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OSCILLATIONS OF A BEAM ON A NON-LINEAR ELASTIC FOUNDATION UNDER PERIODIC LOADS

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Abstract. *Initially, a mathematical model of a flexible beam resting on a non-linear elastic foundation is presented and its non-linear vibrations and instabilities are investigated using several numerical methods. At a second stage, a parametric study is carried out, using analytical and semi-analytical perturbation methods. So, the influence of the various physical and geometrical parameters of the mathematical model on the non-linear response of the beam are evaluated, in particular, the relation between the natural frequency and the vibration amplitude, the first period doubling and saddle-node bifurcation. These two instability phenomena are the two basic mechanism associated with the loss of stability of the beam. Finally Melnikov's method is used to determine an algebraic expression for the boundary that separates a safe from an unsafe region in the force control parameter space. This can be used as a basis for a reliable engineering design criterion.*

***Keywords**: Elastic Foundation, Non-linear Oscillations, Beams, Melnikov Method, Ramberg-Osgood Constitutive Law.*

1. Introduction

Beams on elastic foundation are a well known topic in structural mechanics, with applications in many engineering branches. In the present paper the nonlinear dynamics of an axially loaded beam-foundation system is analyzed, considering both the nonlinearity of the beam and foundation. The foundation's non-linearity is of the softening type and has a marked influence on the vibrations of the structural system. The periodic load combined with the elastic foundation's non-linearity makes the beam oscillations complex, with supercritical and sub-critical bifurcations, cascade period doubling bifurcations, leading to chaos, and saddle-node bifurcations. As a final outcome, the beam, under certain excitation frequencies, can have a load bearing capacity of only a fraction of its normal capacity when compared to static or step loads. On the other hand, at some frequencies the beam load capacity can be many times superior to its static critical load.

The complexity of the beam's response to a periodic load makes the design of this structural element particularly challenging because there is, apparently, no algebraic relation of the load bearing capacity with the geometrical and elastic parameter of the beam in design codes or technical literature.

2. Large Deflection Beam Formulation.

To analyze the beam, a non-linear large-deflection formulation is used together with the Ritz method to obtain a simplified equation of motion, which is in turn solved by approximation using several strategies.

Consider the prismatic beam of Figure 1 with length L and stiffness EI, simply supported and loaded by the axial force P which retains its magnitude and direction as the beam deflects and a time dependent transversal load $q(t)$. Additionally consider that the beam lies on an elastic foundation modeled as a set of discrete nonlinear soil springs. Experimental results (Greimann et al., 1987) show that most soils exhibit a nonlinearity of the softening type. The most frequently used mathematical functions to describe this nonlinearity are parabolas, hyperbolas, splines and the Ramberg-Osgood functions. Here the foundation non-linear restoration force is given by

$$F_{foundation} = k\,w\left(1 - g\,w^2\right) \tag{1}$$

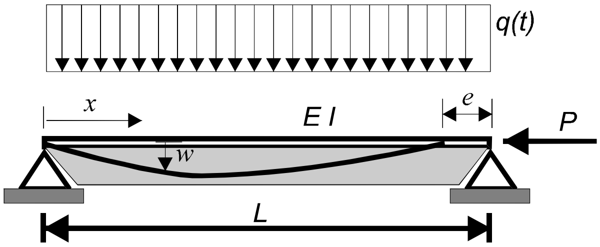


Figure 1. Beam on an Elastic Foundation

The foundation parameters *k* and γ can be described as a function of the soil characteristics. This is accomplished by comparing equation (1) with the cubic Taylor approximation of the four parameter Ramberg-Osgood model for non-linear soils proposed by Greimann *et al.* (1987) with the shape parameter *n*=1. This leads to

$$k = E_{ti} \qquad \text{and} \qquad g = \frac{(E_{ti} - E_{tf})^3}{E_{ti}\, P_u^2} \tag{2}$$

where E_{ti} is the initial tangent modulus, E_{tf} is the final tangent modulus and P_u is the ultimate soil resistance.

The transversal uniformly distributed, time dependant periodic load $q(t)$ is given by

$$q(t) = F\cos(\Omega\, t) \tag{3}$$

The beam is assumed to be inextensional with bending stiffness EI . As the beam deflects each point of the center line positioned at a distance *x* of the left corner moves to an unspecified horizontal position and a vertical position, named *w*. Since the center line is inextensional, the arc length from the left corner is equal to *x*, and the deflected form of the beam is totally specified by the mathematical function $w(x)$ where *x* ranges from 0 to *L*.

The curvature χ is, by definition, the rate of change of angle with arc length, so

$$c = \frac{w''}{\sqrt{1 - w'^2}} \tag{4}$$

The strain energy stored in a beam element of length *dx* is $dU_{beam} = M\, c\, dx/2$, where *M* is the bending moment given by $M = EIc$. So, the total strain energy stored in the beam is

$$U_{beam} = \frac{1}{2} EI \int_0^L \frac{w''^2}{1 - w'^2}\, dx \cong \frac{1}{2} EI \int_0^L w''^2 + w''^2 w'^2\, dx \tag{5}$$

Additionally the strain energy stored in a foundation element of length *dx* is

$$dU_{foundation} = \int_0^w k\, w(1 - g\, w^2)\, dw \;\; dx \tag{6}$$

And the total strain energy stored in the foundation is

$$U_{foundation} = \int_0^L \frac{1}{2} k\, w^2 \left(1 - \frac{g}{2} w^2\right) dx \tag{7}$$

The potential energy, L_P, of the conservative load *P* is given by

$$L_P = -P\, e = -P\left(L - \int_0^L \sqrt{1 - w'^2}\, dx\right) \cong -P\left(\int_0^L \frac{1}{2} w'^2 + \frac{1}{8} w'^4\, dx\right) \tag{8}$$

where *e* is the end-shortening of the column.

Let *m* be the mass per unit length of the beam, then the kinetic energy, *T*, can be expressed as:

$$T = \frac{1}{2} m \int_0^L \dot{w}^2 \, dx \tag{9}$$

The variation of the work of the transversal and damping forces can be written as

$$dW_{nc} = \int_0^L F \cos(\Omega t) - m \, \dot{w}) \, dx \; dw \tag{10}$$

The deformed shape of a simply-supported beam under small to moderately large deflections in the vicinity of the lowest natural frequency can be described fairly well by the one term approximation (Thompson, 1982; Dym and Shames, 1973):

$$w = W(t) \, sin\left(\frac{p \; x}{L} \right) \tag{11}$$

that satisfies all boundary conditions.

Using Hamilton's principle and calculus of variations

$$d \int_{t_1}^{t_2} (T - U_{beam} - U_{foundation} - L_P) dt + d \int_{t_1}^{t_2} W_{nc} dt = 0 \tag{12}$$

one obtains the following differential equation of motion:

$$\frac{L}{2} m \ddot{W} + \frac{2Lm}{p} \dot{W} + \left(\frac{p^4 EI}{2L^3} + \frac{L\,E_{ti}}{2} - \frac{p^2}{2\,L} P \right) W - \left(\frac{3}{8} L \frac{(E_{ti} - E_{tf})^3}{P_u^2} + \frac{3p^4}{16\,L^3} P - \frac{p^6 EI}{4\,L^5} \right) W^3 = L\,F \cos(\Omega t) \tag{13}$$

which can be rewritten in a more concise form as

$$\ddot{W} + c\dot{W} + w_0^2 \, W - b \, W^3 = f \cos(\Omega t) \tag{14}$$

where:

$$c = \frac{4\,m}{p \; m} \qquad w_0^2 = \left(\frac{p}{L} \right)^4 \frac{EI}{m} + \frac{E_{ti}}{m} - \left(\frac{p}{L} \right)^2 \frac{P}{m}$$

$$b = \frac{3}{4} \frac{(E_{ti} - E_{tf})^3}{P_u^2 \; m} + \frac{3}{8} \left(\frac{p}{L} \right)^4 \frac{P}{m} - \frac{1}{2} \left(\frac{p}{L} \right)^6 \frac{EI}{m} \qquad f = \frac{2\,F}{m} \tag{15}$$

3. A Practical Example

Consider a circular cylindrical steel pile with thickness *t= 1 cm*; external diameter *D = 10 cm,* length *L* = 20 *m*, Young modulus E_{steel} = 210 GPA, mass density, *m* = 20 kg/m and bending stiffness $EI = 5672067 \;\; N\,m^2$. Considering a stiff clay, the parameters used for the Ramberg-Osgood Constitutive Law are (Greimann *et al.,* 1987): initial tangent modulus - E_{ti} = 4037 *Pa* ; final tangent modulus - E_{tf} = 0; shape parameter - *n* = 1 and ultimate soil resistance - *Pu* = 50 *N/m*.

Based on these values, one obtains the following reference parameters:

- Lowest natural frequency – 17 *rad/s*;
- Euler buckling load - $P_{euler} = (p/L)^2 \, E\,I = 139953\,N$

For this particular case, the differential equation of motion (14) takes the form

$$\ddot{W} + 0.34 \; \dot{W} + 289\,W - 986886 \; W^3 = f \cos(\Omega t) \tag{16}$$

where a damping parameter of about 1% of the critical damping value is adopted (m = 5,34 kg/m/s).

Considering a linear foundation model, equation (14) reduces to

$$\ddot{W} + 0.34\,\dot{W} + 289W + 1.33\,W^3 = f\cos(\Omega t) \tag{17}$$

3.1 Influence of the foundation nonlinearity on buckling and vibration

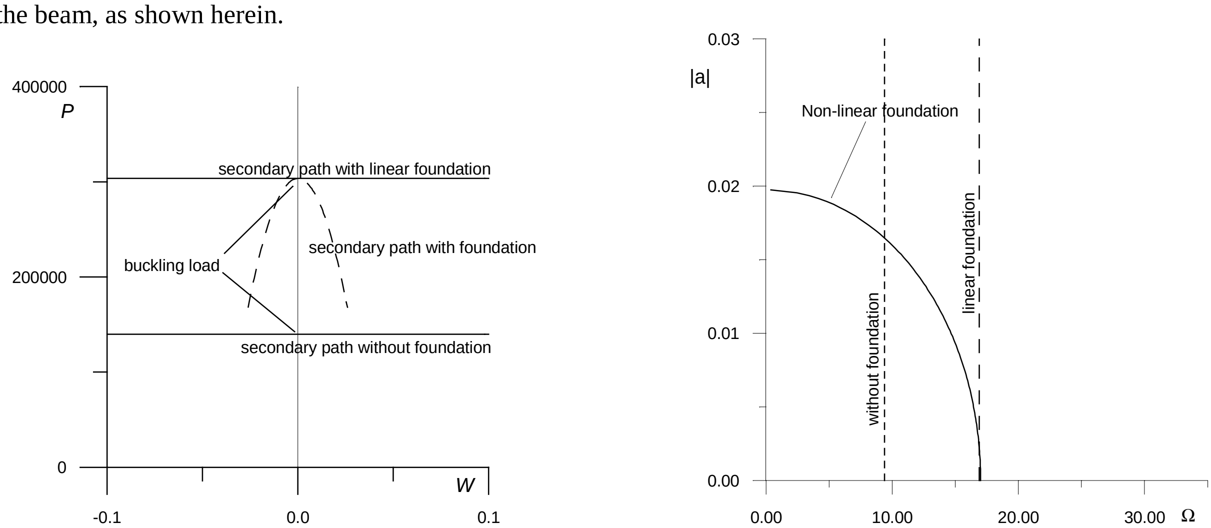
Considering only the static terms in equations (13) and (14) one obtains for the post-buckling path of the column the following equation

$$P = \left(\frac{p}{L}\right)^2 EI + E_{ti}\left(\frac{L}{p}\right)^2 - \left(\frac{L}{p}\right)^2\left[\frac{3}{4}\frac{(E_{ti}-E_{tf})^3}{P_u^2} - \frac{1}{2}\left(\frac{p}{L}\right)^6 EI + \frac{3}{8}\left(\frac{p}{L}\right)^2\left\{\left(\frac{p}{L}\right)^4 EI + E_{ti}\right\}\right]W^2 \tag{18}$$

Now considering the free vibration model, the frequency-amplitude relation is given by

$$w^2 = \frac{1}{m}\left\{\left(\frac{p}{L}\right)^4 EI + \left(\frac{p}{L}\right)^2 P + E_{ti} - \left[\frac{3}{4}\frac{(E_{ti}-E_{tf})^3}{P_u^2} - \frac{1}{2}\left(\frac{p}{L}\right)^6 EI + \frac{3}{8}\left(\frac{p}{L}\right)^2\left\{\left(\frac{p}{L}\right)^4 EI + E_{ti}\right\}\right]W^2\right\} \tag{19}$$

As observed in Figure 2 the degree of non-linearity of the post-buckling path and of the frequency-amplitude response of a beam without foundation or considering a linear foundation model is rather small. In fact the linear foundation only increases the critical load and natural frequency, without affecting, as expected, the non-linear term in Eq. (14). On the other hand, if the softening behavior is considered, the post-buckling path and the frequency-amplitude relation becomes highly non-linear due to the decreasing stiffness of the beam-foundation system. As a result the beam becomes imperfection sensitive. This softening effect is the dominant factor in the local and global non-linear dynamics of the beam, as shown herein.



(a) (b)

Figure 2 – Influence of the foundation on (a) the buckling and post-buckling behavior of the beam and (b) on the frequency-amplitude relation.

4. Elements of non-linear vibration and escape mechanisms

Special cases of Eq. (14) have been extensively studied (Bishop & Franciosi, 1987; Dowell et all., 1994; Hackl, et al., 1993; Lu & Evan-Iwanowski, 1994; Malasoma & Lamarque, 1994; Szemplińska-Stupnicka & Bajkowki, 1986; Szemplińska-Stupnicka, 1987, 1988, 1990 and 1995; Szemplińska-Stupnicka, Plaut & Hsieh, 1989; Virgin, Plaut & Cheng, 1992; Rega, Salvatori & Benedettini, 1995) and it is known that for the laterally unloaded beam (*F*=0) and for axial loads smaller than the buckling load (w_0 >0), the beam on a softening foundation has a trivial stable equilibrium configuration. This trivial stable equilibrium sits between two non trivial unstable configurations. The heteroclinic orbit connecting these solutions is the boundary of the stable solution's basin of attraction (Figure 3). For initial conditions outside this basin of attraction, the solution tends to infinity, indicating that the beam looses its stability or the deflections become larger than the underlying beam theory can accept. We call this phenomena *escape* from the safe pre-buckling well.

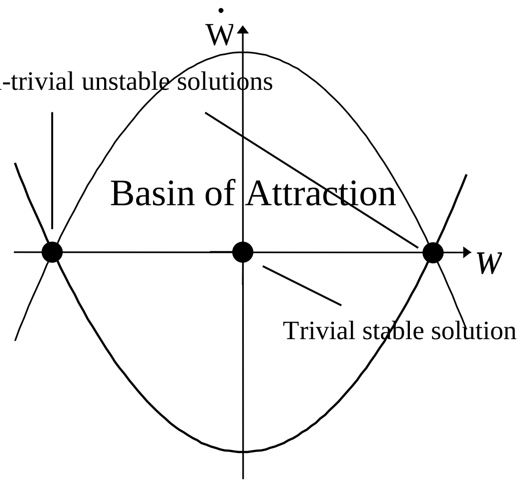


Figure 3 - Basin of Attraction of the trivial solution for the laterally unloaded beam

Besides the initial position and velocity conditions, the lateral cyclic load, at certain forcing levels and frequencies, can also lead to escape from this potential well. When small levels of this load are applied, the solutions tend to be periodic with the same period of the applied load. For the same load level F the vibration amplitude varies with the forcing frequency Ω. It tends to be larger close to the beam's natural frequencies and its super and sub-harmonics. Figure 4 shows an typical resonance curve for the beam on a softening foundation. Note that the peak of the curve bends to the left.

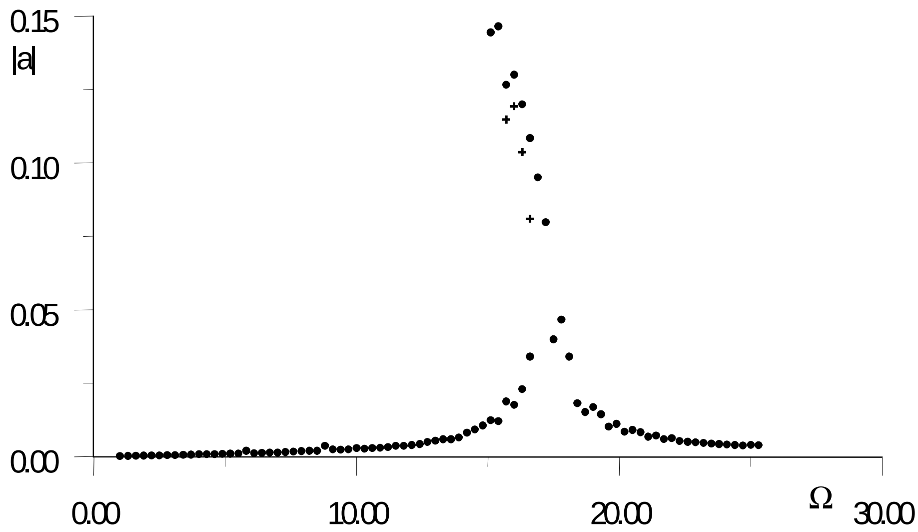


Figure 4 – Typical resonance curve for F=0.01 N

As the load level is increased, the vibration amplitude increases and the overall vibration tends to become more complex. Finally escape occurs. Different excitation frequencies lead to different escape loads. The escape load, F_e, as a function of the excitation frequency, Ω, is called the *escape boundary*. Figure 5 shows the escape boundary obtained for the present example considering a slowly evolving environment. The escape boundary is formed by a sequence of ascending and descending parts. The ascending parts are of fractal nature (Soliman, 1994; Santee, 1999).

Our investigations have shown that, under a periodic load, there are basically two escape mechanisms (Santee, 1999): A simple saddle-node, or fold, bifurcation when the stable solution, with the same frequency of the excitation, meets an unstable solution and disappears. And a supercritical period doubling, or flip, bifurcation sometimes cascading to chaos.

Figure 6 shows an example of the different escape mechanisms. Case a shows that for F=0 there is only one trivial stable solution. As the load level is increased an unstable solution of the same period appears and becomes increasingly close to the stable solution. When the two, stable and unstable, solutions meet they cancel out and disappear. This means that the system does not have a stable configuration and escapes. One should note that when the two solutions are close, the stable solution's basin of attraction is small, thus finite dynamical perturbations to the system can lead to escape. This kind of bifurcation occurs at the escape boundary's descending branches.

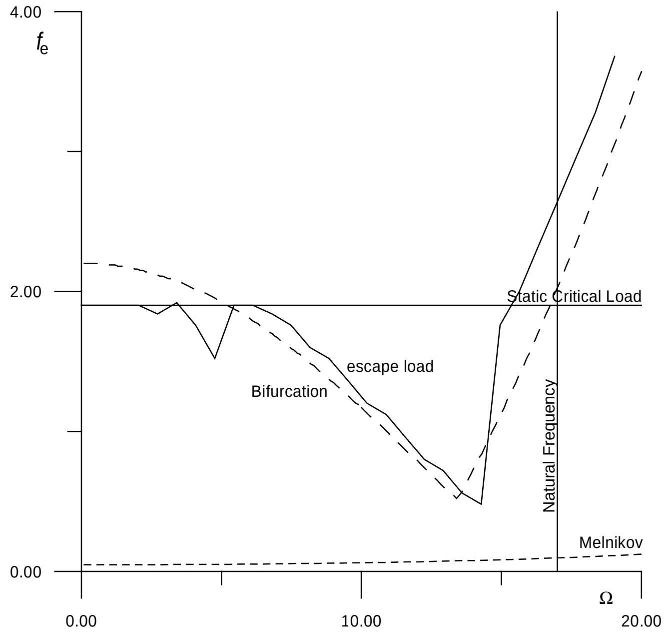
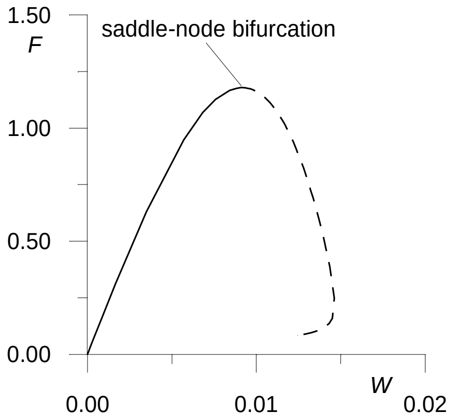
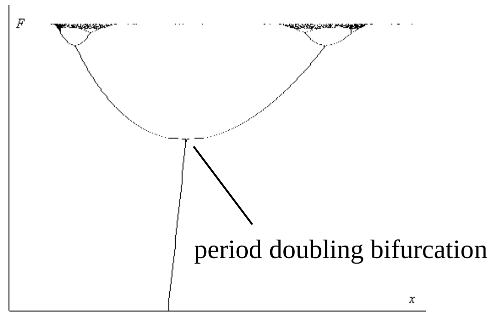


Figure 5 - The Escape Boundary

In case b the solution's behavior prior to escape is conceptually different. At a certain load level the solution doubles its period, but remains stable. As the load increases a succession of period doublings occur at an ever decreasing interval. The solution then becomes chaotic and escape occurs. In this case the basin of attraction's shape may not change as the load increases, but becomes more and more complex, our investigations show that it becomes fractal, such that escape configurations sit very close to stable configurations. This type of bifurcation occurs at the ascending branches of the escape boundary.



a) Saddle-Node (fold) Bifurcation for Ω=10



b) Super Critical Period Doubling (flip) Cascade Bifurcation for Ω=17

Figure 6 - The Escape Mechanisms

Besides the main solution, which starts from the trivial solution of the unloaded beam, there can exist other stable and unstable solutions co-existing at the same load level. In fact numerical studies of these secondary paths have shown that they are quite complex. But the knowledge of these two basic escape mechanisms, the fold and the flip, will be the basis of the development of the theory that will enable one to predict the beams load carrying capacity in spite of the underlying complex dynamics.

5. An Approximate Solution

In order to be sure that the vibration phenomena described above are not restricted to the parameters used in the present numerical simulations, it is important to obtain analytical expressions based on the system parameters. Due to the impossibility of obtaining an exact analytical solution for the non-linear equation (14), an approximation using Lindstedt-Poincaré's method (Nayfeh, 1973) is obtained. Assuming that the response, and its frequency can be expanded up to second order as a function of a small parameter ξ, one can write:

$$W(t,x) = W_0(t) + x\,W_1(t) + x^2\,W_2(t) \tag{20}$$

$$w_0^2 = w^2(1 + x\,a_1 + x^2 a_2) \tag{21}$$

$$\Omega t = t + f \tag{22}$$

The substitution of Eq. (20), Eq. (21) and Eq. (22) into Eq. (14) generates a sequence of ordinary differential equations in W_0, W_1 and W_2. Solving the equations one at a time and adjusting the parameters a_1, a_2 and f to eliminate the secular terms, the following relations are obtained:

$$sin(f) = \frac{a\,c\,\Omega}{f} \tag{23}$$

$$w_0^2 = \Omega^2 + \frac{f\cos(f)}{a} + \frac{3}{4}a^2\,b \tag{24}$$

$$W(t) = a\cos(\Omega t - f) - \frac{1}{32}\frac{b\,a^3}{w_0^2}\cos(3\Omega t - 3f) \tag{25}$$

The amplitude parameter a can be obtained by solving Eq. (23) and Eq. (24). Once this parameter is obtained, the time response of the beam, Eq. (25), is totally defined. Additionally making the damping parameter, c, and the external load, f, zero, in Eq. (24) one obtains the following approximation for the frequency-amplitude relation

$$\frac{a}{w_0} = \sqrt{\frac{4}{3b}\left[1 - \left(\frac{\Omega}{w_0}\right)^2\right]} \tag{26}$$

This is the expression of the backbone relating the beams natural frequency with the vibration amplitude as shown in Figure 2. Note that due to the softening effect of the non-linear term the natural frequency decreases with the vibration's amplitude, as obtained numerically in Figure 4.

5.1 The Solution's Stability

An important element in the non-linear dynamic analysis of the beam is to consider how it looses stability under external excitation. This can be done by assuming a small perturbation $x(t)$<<1 on the solution $W(t)$. Substituting this in Eq. (14), using Eq. (25) as the approximate solution and eliminating the higher order terms of x, the stability equation becomes:

$$\ddot{x} + c\,\dot{x} + \left(l_0 + \sum_{i=1}^{3} l_i\cos(2i\Omega t - 2if)\right)x = 0 \tag{27}$$

where

$$l_0 = w_0^2 - \frac{3b\,a^2}{2}\left(1 + \frac{1}{1024}\frac{b^2}{w_0^4}a^4\right) \qquad l_1 = -\frac{3}{2}b\,a^2\left(1 - \frac{1}{16}\frac{b}{w_0^2}a^2\right)$$

$$l_2 = \frac{3}{32}\frac{b^2}{w_0^2}a^4 \qquad l_3 = -\frac{3}{2048}\frac{b^3}{w_0^4}a^6 \tag{28}$$

Clearly the trivial solution $x \equiv 0$ is a solution of Eq. (27) for every set of parameters. Additionally if $a = 0$ the trivial solution is stable and unique. If for any $a^* = a > 0$, Eq. (27) admits another solution, then there is a bifurcation at this point. If the secondary solution is periodic with the same period of Eq. (28) then the bifurcation is a saddle-node (fold) bifurcation. Thus assuming (Szemplińska-Stupnicka, 1995)

$$x(t) = b_c\cos(\Omega t) + b_s\,sin(\Omega t) \tag{29}$$

Hill's Determinant becomes:

$$\Delta_{Hill} = l_0\left[\Omega^2(c^2 - 2l_0 + \Omega^2) + l_0^2 - \frac{1}{4}l_1^2\right] \tag{30}$$

When Eq. (30) becomes zero a fold bifurcation occurs.

On the other hand, if the secondary solution has twice the period of Eq. (25) (half the frequency) a period doubling (flip) bifurcation occurs. Thus assuming

$$x(t) = b_c \cos\left(\frac{\Omega t}{2}\right) + b_s sin\left(\frac{\Omega t}{2}\right) \tag{31}$$

Hill's determinant becomes

$$\Delta_{Hill} = \left(\frac{\Omega}{2}\right)^2\left[\left(\frac{\Omega}{2}\right)^2 + c^2\right] + l_0\left[l_0 - 2\left(\frac{\Omega}{2}\right)^2\right] \tag{32}$$

Equation (30) together with Eq. (32) allows one to verify if the solution given by Eq. (23) and Eq. (24) is stable. These criteria were used to identify the bifurcation events connected with the stability boundaries shown in Figure 5.

6. A Lower Bound Estimate

The use of the fold and the flip bifurcations as a prediction of the escape load appears initially to be a safe and reliable criterion for design. Actually it is an unsafe prediction. Prior to the fold bifurcation, the stable solution's basin of attraction can be so small that finite small dynamical perturbations can make the beam escape to infinity. Also the basin boundary can become fractal and, in this case, the long term behavior of the beam becomes unpredictable (Soliman & Thompson, 1989; Soliman, 1990; Santee, 1999). These two possible cases are illustrated in Figure 7. So, based on these arguments, the escape boundary as predicted above is better defined as an upper bound.

a) Small b) Fractal

Figure 7 - Basin of Attraction Close to the Escape Load.

Thus, a safer load capacity prediction can be obtained by examining the basin of attraction's boundary. For the laterally unloaded beam this basin of attraction's boundary is the stable manifold of the two saddle points that form the heteroclinic orbits around the stable focus. This is an indication for the use of the Melnikov method to determine the load level that produces the first transversal crossing between the stable and unstable manifolds of the saddle points. When this crossing occurs at a certain point, it occurs at an infinite number of points, making the corresponding basin of attraction fractal.

To apply Melnikov's method one starts with an exact expression of the heteroclinic orbit of the conservative part of Eq. (16) (Moon, 1992). This is

$$W_p(t) = \frac{w_0}{\sqrt{b}}\frac{e^{\sqrt{2}w_0 t} - 1}{e^{\sqrt{2}w_0 t} + 1} \tag{33}$$

By this expression the Melnikov Function, that measures the distance between the stable and unstable manifolds of the saddle point can be written as

$$M(t_0) = \frac{x\ f}{\sqrt{b}} \cos(\Omega\, t_0)\, \frac{\sqrt{2}\, p\ \Omega}{sinh\left(\dfrac{p\ \Omega \sqrt{2}}{2\, w_0}\right)} - \frac{2\sqrt{2}}{3}\, x\ c\, \frac{w_0^2}{b} \tag{34}$$

where x is a perturbation parameter of the method. Therefore a necessary condition for the Melnikov Function to be zero is that

$$f > f^M = \frac{2\, c\, w_0^3}{3 p\ \Omega \sqrt{b}}\, sinh\left(\frac{\sqrt{2}\, p\ \Omega}{2\, w_0}\right) \tag{35}$$

In terms of the system parameters the lower bound is given by

$$f^M = \frac{2c}{3p\ \Omega}\ \frac{\left[\left(\dfrac{p}{L}\right)^4 \dfrac{EI}{m} + \dfrac{E_{ti}}{m} - \left(\dfrac{p}{L}\right)^2 \dfrac{P}{m}\right]^{3/2}}{\sqrt{\dfrac{3}{4}\dfrac{(E_{ti}-E_{tf})^3}{P_u^2\, m} + \dfrac{3}{8}\left(\dfrac{p}{L}\right)^4 \dfrac{P}{m} - \dfrac{1}{2}\left(\dfrac{p}{L}\right)^6 \dfrac{EI}{m}}}\ sinh\left(\frac{\sqrt{2}\, p\ \Omega}{2\sqrt{\left(\dfrac{p}{L}\right)^4 \dfrac{EI}{m} + \dfrac{E_{ti}}{m} - \left(\dfrac{p}{L}\right)^2 \dfrac{P}{m}}}\right) \tag{36}$$

Note the relative simplicity of Eq. (36) which can be easily calculated and used as a design criteria. Assuming that the basin of attraction starts to be eroded from f^M up, one can use this load level as a lower bound estimate for the beam's load carrying capacity. Figure 5 shows the values of f^M for the practical example analyzed in this work. The significant difference between the upper and lower bounds in Figure 5 is due to the softening character and consequent imperfection-sensitivity of the system which may lead to instability loads in practice much lower than the theoretical critical load.

7. Conclusions

The influence of a non-linear elastic foundation on the non-linear dynamic behavior and stability of slender beams has bean analyzed using a simplified model. Experimental results show that this foundation non-linearity for most soils is of the softening type. For practical values of soil parameters the softening term is particularly large, dominating the response: while beams without foundation or resting on a linear elastic foundation exhibit a stable post-buckling path and a frequency-amplitude relation of the hardening type with a rather small degree of non-linearity, columns resting on a non-linear softening foundation exhibits a strong non-linearity and a high imperfection-sensitivity. This has a profound influence on the load carrying capacity of the beam and on the bifurcations connected with the instability boundaries in control space. To take into account the imperfection sensitivity, the use of the Melnikov function as a base for the derivation of a safe lower-bound of the dynamic buckling loads is proposed in the present work.

8. References

Bishop, S. R. and Franciosi, C., 1987, "Use of Rotation Numbers to Predict the Incipient Folding of a Periodic Orbit", Appl. Math Modeling 11, pp. 117-126.

Dowell, E. H., Murphy, K. D. and Katz, A. L., 1994, "Simplified Predictive Criteria for the Onset of Chaos", Nonlinear Dynamics 6, pp.247-263.

Dym, C. L. and Shames, I. H., 1973, "Solid Mechanics: A Variational Approach", McGraw-Hill Kogakuxha LTD., Tokyo, Japan, 556 p.

Greimann, L. F., Wolde-Tinsae, A.M., and Yang, P. S., 1987, "Finite Element Model for Soil-Pile Interaction in Integral Abutment Bridges", Computers and Geotechnics 4 , pp. 127-149.

Hackl, K., Yang, C. Y., Cheng, A. H.-D. and Chajes, M. J., 1993, "Stability, Bifurcation and Chaos of Non-linear Structures with Control - I", Int. J. Non-linear Mechanics 28, pp. 441-454.

Hackl, K., Yang, C. Y., Cheng, A. H.-D. and Chajes, M. J., 1993, "Stability, Bifurcation and Chaos of Non-linear Structures with Control - II", Int. J. Non-linear Mechanics 28, pp. 549-565.

Lu, C.-H. and Evan-Iwanowski, R.-M., 1994, "Period Doubling Bifurcation Problems in the Softening Duffing Oscillator with Nonstationary Excitation", Nonlinear Dynamics

Malasoma, J. -M. and Lamarque, C. -H., 1994, "Chaotic Behavior of a Parametrically Excited Nonlinear Mechanical System", Nonlinear Dynamics 5, pp. 153-160.

Moon, F. C., 1992, "Chaotic and Fractal Dynamics: an Introduction for Applied Scientists and Engineers", Willey, New York.

Nayfeh, A. H., 1973, "Perturbation Methods", Chapter 3, John Wiley & Sons, New York, p. 425.

Rega, G., Salvatori, A., and Benedettini, F., 1995, "Numerical and Geometrical Analysis of Bifurcatin and Chaos for an Asymmetric Elastic Nonlinear Oscillator", Nonlinear Dynamics 7, pp. 249-272.

Santee, D. M., 1999, "Vibrações Não-Lineares e Instabilidades de Elementos Estruturais Sensíveis a Imperfeições", Doctoral Thesis at PUC-Rio, Rio de Janeiro, Brazil, 244 p.
Soliman, M. S., Thompson, J. M. T., 1989, "Integrity Measures Quantifying the Erosion of Smooth and Fractal Basins of Attraction", J. of Sound and Vibration 135(3), pp. 453-475.
Soliman, M. S., 1990, "Advances in the Stability Assessment of Nonlinear Dynamical Systems: Chaotic Transients and Fractal Basin Boundaries" Proceedings of the Royal Society London.
Soliman, M. S., 1994, "Global Transient Dynamics of Nonlinear Parametrically Excited Systems", Nonlinear Dynamics 6, pp. 317-329.
Szemplińska-Stupnicka, W. and Bajkowki, J., 1986, "The 1/2 Subharmonic Resonance and its Transition to Chaotic Motion in a Nonlinear Oscillator", Int. J. Non-Linear Mechanics 21, pp. 401-419.
Szemplińska-Stupnicka, W., 1988, "Bifurcations of Harmonic Solution Leading to Chaotic Motion in the Softening Type Duffing's Oscillator", Int. J. of Non-Linear Mech.
Szemplińska-Stupnicka, W., 1987, "Secondary Resonances and Approximate Models of Routes to Chaotic Motion in Non-Linear Oscillators", J. of Sound and Vibration 113, pp. 155-172.
Szemplińska-Stupnicka, W., 1995, "The Analytical Predictive Criteria for Chaos and Escpae in Nonlinear Oscillators: A Survey", Nonlinear Dynamics 7, pp. 129-147.
Szemplińska-Stupnicka, W., 1990, "The Approximate Approach to Chaos Phonomena in Oscillators Having Single Equilibrium Positions", J. of Sound and Vibration 141, pp. 181-192.
Szemplińska-Stupnicka, W., Plaut, R. H. and Hsieh, J. -C., 1989, "Period Doubling and Chaos in Asymmetric Structures Unper Parametric Excitation", J. of Appl. Mech 56(4), pp. 947-952.
Thompson, J. M. T., 1982, “Instabilities and Catastrophes in Science and Engineering”, John Wiley & Sons, New York, 226 p.
Virgin, L. N. Plaut, R. H. and Cheng, C-C, 1992, "Prediction of Escape from a Potential Well under Harmonic Excitation", Int. J. Non-Linear Mechanics 27(3), pp. 357-365.

9. Responsibility notice

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