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INTEGRAL TRANSFORM SOLUTION OF ONE-DIMENSIONAL DIFFUSION PROBLEMS USING AN ENCLOSING DOMAIN APPROACH

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The Generalized Integral Transform Solution has been shown to be an effective approach for solving a variety of advection-diffusion problems. In spite of advantages of the GITT, handling irregular geometries can still be quite cumbersome. In general, irregular geometries are tackled either by coordinates transformation into a regular geometry or by using simpler eigenvalue problems in defined in the irregular geometry itself (termed a coincident domain transformation). Both approaches are limited since they cannot handle a general arbitrary domain. A new advancement to handle a general class of irregular geometries is to use an enclosing domain technique. This methodology yielded satisfactory results for calculating eigenvalues of a 1D Sturm-Liouville problem using an auxiliary problem defined within an enclosing domain. This paper extends the formerly presented results, solving a 1D diffusion problem using eigenfunction expansions in terms of an eigenvalue problem defined within a domain that encloses the original one. Then, for the purpose of validating the methodology, test case results for a simple case whose exact solution is known are computed and compared with exact values.

Keywords: *Integral transform, Irregular Domains, Diffusion Problem*

1. NOMENCLATURE

$A_{i,j}$	Coefficients matrix
$B_{i,j}$	Coefficients matrix
$D_{i,j}$	Coefficients matrix
$S_{i,j}$	Coefficients matrix
$\mathcal{B}$, $\mathcal{B}^*$	Boundary condition operators
a, b	Boundary of original problem
d	Eigenvalue problem parameter
k	Parameter in diffusion term
N	Norm of eigenfunctions
w	Weight function of original eigenfunctions
w^*	Weight function of auxiliary eigenfunctions
$\phi(x,t)$	Boundary condition function
F_o	Fourier Number

Greek Symbols

α^*, β^*	boundary condition parameters
α, β	boundary condition parameters
$\delta_{i,j}$	Kronecker delta
Ψ_n	original eigenfunctions
Ω_i^*	Normalized auxiliary eigenfunctions
μ_n	eigenvalues of original problem
γ_i	eigenvalues of auxiliary problem
$\varphi(t)$	parameter in the generalized diffusion equation
$\sigma(t)$	parameter in the generalized diffusion equation

2. INTRODUCTION

The analysis and solution of advection-diffusion can involve a considerable computational effort, especially if non-linearity and multidimensional effects are present. Further difficulties arise if there are complexities in the geometries considered. In this context, different approaches have been proposed, ranging from full numerical solutions using traditional discretization techniques to analytical methods, the former being limited to simpler situations. Between these two extremes hybrid methodologies are available, combining the flexibility of numerical solutions with the accuracy of analytical approaches. One such technique is the so-called Generalized Integral Transform (GITT) (Cotta, 1993; Cotta and Mikhailov, 1997; Cotta, 1998). This technique is based on obtaining solutions using orthogonal eigenfunction expansions. Nevertheless, if complex geometries are considered, a strategy for handling domain irregularities becomes necessary. A common option for dealing with this problem was applied to problems involving heat and fluid flow within irregularly shaped channels, heat conduction in fins of arbitrary geometry as well as problems with temperature dependent thermal conductivity (Aparecido *et al.*, 1989; Aparecido and Cotta, 1990, 1992; Cotta and Ramos, 1998; Barbuto and Cotta, 1997;

Guerrero *et al.*, 2000). All these applications, either of elliptic or parabolic mathematical nature, the domain irregularities are handled by adopting individual auxiliary problems in each coordinate direction that maps the irregular domain boundaries exactly.

A different strategy involves employing multidimensional eigenvalue problems defined within the considered irregular domain itself. This transfers the task of handling with complex geometries from the original PDE system to the associated eigenvalue problem, as demonstrated in (Sphaier and Cotta, 2002). As expected, this approach involves the solution of a multidimensional eigenvalue problem in an irregular geometry. The solution of eigenvalue problems in irregular domains represents a challenging task, even for well established numerical methods, especially when higher order eigenfunctions are needed, due to their highly oscillatory nature. Nevertheless, the solution to such a difficult eigenvalue problem can also be obtained using the GITT. As a matter of fact, a general methodology for solving multidimensional eigenvalue problems via integral transforms in a class arbitrary geometries was proposed in (Sphaier and Cotta, 2000). This strategy becomes particularly interesting for linear advection-diffusion problems, since direct analytical solutions can be obtained once the solution to the multidimensional eigenproblem is accomplished. Although the first methodology can seem more suitable for non-linear problems, if the physical nature of the problem require the calculation of eigenvalues defined within arbitrarily shaped regions, an approach similar to the one developed in (Sphaier and Cotta, 2000) needs be applied.

The methodology presented in (Sphaier and Cotta, 2000) consists of using auxiliary eigenfunctions, defined within a domain coinciding with the original irregular one to solve the problem. Although the auxiliary eigenfunctions are defined within an irregular domain, they are constructed in a simple fashion, using one-dimensional eigenfunctions. This approach, termed the *Coincident Domain Approach* (CDA) is capable of solving a class of geometries; however it cannot be applied to certain situations. In order to circumvent this limitation, extending the solution of eigenvalue problems in arbitrary domains to a broader class of geometries, an alternative approach is herein proposed. The idea behind this alternative, termed the *Enclosing Domain Approach* (EDA), is to solve the eigenvalue problem defined within the irregular domain by using an auxiliary problem defined within a regular domain that encloses the original irregular boundaries. While this alternative technique is still under development stage, the purpose of this work is to apply it within a one-dimensional framework, in order to verify its feasibility. A preliminary test was carried-out in (da Silva and Sphaier, 2009). In that study, a one-dimensional eigenvalue problem was solved using an auxiliary eigenproblem defined within a domain that enclosed the original problem. The results demonstrated a viability of the proposed methodology, since the exact values of the original eigenproblem were successfully calculated with the EDA. Difficulties were indeed encountered, as notably higher numerical precision was required, in some cases, if higher series truncation orders were requested. Nevertheless, the findings in (da Silva and Sphaier, 2009) revealed that the results depend on the relation between the enclosing and the original domain, suggesting that an optimum choice of the enclosing domain could reduce the high numerical precision requirement.

The current work extends the analysis performed in (da Silva and Sphaier, 2009), by solving an actual diffusion problem by using an eigenproblem defined within a domain that encloses the original one. The results were compared to the exact solution of the original problem and a very good agreement could be found.

3. MATHEMATICAL FORMULATION FOR THE DIFFUSION PROBLEM

A general one-dimensional diffusion problem can be written as:

$$\varphi(t)\,w(x)\,\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial T(x,t)}{\partial x}\right) + \\ + \big(\sigma(t)\,w(x) - d(x)\big)\,T(x,t) + P(x,t), \qquad \text{for} \qquad a \le x \le b, \quad (1)$$

$$\mathcal{B}\,T(x,t) = \phi(a,t), \qquad \text{for} \qquad x = a, \quad (2)$$

$$\mathcal{B}\,T(x,t) = \phi(b,t), \qquad \text{for} \qquad x = b, \quad (3)$$

$$T(x,0) = f(x), \qquad \text{for} \qquad a \le x \le b \quad (4)$$

where the boundary condition operator is defined as:

$$\mathcal{B} \equiv \left(\alpha(x) + \beta(x)\,k(x)\,\frac{\partial}{\partial x}\right). \quad (5)$$

The exact solution is obtained by the Classical Integral Transform Technique (Mikhailov and Özişik, 1984) and can be written as:

$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{N(\mu_n)}\,\bar{T}_n(t)\,\Psi_n(x) \quad (6)$$

where $\Psi_n(x)$ and μ_n are the eigenfunctions and eigenvalues of the Sturm-Liouville problem, which is defined as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x}\right) + \left(\mu^2\,w(x) - d(x)\right)\Psi(x) = 0, \qquad \text{for} \qquad a \le x \le b, \tag{7}$$

$$\mathcal{B}\,\Psi = 0, \qquad \text{for} \qquad x = a, \tag{8}$$

$$\mathcal{B}\,\Psi = 0, \qquad \text{for} \qquad x = b, \tag{9}$$

It is known that the Sturm-Liouville problem has the following orthogonality property:

$$\int_a^b w(x)\,\Psi_m(x)\,\Psi_n(x)\,\mathrm{d}x = \delta_{m,n}\,N(\mu_m), \tag{10}$$

where $\Psi_m(x)$ and $\Psi_n(x)$ are eigenfunctions associated to eigenvectors μ_m and μ_n, respectively, $\delta_{m,n}$ is the Kronecker delta function and $N(\mu_m)$ in the norm, which is defined as:

$$N(\mu_m) \equiv \int_a^b w(x)\,\Psi_m(x)^2\,\mathrm{d}x. \tag{11}$$

The transformed potentials ($\bar{T}_n(t)$) are obtained from the solution of the decoupled system:

$$\varphi(t)\,\frac{\mathrm{d}\bar{T}_n}{\mathrm{d}t} + \left(\mu_n^2 - \sigma(t)\right)\bar{T}_n(t) = \bar{g}_n(t) \tag{12}$$

$$\bar{T}_n(0) = \bar{f}_n, \tag{13}$$

for $n = 1, \ldots, \infty$. The transformed initial condition and the transformed terms are given by:

$$\bar{f}_n = \int_a^b w(x)\,f(x)\,\Psi_n(x)\,\mathrm{d}x \tag{14}$$

$$\bar{g}_n(t) = \int_a^b P(x,t)\Psi_n(x)\,\mathrm{d}x + \left[\phi(x,t)\left(\frac{\Psi_n(x) \pm k(x)\,\Psi_n'(x)}{\alpha(x)+\beta(x)}\right)\right]_{x=a}^{x=b} \tag{15}$$

The solution of the transformed system is easily obtained as:

$$\bar{T}_n(t) = \left(\bar{f}_n + \int_0^t \bar{g}_n(\tau)\,\mathrm{e}^{\gamma_n(\tau)}\right)\mathrm{e}^{-\gamma_n(t)}\,\mathrm{d}\tau \tag{16}$$

in which:

$$\gamma_n(t) = \int_0^t \frac{\mu_n^2 - \sigma(\tau)}{\varphi(\tau)}\mathrm{d}\tau \tag{17}$$

Therefore, the solution for the generalized analytical diffusion problem (1,2,3, 4) is obtained directly once the eigenvalue problem (7,8,9) is known.

4. SOLUTION OF DIFFUSION PROBLEM WITH THE ENCLOSED DOMAIN APPROACH

The object of the present methodology is to use the auxiliary eigenfunction, described by the Sturn-Lioville problem and defined in a domain that encloses the original domain, thereby within the solution of the original problem as an expansion in terms of auxiliary eigenfunctions as shown bellow:

$$T(x,t) = \sum_{i=1}^{\infty} \bar{T}_i(t)\,\Omega_i^*(x), \qquad \text{for} \qquad a \le x \le b\,. \tag{18}$$

This expression is called the inversion formula. The auxiliary eigenvalue problem is defined as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Omega^*(x)}{\mathrm{d}x}\right) + \left(\lambda^2\,w^*(x) - d(x)\right)\Omega^*(x) = 0, \qquad \text{for} \qquad 0 \le x \le 1, \tag{19}$$

$$\mathcal{B}\,\Omega^*(x) = 0, \qquad \text{for} \qquad x = 0, \tag{20}$$

$$\mathcal{B}\,\Psi^*(x) = 0, \qquad \text{for} \qquad x = 1, \tag{21}$$

Notice that the auxiliary eigenfunction $\Omega^*(x)$ was defined to satisfy the following relation:

$$\int_0^1 w^*(x)\,\Omega_n^*(x)\,\Omega_m^*(x)\,\mathrm{d}x \;=\; \delta_{m,n} \tag{22}$$

Therefore, the above expression (22) implies:

$$\Omega_n^*(x) \;=\; \frac{\Omega_n(x)}{\sqrt{N(\lambda_n)}}, \tag{23}$$

Based on equation (18), the following integral transform is obtained:

$$\bar{T}_i(t) \;=\; \int_0^1 w^*(x)\,T(x,t)\,\Omega_i(x)\,\mathrm{d}x\;. \tag{24}$$

This integral transform is different than the one employed in the exact solution, in which the eigenfunction is defined in the domain of the original problem(7,8,9). The traditional transformation is termed a *coincident domain transformation*. In the form of the previously equation, the transformation (24) is termed a *enclosing domain transformation*.

4.1 Transformation of the generalized one dimensional diffusion problem

Although the integral transform is defined in terms of the enclosing domain, the governing PDE (eq. (1)) is transformed by integrating within the original domain:

$$\begin{aligned}\varphi(t)\int_a^b w(x)\,\frac{\partial T(x,t)}{\partial t}\,\Omega_i^*(x)\,\mathrm{d}x \;&=\; \int_a^b P(x,t)\,\Omega_i^*(x)\,\mathrm{d}x\,+\\ +\int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial T(x,t)}{\partial x}\right)\Omega_i^*(x)\,\mathrm{d}x \;&+\; \sigma(t)\int_a^b w(x)\,T(x,t)\,\Omega_i^*(x)\,\mathrm{d}x \;-\; \int_a^b d(x)\,T(x,t)\,\Omega_i^*(x)\,\mathrm{d}x,\end{aligned} \tag{25}$$

The diffusion term can be transformed using Green's second formula:

$$\begin{aligned}\int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial T(x,t)}{\partial x}\right)\Omega_i^*(x)\,\mathrm{d}x \;&=\; \int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \Omega_i^*(x)}{\partial x}\right)T(x,t)\,\mathrm{d}x\,+\\ &+\left[k(x)\,\left(\frac{\partial T(x,t)}{\partial x}\,\Omega_i^*(x) - T(x,t)\,\Omega_i^{*'}(x)\right)\right]_{x=a}^{x=b}\end{aligned} \tag{26}$$

Simplifications can be obtained in the contour term using the boundary condition of the original problem:

$$\begin{aligned}&\left[k(x)\left(\frac{\partial T(x,t)}{\partial x}\,\Omega_i^*(x) \;-\; T(x,t)\,\Omega_i^{*'}(x)\right)\right]\Bigg|_{x=a}^{x=b} \;=\\ &\qquad=\;\left[\frac{k(x)\,(\frac{\partial T(x,t)}{\partial x} - T(x,t))\left(\alpha(x)\,\Omega_i^*(x) \;+\; \beta(x)\,k(x)\,\Omega_i^{*'}(x)\right)}{\alpha(x) \;+\; \beta(x)\,k(x)}\right]\Bigg|_{x=a}^{x=b},\end{aligned} \tag{27}$$

Substituting the inversion formula in equations (25) and (27) yields:

$$\begin{aligned}\sum_{j=1}^{\infty}\varphi(t)\left(\int_a^b w(x)\,\Omega_i^*(x)\,\Omega_j^*(x)\,\mathrm{d}x\right)\bar{T}_j'(t) \;&=\; \bar{P}_i(t) + \sum_{j=1}^{\infty}\left(\int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \Omega_j^*(x)}{\partial x}\right)\Omega_i^*(x)\,\mathrm{d}x\,+\right.\\ &\left.+\,\sigma(t)\int_a^b w(x)\,\Omega_i^*(x)\,\Omega_j^*(x)\,\mathrm{d}x \;-\; \int_a^b d(x)\,\Omega_i^*(x)\,\Omega_j^*(x)\,\mathrm{d}x\right)\bar{T}_j(t),\end{aligned} \tag{28}$$

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \Omega_j^*(x)}{\partial x}\right)\Omega_i^*(x)\,\mathrm{d}x \;-\; \int_a^b \frac{\partial}{\partial x}\left(k\,\frac{\partial \Omega_i^*(x)}{\partial x}\right)\Omega_j^*(x)\,\mathrm{d}x\right)\bar{T}_j(t) \;&=\\ =\;\sum_{j=1}^{\infty}\left[k(x)\left(\Omega_j^{*'}(x)\,\Omega_i^*(x) \;-\; \Omega_j^*(x)\,\Omega_i^{*'}(x)\right)\right]\Bigg|_{x=a}^{x=b}\bar{T}_j(t),&\end{aligned} \tag{29}$$

where

$$\bar{P}_i(t) \;=\; \int_a^b P(x,t)\,\Omega^*(x)\,\mathrm{d}x \tag{30}$$

It is useful, at this point, to define matrix coefficients as follows:

$$A_{i,j} \;=\; \int_a^b \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial \Omega_j^*(x)}{\partial x}\right)\Omega_i^*(x)\,\mathrm{d}x, \qquad B_{i,j} \;=\; \int_a^b w(x)\,\Omega_i^*(x)\,\Omega_j^*(x)\,\mathrm{d}x, \tag{31}$$

$$D_{i,j} \;=\; \int_a^b d(x)\,\Omega_i^*(x)\,\Omega_j^*(x)\,\mathrm{d}x, \qquad S_{i,j} \;=\; \left[k(x)\left(\Omega_j^{*'}(x)\,\Omega_i^*(x) - \Omega_j^*(x)\,\Omega_i^{*'}(x)\right)\right]\Big|_{x=a}^{x=b}. \tag{32}$$

As the boundary condition of the diffusion problem could be non-homogeneous, an independent term (b_i) is introduced in matrix $S_{i,j}$:

$$\left[k(x)\left(\frac{\partial T(x,t)}{\partial x}\,\Omega_i^*(x) \;-\; T(x,t)\,\Omega_i^{*'}(x)\right)\right]\Bigg|_{x=a}^{x=b} \;=\; \sum_{j=1}^{\infty} S_{i,j}\,\bar{T}_j(t) \;+\; \bar{b}_i(t). \tag{33}$$

Then, by introducing the coefficient matrix given by the equations (31,32), equations (28, 29) can be rewritten as:

$$\varphi(t)\sum_{j=1}^{\infty} B_{i,j}\bar{T}_j'(t) \;=\; \sum_{j=1}^{\infty}\left(A_{i,j} \;+\; \sigma(t)\,B_{i,j} \;-\; D_{i,j}\right)\bar{T}_j(t) \;+\; \bar{P}_i(t), \tag{34}$$

$$\sum_{j=1}^{\infty}\left(A_{i,j} - A_{j,i}\right)\bar{T}_j(t) \;=\; \sum_{j=1}^{\infty} S_{i,j}\,\bar{T}_j(t) \;+\; \bar{b}_i(t), \tag{35}$$

In the vector form, the above equations are given by:

$$\varphi(t)\,\boldsymbol{B}\,\bar{\boldsymbol{T}}'(t) \;=\; \big(\mathbf{A} \;+\; \sigma(t)\,\boldsymbol{B} \;-\; \mathbf{D}\big)\,\bar{\boldsymbol{T}}(t) \;+\; \bar{\boldsymbol{P}}(t), \tag{36}$$

$$\big(\mathbf{A} - \mathbf{A}^T - \boldsymbol{S}\big)\,\bar{\boldsymbol{T}}(t) \;=\; \boldsymbol{b}(t). \tag{37}$$

In addition, equation (37) implies that:

$$\mathbf{A}\,\bar{\boldsymbol{T}}(t) \;=\; (\mathbf{A}^T + \boldsymbol{S})\,\bar{\boldsymbol{T}}(t) \;+\; \boldsymbol{b}(t), \tag{38}$$

Allowing equation (36) to be rewritten as:

$$\varphi(t)\,\boldsymbol{B}\,\bar{\boldsymbol{T}}'(t) \;=\; \big(\mathbf{A}^T + \boldsymbol{S} + \sigma(t)\,\boldsymbol{B} - \mathbf{D}\big)\,\bar{\boldsymbol{T}}(t) \;+\; \bar{\boldsymbol{P}}(t) \;+\; \boldsymbol{b}(t), \tag{39}$$

which must be solved with the following initial conditions:

$$\bar{\boldsymbol{T}}(0) \;=\; \bar{\boldsymbol{f}}, \tag{40}$$

where the coefficients of $\bar{\boldsymbol{f}}$ are given by:

$$\bar{f}_i \;=\; \int_a^b w^*(x)\,f(x)\Omega_i(x)\,\mathrm{d}x \tag{41}$$

With some matrix operations, equation (39) can be rewritten as:

$$\bar{\boldsymbol{T}}'(t) \;=\; \boldsymbol{B}^{-1}\mathbf{M}\,\bar{\boldsymbol{T}}(t) \;+\; \bar{\boldsymbol{g}}(t), \tag{42}$$

where $\mathbf{M}$ and $\bar{\boldsymbol{g}}$ are given by:

$$\mathbf{M} \;=\; \frac{1}{\varphi(t)}\big(\mathbf{A}^T + \boldsymbol{S} + \sigma(t)\,\boldsymbol{B} - \mathbf{D}\big) \tag{43}$$

$$\bar{\boldsymbol{g}}(t) \;=\; \frac{1}{\varphi(t)}\boldsymbol{B}^{-1}\big(\bar{\boldsymbol{P}}(t) \;+\; \boldsymbol{b}(t)\big) \tag{44}$$

System (42,40) allows a closed-form analytical solution:

$$\bar{\boldsymbol{T}}(t) \;=\; \mathbf{C}(t)\,\left(\bar{\boldsymbol{f}} \;+\; \int_0^t \mathbf{C}^{-1}(\tau)\,\bar{\boldsymbol{g}}(t)\,\mathrm{d}\tau\right) \tag{45}$$

Where the involved matrices are given in terms of matrix exponentials(Greenberg, 1998):

$$\mathbf{C}(t) \;=\; \exp\left(-\boldsymbol{B}^{-1}\mathbf{M}\,t\right) \qquad \text{and} \qquad \mathbf{C}^{-1}(t) \;=\; \exp\left(\boldsymbol{B}^{-1}\mathbf{M}\,t\right) \tag{46}$$

Although the above solution consists of an elegant analytical closed-form, the matrix exponential evaluation could be problematic for higher truncation orders. Therefore, a direct numerical solution of the ordinary differential equations (39,40) are also implemented and compared with the computation of the analytical form. Once the transformed potential are obtained, the temperature field is evaluated using the inversion formula (18).

5. TEST CASE

In order to illustrate the present methodology, a simplified version of the generalized one dimensional diffusion problem, with Dirichlet boundary conditions is considered:

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}, \qquad \text{for} \quad a \le x \le b, \tag{47}$$

$$T(0,t) = \phi(a,t) = 0, \qquad \text{for} \quad x = a, \tag{48}$$

$$T(1,t) = \phi(b,t) = 0, \qquad \text{for} \quad x = b, \tag{49}$$

$$T(x,0) = f(x), \qquad \text{for} \quad a \le x \le b, \tag{50}$$

This problem has a well-known analytical solution in the form:

$$T(x,t) = \sum_{i=1}^{\infty} \frac{\left(\int_0^1 f(x)\,\Psi_i(x)\,\mathrm{d}x\right)}{N(\mu_i)} \exp^{-\mu_i^2\,\alpha\,t}\;\Psi_i(x). \tag{51}$$

In order to solve the problem by the proposed methodology an auxiliary eigenvalue problem in a form similar to the original one is chosen:

$$\Omega''(x) + \gamma^2\,\Omega(x) = 0, \qquad \text{for} \quad 0 \le x \le 1, \tag{52}$$

$$\mathcal{B}^*\,\Omega = 0, \qquad \text{for} \quad x = 0, \tag{53}$$

$$\mathcal{B}^*\,\Omega = 0, \qquad \text{for} \quad x = 1, \tag{54}$$

where the operator $\mathcal{B}^*$ is defined as:

$$\mathcal{B}^* \equiv \left(\alpha^*(x) + \beta^*(x)\,k^*(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{55}$$

Although equation (52) has the same form as the equation that gives the functions Ψ, different combination of the boundary conditions parameters for Ω are analyzed for comparison purposes. Regardless of the boundary conditions, for the selected test-case, some coefficients are simplified, yielding:

$$B_{i,j} = \int_a^b \Omega_j\,\Omega_i\,\mathrm{d}x, \qquad A_{i,j} = -\gamma_j^2\,B_{i,j}, \qquad D_{i,j} = 0, \qquad S_{i,j} = \left[(\Omega_j' - \Omega_j)\Omega_i\right]_{x=a}^{x=b}. \tag{56}$$

The different boundary conditions and the resulting auxiliary eigenfunction, for the analyzed cases are described below:

- Case 1: $\Omega(0) = \Omega(1) = 0$.

$$\Omega_i(x) = \sqrt{2}\,\sin(\gamma_i\,x), \qquad \gamma_i = n\,\pi, \qquad n = 1, 2, \ldots \tag{57}$$

- Case 2: $\Omega'(0) = \Omega(1) = 0$.

$$\Omega_i(x) = \sqrt{2}\,\cos(\gamma_i\,x), \qquad \gamma_i = (n - 1/2)\,\pi, \qquad n = 1, 2, \ldots \tag{58}$$

- Case 3: $\Omega(0) = \Omega'(1) = 0$.

$$\Omega_i(x) = \sqrt{2}\,\sin(\gamma_i\,x), \qquad \gamma_i = (n - 1/2)\,\pi, \qquad n = 1, 2, \ldots \tag{59}$$

6. RESULTS AND DISCUSSIONS

The solutions given in the previous sections were implemented in the Mathematica system (Wolfram, 2003) and are now presented. In order to eliminate the influence of using smaller numerical precision, a high numerical precision was used (100 digits). An investigation of the variation of the number of digits used in the calculations is out of the scope of this work. The results were calculated for different truncation orders ($i_{\max}$), where this number is the actual number of terms used to construct the matrices involved in equation (43). However, the error in the calculated eigenvalues and associated eigenvectors is significantly higher for the last two eigenvalues. Hence, when evaluating the matrix exponentials, the last two eigenvalues (and associated eigenvectors) needed to be discarded. When this is not done, this error is greatly amplified by the exponentials. Hence a smaller truncation order was used in the calculation of the final solution (involving the calculation of matrix exponentials), namely i_d. These two different truncation orders are termed the *eigenvalue problem truncation order* ($i_{\max}$) and the *diffusion problem truncation order* (i_d) .

Table 1 presents the results for cases 1, 2 and 3, using $a = 0.25$, $b = 0.75$, $x = 0.5$ and $t = 2.5$, varying the eigenvalue problem truncation order, the diffusion problem truncation order and using $\alpha = 10^-2$ and $\alpha = 10^-4$. As can be seen, the convergence rate of the temperature field is similar for all cases. This is seen for both values of the thermal diffusivity. Nevertheless, when comparing the results between the different values of α, it is seen that for smaller thermal diffusivity, the convergence rate is better.

Next, table 2 presents the results for the same three cases, but setting $a = 0.1$ and $b = 0.9$. As one can observe, the previous observations still hold in this table. However, in this table, for certain values the truncation orders, the calculated solution present significant numerical errors. When comparing tables 1 and 2, it is clear that the convergence of the solution using an enclosing domain with boundaries farther from the ones of the original problem a noticeably worse convergence rate is found.

The obtained results are in accordance with the findings of (da Silva and Sphaier, 2009), where the solution with an enclosing boundary closer to the original domain presents better convergence rates. Nevertheless, the better convergence rate of case 3 for the calculation of eigenvalues (as demonstrated in the above mentioned study), has no effect in the solution of the diffusion problem. This result seems to suggests that using different boundary conditions for the auxiliary eigenproblem has little or no influence on the solution of the diffusion problem.

Table 1. Temperature convergence for $x = 0.5$ and $t = 2.5$

$a = 0.25$, $b = 0.75$, $\alpha = 10^{-2}$					
i_{max}	i_d	Case 1	Case 2	Case 3	Analytical
14	12	0.486562	0.487192	0.487191	0.474487
18	16	0.481320	0.481594	0.481593	0.474487
22	20	0.478871	0.479029	0.479029	0.474487
26	24	0.477575	0.477661	0.477661	0.474487
30	28	0.476764	0.476823	0.476823	0.474487
34	32	0.476246	0.476284	0.476284	0.474487
38	36	0.475881	0.475909	0.475909	0.474487
42	40	0.475623	0.475642	0.475642	0.474487
46	44	0.475428	0.475443	0.475443	0.474487
50	48	0.475281	0.475292	0.475292	0.474487
54	52	0.475164	0.475174	0.475174	0.474487
58	56	0.475073	0.475080	0.475080	0.474487
62	60	0.474998	0.475004	0.475004	0.474487
$a = 0.25$, $b = 0.75$, $\alpha = 10^{-4}$					
14	12	1.204260	1.185200	1.185150	0.987755
18	16	0.791526	0.810564	0.810635	0.997017
22	20	0.864013	0.883806	0.883791	0.999439
26	24	1.185710	1.172710	1.172710	0.99992
30	28	0.933781	0.935366	0.935372	0.999992
34	32	0.960559	0.966743	0.966741	0.999999
38	36	1.049300	1.045460	1.045460	1.000000
42	40	0.973921	0.975167	0.975168	1.000000
46	44	1.006650	1.006810	1.006810	1.000000
50	48	1.000250	0.999963	0.999962	1.000000
54	52	0.998758	0.998921	0.998920	1.000000
58	56	1.000690	1.000620	1.000620	1.000000
62	60	0.999752	0.999640	0.999764	1.000000

7. CONCLUSIONS

This paper presented an alternative methodology for solving diffusion problems via the Generalized Integral Transform Technique (GITT). The alternative scheme, termed the Enclosing Domain Approach (EDA), consists of using an auxiliary eigenproblem defined within a region that encloses the original domain. Although the auxiliary eigenvalue problem is defined within an enclosing domain, the transformation of the diffusion problem is carried out by integration within

Table 2. Temperature convergence for $x = 0.5$ and $t = 2.5$

i_{max}	i_d	Case 1	Case 2	Case 3	Analytical
		$a = 0.1$, $b = 0.9$, $\alpha = 10^{-2}$			
14	12	0.868597	6.745×10^{25}	3.299×10^{27}	0.852724
18	16	0.859274	0.858500	0.858481	0.852724
22	20	0.856240	0.856259	0.856256	0.852724
26	24	0.854960	-0.903011	-2.647600	0.852724
30	28	0.000000	0.854161	0.854161	0.852724
34	32	0.853866	0.853891	0.853891	0.852724
38	36	0.853612	0.853588	0.853588	0.852724
42	40	0.853444	0.853469	0.853469	0.852724
46	44	0.853321	-7.123720	-22.90680	0.852724
50	48	0.000000	0.853205	0.853205	0.852724
54	52	0.853147	0.853155	0.853155	0.852724
58	56	0.853088	0.853082	0.853082	0.852724
62	60	0.853042	0.853050	0.853050	0.852724
		$a = 0.1$, $b = 0.9$, $\alpha = 10^{-4}$			
14	12	1.028500	1.177950	1.209120	0.969998
18	16	1.033550	1.029390	1.029100	0.985396
22	20	1.018190	1.014940	1.014870	0.993314
26	24	1.000400	-6.445850	-13.79860	0.997186
30	28	0.000000	0.994855	0.994858	0.998924
34	32	0.995641	0.996663	0.996664	0.999629
38	36	0.998756	0.999140	0.999140	0.999885
42	40	0.999975	1.000030	1.000030	0.999968
46	44	1.000050	1.510520	2.521430	0.999992
50	48	0.000000	0.999976	0.999976	0.999998
54	52	0.999970	0.999977	0.999977	1.000000
58	56	0.999991	0.999993	0.999993	1.000000
62	60	0.999999	0.999999	0.999999	1.000000

the actual domain. As a result, all terms in the transformed system are coupled.

In order to obtain a preliminary validation of the proposed methodology a the solution of one-dimensional test problem, with known analytical solution, is implemented. Three different boundary condition combinations, and two different domains were tested. The results showed that the convergence rate depends not only on the truncation order, but also on the proximity of the boundaries of the enclosing and original domains.

The couplings in all terms of the transformed system lead to a much more complex system when compared to the traditionally used methodology, termed the Coincident Domain Approach (CDA). This may seem, at first, that there is little advantage in using the EDA over the CDA.However the EDA can potentially be applied to a class of domains previously unsuitable for integral transformation via the CDA. Hence, the ideas herein developed can open a new direction for developing the GITT in irregular geometries.

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