot{q}_6\right)-\left(\dot{q}_7-\dot{q}_2+a_f\dot{q}_5-b_f\dot{q}_6\right)\right]+\\ C_{st}\left[\left(\dot{q}_7-\dot{q}_3+a_f\dot{q}_4-a_f\dot{q}_5+b_t\dot{q}_6\right)-\left(\dot{q}_7-\dot{q}_3-a_f\dot{q}_4+a_f\dot{q}_5+b_t\dot{q}_6\right)\right]\end{matrix}\right\}\\
Q_6 &= b_f\left\{\begin{matrix}C_{sf}\left[\left(\dot{q}_7-\dot{q}_1-a_f\dot{q}_5-b_f\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_2+a_f\dot{q}_5-b_f\dot{q}_6\right)\right]+\\ -b_tC_{st}\left[\left(\dot{q}_7-\dot{q}_3+a_f\dot{q}_4-a_f\dot{q}_5+b_t\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_3-a_f\dot{q}_4+a_f\dot{q}_5+b_t\dot{q}_6\right)\right]\end{matrix}\right\}\\
Q_7 &= -C_{sf}\left[\left(\dot{q}_7-\dot{q}_1-a_f\dot{q}_5-b_f\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_2+a_f\dot{q}_5-b_f\dot{q}_6\right)\right]+\\
&\qquad -C_{st}\left[\left(\dot{q}_7-\dot{q}_3+a_f\dot{q}_4-a_f\dot{q}_5+b_t\dot{q}_6\right)+\left(\dot{q}_7-\dot{q}_3-a_f\dot{q}_4+a_f\dot{q}_5+b_t\dot{q}_6\right)\right]
\end{aligned}
\tag{12}
$$

Leading to the Damping Matrix

$$
C=\begin{bmatrix}
C_{pf}+C_{sf} & 0 & 0 & 0 & a_fC_{sf} & b_fC_{sf} & -C_{sf}\\
0 & C_{pt}+C_{sf} & 0 & 0 & -a_fC_{sf} & b_fC_{sf} & -C_{sf}\\
0 & 0 & 2\left(C_{pt}+C_{st}\right) & 0 & 0 & -b_tC_{st} & -2C_{st}\\
0 & 0 & 0 & 2\left(a_t^2C_{pt}+a_f^2C_{st}\right) & -a_f^2C_{st} & 0 & 0\\
a_fC_{sf} & -a_fC_{sf} & 0 & -2a_t^2C_{st} & a_f^2\left(C_{sf}+C_{st}\right) & 0 & 0\\
b_fC_{sf} & b_fC_{sf} & -2b_tC_{st} & 0 & 0 & 2\left(b_f^2C_{sf}+b_t^2C_{st}\right) & 2\left(b_tC_{st}-b_fC_{sf}\right)\\
-C_{sf} & -C_{sf} & -2C_{st} & 0 & 0 & 2\left(b_tC_{st}-b_fC_{sf}\right) & 2\left(C_{sf}-C_{st}\right)
\end{bmatrix}
\tag{13}
$$

2.2.8 Load vectors

Vector of conservative loads

$$\mathbf{p}_c = \begin{bmatrix} K_{pf} w_1 - m_1 g \\ K_{pf} w_2 - m_2 g \\ K_{pt}(w_3 + w_4) - m_3 g \\ a_t K_{pt}(w_3 + w_4) \\ 0 \\ 0 \\ m_7 g \end{bmatrix} \tag{14}$$

Vector of non-conservative loads

$$\mathbf{p}_{nc} = \begin{bmatrix} C_{pf} \dot{w}_1 \\ C_{pf} \dot{w}_2 \\ C_{pt}(\dot{w}_1 + \dot{w}_2) \\ a_t C_{pt}(\dot{w}_1 + \dot{w}_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

Full Loads Vector

$$\mathbf{p} = \mathbf{p}_c + \mathbf{p}_{nc} \tag{16}$$

$$\mathbf{p} = \begin{bmatrix} K_{pf} w_1 - m_1 g + C_{pf} \dot{w}_1 \\ K_{pf} w_2 - m_2 g + C_{pf} \dot{w}_2 \\ K_{pt}(w_3 + w_4) - m_3 g + C_{pt}(\dot{w}_1 + \dot{w}_2) \\ a_t K_{pt}(w_3 + w_4) + a_t C_{pt}(\dot{w}_1 + \dot{w}_2) \\ 0 \\ 0 \\ m_7 g \end{bmatrix} \tag{17}$$

3. A Monte Carlo type Simulation of Road Surface Irregularities

The ground profile of a road may be regarded as a random superposition of harmonic functions. The practical acquisition of this information is usually performed using accelerometers mounted in test vehicles. The generated electric analog signal is afterwards transformed into digital signals to generate power spectra.

A power spectrum, or, more exactly, a power spectral density function is, by definition, the Fourier Transform of the Autocorrelation function of the signal. It allows for a practical evaluation of the frequency that certain amplitude levels repeat in the signal.

For our application, road irregularities, one of the most used spectra is the one given by Wong (1978), defined as:

$$S(\Omega) = C_{sp} \Omega^{-N_{sp}} \tag{18}$$

where $S(\Omega)$ is the spectral density in *m²/cycle/m*, Ω is the frequency (in cycles/m), and C_{sp} and N_{sp} are constants related to the surface type. The domain of the function is also related to the type of surface.

One must be careful to note that Ω, in this application, is the *spatial frequency* or *wave number*, related to the *time frequency f* (in Hz) or *circular frequency* ω (in rad/s) by

$$f = \Omega V_x \tag{19}$$

$$\omega = 2\pi\Omega V_x \tag{20}$$

where V_x is the velocity of the vehicle (in m/s).

The discretization algorithm we use to simulate the excitation histories is a superposition of harmonic functions in the form:

$$w(x) = \sum_{i=1}^{nh} A_i \operatorname{sen}(\Omega_i x + \phi_i) \tag{21}$$

where *w(x)* is the composed function that simulates the surface vertical irregularities at a *x* position along the road, A_i is the amplitude obtained from the spectrum for each one of the *nh* discrete frequencies Ω_i we arbitrarily divide the spectrum, and ϕ_i is the randomly drawn phase of a particular harmonic function.

The amplitude for each frequency band the spectrum is arbitrarily divided is given by:

$$A_i^2 = \int_{\Omega_{i1}}^{\Omega_{i2}} S(\Omega)d\Omega \tag{22}$$

$$A_i^2 = \int_{\Omega_{i1}}^{\Omega_{i2}} C_{sp}\Omega^{-N_{sp}} d\Omega \tag{23}$$

$$A_i = \sqrt{\frac{C_{sp}}{-N_{sp}+1}(\Omega_{i2}^{-N_{sp}+1} - \Omega_{i1}^{-N_{sp}+1})} \tag{24}$$

where Ω_{i1} and Ω_{i2} are, respectively, the lower and upper frequencies of the band. The average value of the frequency of the particular harmonic function is

$$\Omega_i = \sqrt{\Omega_{i1}\Omega_{i2}} \tag{25}$$

As suggested by Franco (1993) we have adopted 12 bands.

To avoid undesired resonance, phase angles are pseudo-randomly set for each function and for the left and right sides of the vehicle.

4. Numerical integration

Due to the complexity of the problem we chose to use numerical step-by-step integration of the differential equations. We adopted the Constant Average Acceleration Newmark Method, which, for linear systems, is unconditionally stable.

To assure accuracy, one must take special care in setting the integration time step. Further, we must be sure to capture the effect of the chosen higher frequency harmonic loading function. A practical rule of thumb given by authors, such as Clough and Penzien (1994), is to use at least 10 steps to reproduce the period of the higher frequency, that is, to adopt at least one tenth of this period as our time step.

5. A complete example

We present a sample analysis of a lightweight road vehicle produced by Agrale SA, Caxias do Sul, RS, Brazil. Total suspended mass is $m_7 = 3738$ kg comprising self-weight, payload, motor, fuel and other masses. The masses of the two front suspensions and the rear axis, including springs, are, respectively, $m_1 = 140kg$, $m_2 = 140kg$ and $m_3 = 398kg$.

Moments of inertia of the rear axis about the X global axis and of the suspended mass about the X and Y global axis are, respectively, $I_4 = 206{,}4kgm^2$, $I_5 = 1712kgm^2$, and $I_6 = 8086kgm^2$.

The springs stiffness and the damping coefficients of the dampers, as given by the manufactures are: K_{sf} *= 120000 N/m,* K_{st} *= 140000 N/m,* C_{sf} *= 16192 Ns/m,* C_{st} *= 17400 Ns/m.*

The stiffness and damping coefficients of the tires, as given by the manufactures are: K_{pf} *= 530000 N/m,* K_{pt} *= 530000 N/m,* C_{pf} *= 1000 Ns/m,* C_{pt} *= 1000 Ns/m.*

Dimensions of the vehicle are: a_f - 0,86 m, a_t - 0,86 m, b_f - 1,76 m, b_t - 1,04 m.

For this class of vehicle and considering unpaved roads, we adopt the following parameters for the spectrum given by Wong (1978):

$$C_{sp} = 4{,}4x10^{-6}$$

$$N_{sp} = 2{,}1$$

in a band of wave numbers $0{,}12 \leq \Omega \leq 1{,}1$. We adopted for the analysis a vehicle velocity of 50 km/h.

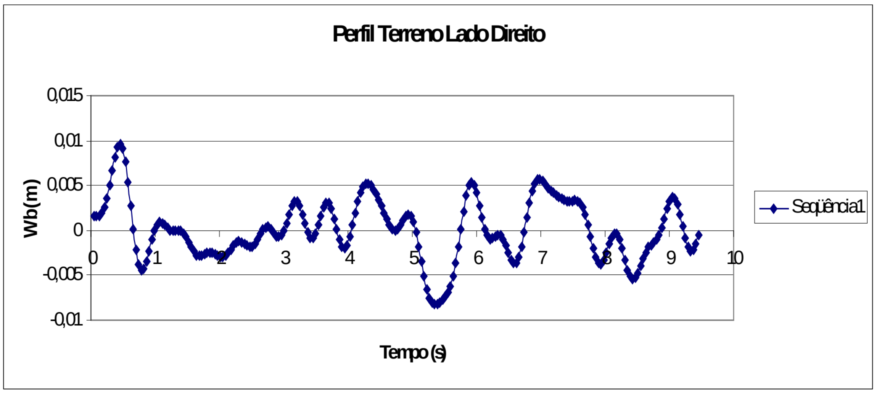

Figure 5: sample simulation of road irregularities, right side of the vehicle

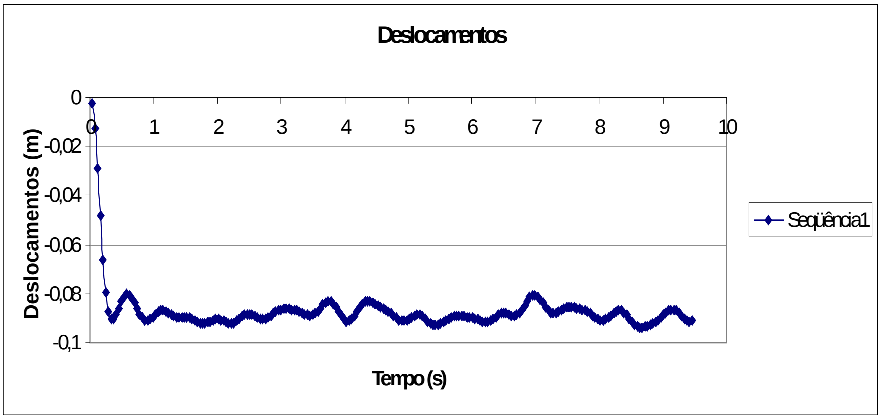

Figure 6: sample displacements response in the 7 generalized coordinate (vertical motion of the suspended mass)

We considered 230 values. Response statistics are:
Average = -0.087228m
Standard deviation = 0.009429m
If we adopt a certain probability distribution, such as the Gaussian one, we are able to determine the probability of occurrence of certain response values.
It is interesting to notice that as we performed 10 other simulations, varying the phase angles setting, little variation was detected in the average and standard deviation values. This is an evidence of an ergodic process.

6. Acknowledgements

The first author acknowledges support by FAPES and CNPq, both Brazilian research-funding agencies. The second author had a Scholarship granted by CAPES for his Master Degree and full support of Agrale SA, Caxias do Sul, RS, Brazil.

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8. Responsibility notice

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