# A NEW APPROACH FOR DESCRIBING THE FLOW THROUGH RIGID POROUS MEDIA ACCOUNTING FOR THE TRANSITION SATURATED/UNSATURATED

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***Abstract.*** *This work proposes a mathematical model to study the filling up of an unsaturated rigid porous medium by a liquid identifying the transition from unsaturated to saturated flow. A mixture theory approach is employed consisting of three overlapping continuous constituents, representing the porous matrix (solid constituent), the fluid (liquid constituent) and an inert gas included to account for the compressibility of the mixture as a whole. The mathematical description gives rise to a nonlinear hyperbolic system in which the fluid fraction must satisfy an inequality – an upper bound – in order to give rise to a physically realistic model. Unsaturated flows through porous media are characterized by a strong dependence of the motion on the saturation, since a force depending on the saturation gradient gives rise to the fluid flow. The model introduced in this work accounts for the physical upper bound of the fluid fraction (and the saturation) that depends on the volume of the pores.*

***Keywords****: Flow through unsaturated porous media, transition saturated/unsaturated flow, constrained flow, shock waves.*

## 1. INTRODUCTION

The study of transport in porous media dates from the 1920s, according to a comprehensive review, by Alazmi and Vafai (2000), comparing different models for complex problems. However, adequate description of the transition from saturated to unsaturated flows through porous media remains an open subject.

This work presents a physically realistic mathematical model to represent the filling up of an unsaturated rigid porous matrix by a fluid, identifying the transition from unsaturated to saturated flow, by imposing a constraint (an upper bound) on the saturation. The mechanical modeling uses a mixture theory approach (Atkin and Craine, 1976; Rajagopal and Tao, 1995) – a convenient method for modeling multicomponent systems – supported by a local theory with thermodynamic consistency which generalizes the classical Continuum Mechanics (Germain and Muller, 1986).

The unsaturated porous medium is modeled as a mixture of three overlapping continuous constituents: a solid (a rigid, homogeneous and isotropic porous matrix), a liquid (an incompressible fluid) and an inert gas, assumed with very low mass density; which was included to account for the compressibility of the system as a whole.

## 2. MECHANICAL MODEL

Since the chemically non reacting mixture consists of a rigid solid constituent at rest, a liquid constituent – from now on denoted as fluid constituent and an inert gas, playing the role of the third constituent, it suffices to solve mass and momentum balance equations for the fluid constituent only, as presented below, combined with constitutive assumptions, to build the mechanical model.

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \nabla \cdot (\rho_F \mathbf{v}_F) = 0 \\ &\rho_F \left[ \frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{b}_F \end{aligned} \tag{1}$$

where $\rho_F$ stands for the fluid constituent mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture, $\mathbf{v}_F$ is the fluid constituent velocity in the mixture, $\mathbf{T}_F$ represents the partial stress tensor – analogous to Cauchy stress tensor in Continuum Mechanics – associated with the fluid constituent, $\mathbf{b}_F$ stands for the body force (per unit mass) and $\mathbf{m}_F$ for the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. The ratio between the fluid fraction $\phi$ and the porous matrix porosity ε is defined as the saturation $\psi$, so that $\psi = \phi / \varepsilon = \rho_F / \varepsilon \rho_f$ with $0 < \psi \leq 1$ everywhere, in which $\rho_f$ is the actual mass density of the fluid – regarded from a Continuum Mechanics viewpoint, in contrast to $\rho_F$ defined as the fluid constituent mass density.

Constitutive relations are now presented for the partial stress tensor associated with the fluid constituent and for the momentum supply acting on the fluid constituent. The former is modeled under the simplifying assumption proposed by Allen (1986) considering the normal fluid stresses dominant over shear stresses and interphase tractions. The momentum source usually accounts for a term related to the fluid constituent velocity as well as for a term related to the saturation gradient, characterizing the strong dependence of the motion on the saturation. (See Martins-Costa and Saldanha da Gama (2001) for a detailed discussion.)

$$\mathbf{m}_F = -\frac{\mu_f}{K}\phi^2\mathbf{v}_F - \frac{\mu_f\mathcal{D}}{K}\nabla\phi \qquad\qquad \mathbf{T}_F = -\phi\overline{p}\mathbf{I} \tag{2}$$

where $\mu_f$ represents the fluid viscosity (measured considering a Continuum Mechanics viewpoint), $K$ the porous matrix specific permeability, $\mathcal{D}$ a diffusion coefficient – analogous to the usual mass diffusion coefficient, $\overline{p}$ is a pressure (assumed constant while the flow is unsaturated) and $\mathbf{I}$ is the identity tensor. The first term of the momentum source $\mathbf{m}_F$, usually called Darcian term, will be neglected in the present work.

Assuming all the quantities depending only on the time $t$ and on the position $x$ and that $v$ is the only non-vanishing component of the fluid constituent velocity $\mathbf{v}_F$, then the balance equations, Eq. (1), combined with the constitutive relations in Eq. (2) give rise to

$$\begin{gathered}\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x}(\phi v) = 0 \\ \rho_f\left[\phi\frac{\partial v}{\partial t} + \phi v\frac{\partial v}{\partial x}\right] = -\frac{\partial}{\partial x}(\phi\overline{p}) - \frac{\mathcal{D}\mu_f}{K}\frac{\partial\phi}{\partial x}\end{gathered} \tag{3}$$

The nonlinear system presented in Eq. (3) may be rewritten in a more convenient form by redefining the pressure $p = \hat{p}(\phi)$ as $p = \overline{p}\phi + (\mu_f\mathcal{D}/K)\phi$, so that the following nonlinear hyperbolic system represents mathematically a mixture theory description of a one dimensional flow of a fluid through an unsaturated rigid porous matrix at rest

$$\begin{gathered}\frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x}(\phi v) = 0 \\ \frac{\partial}{\partial t}(\phi v) + \frac{\partial}{\partial x}\left(p + \phi v^2\right) = 0\end{gathered} \tag{4}$$

This hyperbolic system may not admit continuous solutions, requiring an enlargement of the space of admissible functions in order to admit generalized (discontinuous) solutions. The system presented in Eq. (4) may also represent other relevant Engineering problems such as the dynamical behavior of an elastic rod in the current configuration (Saldanha da Gama, 1990) or the dynamics of an ideal gas (Marchesin and Paes Leme, 1983).

Considering the description of the flow through an unsaturated porous medium presented in Eq. (4), the unknowns $\phi$ and $v$ represent, respectively, the fluid fraction and the fluid constituent velocity. The function $p$ is a ratio between pressure and density, from now on simply referred to as "pressure".

Both $\phi$ and $v$ depend on the position $x$ and on the time $t$. While the velocity can assume any real value, the fluid fraction $\phi$ must be is positive valued and smaller than (or equal to) the porosity ε, in order to be physically meaningful. In other words,

$$0 < \phi \le \varepsilon, \quad \text{for all } t > 0, \text{ for all } x \tag{5}$$

The inequality presented in Eq. (5) clearly shows that the volume of the fluid can not exceed the volume of the pores and that there exists fluid in the pores (positiveness).

As it will be shown later, the positiveness of the fluid fraction $\phi$ is ensured by a convenient constitutive relation between $p$ and $\phi$. On the other hand, the (physical) upper bound for the fluid fraction must be imposed during any simulation, in order to avoid a fluid fraction greater than the porosity, obviously not physically admissible, and in order to properly describe the transition from unsaturated flow ($\phi<\varepsilon$) to saturated flow ($\phi=\varepsilon$).

The constraint $\phi\le\varepsilon$ must be verified for all position and time, otherwise, depending on the initial data, the results may present regions without physical meaning – in which $\phi>\varepsilon$. Some of these cases will be illustrated in this work.

It is important to notice that initial data may be conveniently chosen in order to automatically ensure the inequality (5). On the other hand, some initial data may give rise to mathematical descriptions without physical meaning (in which Eq. (5) is not always satisfied), except when the constraint is imposed during the simulation. This is the case of the saturation process that can not be simulated without employing the constraint $\phi\le\varepsilon$.

Since a rigid homogeneous porous matrix is considered in this work, the pressure $p$ is assumed to be a linear and increasing function of the fluid fraction $\phi$, provided that the porous medium is not saturated.

This work main subject is to adequately model the one dimensional flow of a fluid through a rigid porous medium, with uniform porosity $\varepsilon$, which is mathematically represented by the following system

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi v) = 0 \\ \frac{\partial}{\partial t}(\phi v) + \frac{\partial}{\partial x}\left(p + \phi v^2\right) = 0 \end{aligned}\right\rangle \quad \text{with} \quad 0 < \phi \le \varepsilon, \quad \text{for all } t > 0, \text{ for all } x \tag{6}$$

### 2.1. Relationship between pressure $p$ and fluid fraction $\phi$

In order to clearly identify the transition from unsaturated to saturated flow, the relationship between pressure $p$ and fluid fraction $\phi$ is analysed. The unsaturated flow of a liquid through a rigid porous medium may be regarded as a mixture of three overlapping continuous constituents: one liquid (representing an incompressible fluid), one solid (representing the porous medium) and an inert gas (with very low mass density) that provides the compressibility of the system. The existence of this gas can be assumed while the fluid (liquid) fraction $\phi$ is smaller than the porosity $\varepsilon$ – in other words while the flow is unsaturated. In such cases it is possible to assume that the (partial) pressure $p$ is a constitutive function of the fluid fraction. In this work, the following relation is assumed

$$p = c^2\phi \quad \text{provided} \quad 0 \le \phi < \varepsilon \tag{7}$$

where $c$ is a positive constant.

Regardless of the constitutive choice for the pressure $p$, it makes sense only for $\phi$ within the open interval (0, $\varepsilon$). In fact, $p$ can not be evaluated from Eq. (7) when $\phi = \varepsilon$, since there exists a geometrical bound (rigid porous medium) that allows a pressure increasing with a fixed fluid fraction $\phi$. When the fluid fraction equals the porosity, the flow is saturated ($\phi = \varepsilon \rightarrow$ saturation). For a rigid and homogeneous porous medium, the following must hold

$$\begin{aligned} &p = \hat{p}(\phi) && \text{for} \quad 0 < \phi < \varepsilon \quad \rightarrow \quad \text{unsaturated flow} \\ &\hat{p}(\varepsilon) \le p < \infty && \text{for} \quad \phi = \varepsilon \quad \rightarrow \quad \text{saturated flow} \end{aligned} \tag{8}$$

It is important to notice that the simulation of transition from unsaturated flow to saturated flow must take into account relation (8).

## 3. THE RIEMANN PROBLEM ASSOCIATED TO SYSTEM (4) – UNCONSTRAINED

The Riemann problem associated to system (4) is built in by assuming, for all $x \in (-\infty, \infty)$, the following initial data

$$\phi = \begin{cases} \phi_L & \text{for} \quad t = 0, \; -\infty < x < 0 \\ \phi_R & \text{for} \quad t = 0, \quad 0 < x < \infty \end{cases} \qquad v = \begin{cases} v_L & \text{for} \quad t = 0, \; -\infty < x < 0 \\ v_R & \text{for} \quad t = 0, \quad 0 < x < \infty \end{cases} \tag{9}$$

where $\phi_L$, $\phi_R$, $v_L$ and $v_R$ are constants.

The solution (in a generalized sense) of this Riemann problem depends only on the ratio $x/t$ being obtained by connecting the left state ($\phi_L$, $v_L$) and the right state ($\phi_R$, $v_R$) to an intermediate state ($\phi_*$, $v_*$) by means of rarefactions and shocks (Smoller, 1983, Martins-Costa and Saldanha da Gama, 2001)

The two eigenvalues of system (4) are given, in increasing order, by $\lambda_1 = \hat{\lambda}_1(\phi, v) = v - \sqrt{p'} = v - c$ and $\lambda_2 = \hat{\lambda}_2(\phi, v) = v + \sqrt{p'} = v + c$, where $p'$ represents the first derivative of $p$ with respect to $\phi$, given by $c^2$.

The Riemann invariants $R_1$ and $R_2$, associated to the eigenvalues $\lambda_1$ and $\lambda_2$ are obtained from differential equations, arising from system (4); being expressed as follows: $R_1 = c\ln\phi + \lambda$ and $R_2 = -c\ln\phi + \lambda_2$.

The left state is connected to the intermediate state by a 1-rarefaction if, and only if, between these two states, the first eigenvalue is given by $\lambda_1 = x/t$. Analogously, the right state is connected to the intermediate state by a 2-rarefaction if, and only if, between these two states, the second eigenvalue is given by $\lambda_2 = x/t$.

The intermediate state is obtained from Riemann invariants $R_1$ and $R_2$ as follows:

$$v_* = \frac{1}{2}\left\{ v_L + v_R + c \ln \frac{\phi_L}{\phi_R} \right\} \qquad \text{and} \qquad \phi_* = \sqrt{\phi_L \phi_R} \exp\left\{ \frac{v_L - v_R}{2c} \right\} \tag{10}$$

Therefore, if the states ($\phi_L$, $v_L$) and ($\phi_R$, $v_R$) are connected by a 1-rarefaction/2-rarefaction it comes that

$$c\left| \ln \frac{\phi_L}{\phi_R} \right| < v_R - v_L \tag{11}$$

If $\phi_*=\phi_L$, the states ($\phi_L$, $v_L$) and ($\phi_R$ $v_R$) are connected by a 2-rarefaction (there is no 1-rarefaction) while, if $\phi_*=\phi_R$, the states ($\phi_L$, $v_L$) and ($\phi_R$ $v_R$) are connected by a 1-rarefaction (there is no 2-rarefaction). In such cases, inequality (11) becomes equality.

It is quite obvious that, in many cases, the left-hand side of Eq. (11) may be greater than $v_R$-$v_L$. In these cases, there is no continuous solution for the associated Riemann problem. These cases require a larger space for the solution. The enlargement of the space of admissible solutions allows discontinuous solutions – in a generalized sense – for the associated Riemann problem.

When two states are connected by a discontinuity, they must satisfy the Rankine-Hugoniot jump conditions given by

$$\frac{[\phi v]}{[\phi]} = \frac{[\phi v^2 + p]}{[\phi v]} = s \tag{12}$$

where "[ ]" denotes the jump and $s$ denotes the shock (discontinuity) speed. Also, the entropy conditions must be satisfied – ensuring that the states cannot be connected by a rarefaction (Smoller, 1983).

The states ($\phi_L$, $v_L$) and ($\phi_*$, $v_*$) are connected by a 1-shock, while the states ($\phi_*$, $v_*$) and ($\phi_R$, $v_R$) are connected by a 2-shock if they satisfy the jump conditions (12) and the entropy conditions given, respectively, by

$$\begin{aligned} &s_1 < \hat{\lambda}_1(\phi_L, v_L) \quad \text{and} \quad \hat{\lambda}_1(\phi_*, v_*) < s_1 < \hat{\lambda}_2(\phi_*, v_*), \quad \text{for 1-shock} \\ &s_2 > \hat{\lambda}_2(\phi_R, v_R) \quad \text{and} \quad \hat{\lambda}_2(\phi_*, v_*) > s_2 > \hat{\lambda}_1(\phi_*, v_*), \quad \text{for 2-shock} \end{aligned} \tag{13}$$

The behavior of $p$ ensures the entropy conditions, provided the Rankine-Hugoniot jump conditions hold and $\phi_L < \phi_*$ (for the 1-shock) or $\phi_* > \phi_R$ (for the 2-shock).

The above results allow concluding that there are four possible solutions: 1-rarefaction/2-rarefaction, 1-shock/2-shock, 1-rarefaction/2-shock and 1-shock/2-rarefaction.

If the states ($\phi_L$, $v_L$) and ($\phi_*$, $v_*$) are connected by a 1-shock while the states ($\phi_*$, $v_*$) and ($\phi_R$, $v_R$) are connected by a 2-shock, then the jump conditions become

$$\frac{\phi_* v_* - \phi_L v_L}{\phi_* - \phi_L} = \frac{\phi_* v_*^2 - \phi_L v_L^2 + c^2(\phi_* - \phi_L)}{\phi_* v_* - \phi_L v_L} = s_1 \qquad \text{and} \qquad \frac{\phi_R v_R - \phi_* v_*}{\phi_R - \phi_*} = \frac{\phi_R v_R^2 - \phi_* v_*^2 + c^2(\phi_R - \phi_*)}{\phi_R v_R - \phi_* v_*} = s_2 \tag{14}$$

in which $s_1$ is the 1-shock speed, while the 2-shock speed is $s_2$. The above Eqs. (14) and the entropy conditions (13) lead to the following intermediate state ($\phi_*$, $v_*$)

$$v_* = v_L - c(\phi_* - \phi_L)\sqrt{\frac{1}{\phi_*\phi_L}} = v_L - c\left(\sqrt{\frac{\phi_*}{\phi_L}} - \sqrt{\frac{\phi_L}{\phi_*}}\right) \qquad v_* = v_R - c(\phi_R - \phi_*)\sqrt{\frac{1}{\phi_R\phi_*}} = v_R - c\left(\sqrt{\frac{\phi_R}{\phi_*}} - \sqrt{\frac{\phi_*}{\phi_R}}\right)$$

$$\text{and} \qquad \phi_* = \frac{\phi_R \phi_L}{\left(\sqrt{\phi_R} + \sqrt{\phi_L}\right)^2}\left\{ \frac{v_L - v_R}{2c} + \sqrt{\left(\frac{v_L - v_R}{2c}\right)^2 + 2 + \sqrt{\frac{\phi_R}{\phi_L}} + \sqrt{\frac{\phi_L}{\phi_R}}} \right\}^2, \tag{15}$$

which is the unique nonnegative root of $v_R - c(\phi_R - \phi_*)\sqrt{\frac{1}{\phi_R\phi_*}} = v_L - c(\phi_* - \phi_L)\sqrt{\frac{1}{\phi_*\phi_L}}$

Since the solution 1-shock/2-shock occurs if, and only if, $\phi_L < \phi_* > \phi_R$, it may be concluded that the solution is 1-shock/2-shock if $v_L > v_* > v_R$. This solution is ensured by the following inequality: $c\left|\sqrt{\phi_R/\phi_L} - \sqrt{\phi_L/\phi_R}\right| < v_L - v_R$ .

It is to be noticed that, when the left-hand side of the above inequality is equal to $v_L$-$v_R$, then ($\phi_L$, $v_L$) and ($\phi_R$, $v_R$) are connected by a 1-shock (if $\phi_*=\phi_R$) or by a 2-shock (if $\phi_*=\phi_L$).

If the states ($\phi_L$, $v_L$) and ($\phi_*$, $v_*$) are connected by a 1-rarefaction while the states ($\phi_*$, $v_*$) and ($\phi_R$, $v_R$) are connected by a 2-shock, then $\phi_L > \phi_* > \phi_R$ and the intermediate state ($\phi_*$, $v_*$) is obtained from

$$c\ln\phi_L + v_L = c\ln\phi_* + v_* \qquad v_* = v_R - c\left(\phi_R - \phi_*\right)\sqrt{\frac{1}{\phi_R\phi_*}}$$
$$\text{with } \phi_* \text{ being the unique root of } \quad v_L - v_R = -c\left(\phi_R - \phi_*\right)\sqrt{\frac{1}{\phi_R\phi_*}} + c\ln\phi_* - c\ln\phi_L \tag{16}$$

If the states ($\phi_L$, $v_L$) and ($\phi_*$, $v_*$) are connected by a 1-shock while the states ($\phi_*$, $v_*$) and ($\phi_R$, $v_R$) are connected by a 2-rarefaction, then $\phi_L < \phi_* < \phi_R$ and the intermediate state ($\phi_*$, $v_*$) is obtained from

$$v_* = v_L - c\left(\phi_* - \phi_L\right)\sqrt{\frac{1}{\phi_*\phi_L}} \qquad v_* = v_R - c\ln\phi_R + c\ln\phi_*$$
$$\text{with } \phi_* \text{ being the unique root of } \quad v_R - v_L = c\ln\phi_R - c\ln\phi_* - c\left(\phi_* - \phi_L\right)\sqrt{\frac{1}{\phi_*\phi_L}} \tag{17}$$

Table 1 states the conditions for all the possible solutions of the Riemann problem, provided the intermediate state differs from the left and from the right ones.

Table 1. Conditions for each of the four possible solutions for the Riemann problem.

| CONDITION | Solution Type |
|---|---|
| $v_R - v_L > c\left\|\ln\frac{\phi_L}{\phi_R}\right\|$ | 1-rarefaction/2-rarefaction |
| $-c\left\|\sqrt{\frac{\phi_R}{\phi_L}} - \sqrt{\frac{\phi_L}{\phi_R}}\right\| > v_R - v_L$ | 1-shock/2-shock |
| $c\left[\sqrt{\frac{\phi_R}{\phi_L}} - \sqrt{\frac{\phi_L}{\phi_R}}\right] < v_R - v_L < c\ln\frac{\phi_L}{\phi_R}$ | 1-rarefaction/2-shock |
| $-c\left[\sqrt{\frac{\phi_R}{\phi_L}} - \sqrt{\frac{\phi_L}{\phi_R}}\right] < v_R - v_L < -c\ln\frac{\phi_L}{\phi_R}$ | 1-shock/2-rarefaction |

## 4. AN EXAMPLE (UNCONSTRAINED)

Considering the flow through a rigid porous medium with constant porosity $\varepsilon$, fluid fraction $\phi$ and fluid constituent velocity $v$, the following associated Riemann problem may be stated

$$\left.\begin{aligned} \frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x}(\phi v) = 0 \\ \frac{\partial}{\partial t}(\phi v) + \frac{\partial}{\partial x}\left(c^2\phi + \phi v^2\right) = 0 \end{aligned}\right\rangle \qquad (\phi, v) = \begin{cases} \left(\bar{\phi}, \bar{v}\right) & \text{for} \quad t = 0, \quad -\infty < x < 0 \\ \left(\bar{\phi}, -\bar{v}\right) & \text{for} \quad t = 0, \quad 0 < x < \infty \end{cases} \tag{18}$$

where $\bar{\phi}$ and $\bar{v}$ are positive constants, such that $\bar{\phi} < \varepsilon$ . In this case, it is easy to see that the solution (in a generalized sense) is 1-shock/2-shock. So,

$$(\phi, v) = \begin{cases} \left(\phi_L, v_L\right) & \text{if} \quad -\infty < x/t < s_1 \\ \left(\phi_*, v_*\right) & \text{if} \quad s_1 < x/t < s_2 \\ \left(\phi_R, v_R\right) & \text{if} \quad s_2 < x/t < \infty \end{cases} \tag{19}$$

in which the intermediate state, the pressure $p$ and the shock speeds are, respectively, given by

$$\phi_* = \bar{\phi}\left\{\frac{\bar{v}}{2c} + \sqrt{\left(\frac{\bar{v}}{2c}\right)^2 + 1}\right\}^2 \quad \text{and} \quad v_* = 0 \tag{20}$$

$$p = \begin{cases} c^2\bar{\phi} & \text{if} \quad -\infty < x/t < s_1 \\ \left\{\frac{\bar{v}}{2} + \sqrt{\left(\frac{\bar{v}}{2}\right)^2 + c^2}\right\}^2 \bar{\phi} & \text{if} \quad s_1 < x/t < s_2 \\ c^2\bar{\phi} & \text{if} \quad s_2 < x/t < \infty \end{cases} \tag{21}$$

$$s_2 = -s_1 = \frac{\bar{v}}{\left\{\frac{\bar{v}}{2c} + \sqrt{\left(\frac{\bar{v}}{2c}\right)^2 + 1}\right\}^2 - 1} \tag{22}$$

The solution of the associated Riemann problem (18), given by Eqs. (19)-(22), is physically realistic provided that the intermediate fluid fraction $\phi_*$ is always smaller than or equal to the porosity $\varepsilon$. In other words, the following relation must hold

$$\phi_* = \bar{\phi}\left\{\frac{\bar{v}}{2c} + \sqrt{\left(\frac{\bar{v}}{2c}\right)^2 + 1}\right\}^2 \le \varepsilon \tag{23}$$

The inequality (23) holds if, and only if,

$$\bar{v} \le c\left[\sqrt{\frac{\varepsilon}{\bar{\phi}}} - \sqrt{\frac{\bar{\phi}}{\varepsilon}}\right] \tag{24}$$

It is important to notice that many initial data do not satisfy (23), leading to results without physical meaning. On the other hand, if adequate initial data is chosen, assuring $\phi < \varepsilon$, the nonlinear hyperbolic system (6) may be approximated by Glimm's method, implemented by employing the solution of a certain number of associated Riemann problems (Martins-Costa and Saldanha da Gama, 2005; 2003; 2001).

## 5. THE RIEMANN PROBLEM AND THE CONSTRAINT $\phi \le \varepsilon$

Since $\phi_L$ and $\phi_R$ satisfy the inequality $\phi \le \varepsilon$ then, taking into account the conditions presented in Table 1, this inequality is ensured everywhere, provided that the solution is 1-rarefaction/2-rarefaction, 1-rarefaction/2-shock or 1-shock/2-rarefaction.

However, when the solution is 1-shock/2-shock, one may have $\phi > \varepsilon$, for $x/t$ between $s_1$ and $s_2$, if $p$ is considered as a function of $\phi$. In such cases it must be taken into account that the pressure $p$ is not constitutive, for $\phi = \varepsilon$. In other words, $p$ may assume any value greater than (or equal to) $p = c^2\varepsilon$, provided the Rankine-Hugoniot jump conditions are satisfied as well as the entropy conditions.

So, if the root of the equation in the third line of Eq. (15) – which is employed to compute $\phi_*$ – is such that $\phi_* > \varepsilon$, the physical meaning of the phenomenon has been lost. In these cases, the intermediate state must satisfy the jump conditions with $\phi_* = \varepsilon$. Hence $p_*$ and $v_*$ will be evaluated from

$$\frac{\varepsilon v_* - \phi_L v_L}{\varepsilon - \phi_L} = \frac{\varepsilon v_*^2 - \phi_L v_L^2 + p_* - c^2\phi_L}{\varepsilon v_* - \phi_L v_L} = s_1 \quad \text{and} \quad \frac{\phi_R v_R - \varepsilon v_*}{\phi_R - \varepsilon} = \frac{\phi_R v_R^2 - \varepsilon v_*^2 + c^2\phi_R - p_*}{\phi_R v_R - \varepsilon v_*} = s_2 \tag{25}$$

Thus, the intermediate pressure is obtained from the following equation

$$\sqrt{\left(p_* - c^2\phi_L\right)\left(\frac{1}{\phi_L} - \frac{1}{\varepsilon}\right)} + \sqrt{\left(c^2\phi_R - p_*\right)\left(\frac{1}{\varepsilon} - \frac{1}{\phi_R}\right)} = v_L - v_R \tag{26}$$

Since

$$\sqrt{\left(c^2\phi_R - c^2\phi_L\right)\left(\frac{1}{\phi_L} - \frac{1}{\phi_R}\right)} < v_L - v_R \tag{27}$$

it is ensured that $p \geq c^2\varepsilon$. The intermediate velocity $v_*$ may be obtained from

$$v_* = \frac{1}{2}\left( v_L + v_R - \left(\varepsilon - \phi_L\right)\sqrt{\left(\frac{p_* - c^2\phi_L}{\varepsilon - \phi_L}\right)\frac{1}{\varepsilon\phi_L}} - \left(\phi_R - \varepsilon\right)\sqrt{\left(\frac{c^2\phi_R - p_*}{\phi_R - \varepsilon}\right)\frac{1}{\phi_R\varepsilon}} \right) \tag{28}$$

From Eq. (26) it is easy to know, a priori, if the conditions $\phi_* = \varepsilon$ and $p > c^2\varepsilon$ are fulfilled. It is quite obvious that such cases take place when the following inequality holds

$$\sqrt{\varepsilon}\left[\frac{1}{\sqrt{\phi_R}} + \frac{1}{\sqrt{\phi_L}}\right] - \frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\phi_R} + \sqrt{\phi_L}\right] < \frac{v_L - v_R}{c} \tag{29}$$

Anyway, in order to ensure $\phi_* \leq \varepsilon$, when inequality $c\left|\sqrt{\phi_R / \phi_L} - \sqrt{\phi_L / \phi_R}\right| < v_L - v_R$ holds, $\phi_*$ is given by

$$\phi_* = \frac{1}{2}\left(\phi_\# + \varepsilon - \left|\phi_\# - \varepsilon\right|\right), \quad \text{with} \quad \phi_\# = \frac{\phi_R\phi_L}{\left(\sqrt{\phi_R} + \sqrt{\phi_L}\right)^2}\left\{\frac{v_L - v_R}{2c} + \sqrt{\left(\frac{v_L - v_R}{2c}\right)^2 + 2 + \sqrt{\frac{\phi_R}{\phi_L}} + \sqrt{\frac{\phi_L}{\phi_R}}}\right\}^2 \tag{30}$$

while $v_*$ and $p_*$ are obtained from

$$\begin{gathered} \sqrt{\left(p_* - c^2\phi_L\right)\left(\frac{1}{\phi_L} - \frac{1}{\phi_*}\right)} + \sqrt{\left(c^2\phi_R - p_*\right)\left(\frac{1}{\phi_*} - \frac{1}{\phi_R}\right)} = v_L - v_R \\ v_* = \frac{1}{2}\left\{ v_L + v_R - \left(\sqrt{\frac{\phi_*}{\phi_L}} - \sqrt{\frac{\phi_L}{\phi_*}}\right)\sqrt{\left(\frac{p_* - c^2\phi_L}{\phi_* - \phi_L}\right)} - \left(\sqrt{\frac{\phi_R}{\phi_*}} - \sqrt{\frac{\phi_*}{\phi_R}}\right)\sqrt{\left(\frac{c^2\phi_R - p_*}{\phi_R - \phi_*}\right)} \right\} \end{gathered} \tag{31}$$

## 6. AN EXAMPLE (CONSTRAINED)

Now, considering the flow through a rigid porous medium with constant porosity $\varepsilon$, fluid fraction $\phi$ and fluid constituent velocity $v$, taking into account the physical constraint $\phi \leq \varepsilon$ everywhere, the following associated Riemann problem may be stated

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left(\phi v\right) = 0 \\ \frac{\partial}{\partial t}\left(\phi v\right) + \frac{\partial}{\partial x}\left(c^2\phi + \phi v^2\right) = 0 \\ \phi \leq \varepsilon \end{aligned}\right\rangle \qquad \left(\phi, v\right) = \begin{cases} \left(\overline{\phi}, \overline{v}\right) & \text{for} \quad t = 0, \quad -\infty < x < 0 \\ \left(\overline{\phi}, -\overline{v}\right) & \text{for} \quad t = 0, \quad 0 < x < \infty \end{cases} \tag{32}$$

where $\overline{\phi}$ and $\overline{v}$ are positive constants, such that $\overline{\phi} < \varepsilon$ . If inequality (24) holds, then Eqs. (19)-(22) – representing the generalized solution of the unconstrained Riemann problem (18) – also hold. On the other hand, if Eq. (18) does not hold, the solution is given by Eq. (19), with $\phi_* = \varepsilon$ and $v_* = 0$. The pressure $p_*$ is calculated from

$$\sqrt{\left(p_* - c^2\overline{\phi}\right)\left(\frac{1}{\overline{\phi}} - \frac{1}{\varepsilon}\right)} + \sqrt{\left(c^2\overline{\phi} - p_*\right)\left(\frac{1}{\varepsilon} - \frac{1}{\overline{\phi}}\right)} = \overline{v} + \overline{v} \;, \quad \text{being given by} \quad p_* = \left[c^2 + \frac{\varepsilon\overline{v}^2}{\varepsilon - \overline{\phi}}\right]\overline{\phi} \tag{33}$$

while the shock speeds are

$$s_1 = \frac{-\bar{\phi}\bar{v}}{\varepsilon - \bar{\phi}} = \frac{-\bar{\phi}\bar{v}^2 + p_* - c^2\bar{\phi}}{-\bar{\phi}\bar{v}} \quad \text{and} \quad s_2 = \frac{-\bar{\phi}\bar{v}}{\bar{\phi} - \varepsilon} = \frac{\bar{\phi}\bar{v}^2 + c^2\bar{\phi} - p_*}{\bar{\phi}\bar{v}} \tag{34}$$

Hence, the solution of the constrained associated Riemann problem (32) may be represented as follows

$$(\phi, v) = \begin{cases} (\bar{\phi}, \bar{v}) & \text{if} \quad -\infty < x/t < \dfrac{-\bar{\phi}\bar{v}}{\phi_* - \bar{\phi}} \\ (\phi_*, 0) & \text{if} \quad \dfrac{-\bar{\phi}\bar{v}}{\phi_* - \bar{\phi}} < x/t < \dfrac{\bar{\phi}\bar{v}}{\phi_* - \bar{\phi}} \\ (\bar{\phi}, -\bar{v}) & \text{if} \quad \dfrac{\bar{\phi}\bar{v}}{\phi_* - \bar{\phi}} < x/t < \infty \end{cases} \tag{35}$$

in which the intermediate fluid fraction $\phi_*$ is obtained from

$$\phi_* = \frac{1}{2}\left\{\left[\bar{\phi}\left(\frac{\bar{v}}{2c} + \sqrt{\left(\frac{\bar{v}}{2c}\right)^2 + 1}\right)^2 + \varepsilon\right] - \left|\bar{\phi}\left(\frac{\bar{v}}{2c} + \sqrt{\left(\frac{\bar{v}}{2c}\right)^2 + 1}\right)^2 - \varepsilon\right|\right\} \tag{36}$$

Once $\phi_*$ is known, the pressure $p_*$ and the shock speeds may be obtained from

$$p_* = \left[c^2 + \frac{\phi_*\bar{v}^2}{\phi_* - \bar{\phi}}\right]\bar{\phi} \qquad \text{and} \qquad s_2 = -s_1 = \frac{\bar{\phi}\bar{v}}{\phi_* - \bar{\phi}} \tag{37}$$

Now considering the following three particular situations:

- CASE 1: $\bar{v} = c$, $\bar{\phi} = 0.2$ and $\varepsilon = 0.8$ (Eqs. (41)-(42) hold)
- CASE 2: $\bar{v} = 2c$, $\bar{\phi} = 0.2$ and $\varepsilon = 0.8$ (Eqs. (41)-(42) do not hold)
- CASE 3: $\bar{v} = c$, $\bar{\phi} = 0.6$ and $\varepsilon = 0.8$ (Eqs. (41)-(42) do not hold)

The intermediate fluid fraction $\phi_*$, the ratio $p_*/c^2$ and the ratios $s_2/c = -s_1/c$ for the above cases are presented in Table 2, obtained from Eq. (18) and Eq. (32), respectively, unconstrained and constrained Riemann problem.

Table 2. Comparison between unconstrained and constrained descriptions.

| | UNCONSTRAINED | | CONSTRAINED | |
|---|---|---|---|---|
| case (1) | $\phi_* = 0.5236$<br>$p_*/c^2 = 0.5236$<br>$s_2/c = -s_1/c = 0.6180$ | (**1U**) | $\phi_* = 0.5236$<br>$p_*/c^2 = 0.5236$<br>$s_2/c = -s_1/c = 0.6180$ | (**1C**) |
| case (2) | $\phi_* = 1.1657$<br>$p_*/c^2 = 1.1657$<br>$s_2/c = -s_1/c = 0.4142$ | (**2U**) | $\phi_* = 0.8000$<br>$p_*/c^2 = 1.2667$<br>$s_2/c = -s_1/c = 0.6667$ | (**2C**) |
| case (3) | $\phi_* = 1.5708$<br>$p_*/c^2 = 1.5708$<br>$s_2/c = -s_1/c = 0.1459$ | (**3U**) | $\phi_* = 0.8000$<br>$p_*/c^2 = 3.0000$<br>$s_2/c = -s_1/c = 3.0000$ | (**3C**) |

It is important to notice that unconstrained examples of cases (2) and (3) – represented by (**2U**) and (**3U**) have no physical meaning, since that the fluid fraction is greater than the porosity.

Some results are now presented. In Fig. 1 the solutions in the plane *x–t* for the three cases addressed in Table 2, constrained and unconstrained, are exhibited. Case (2) is analyzed in Figure 2, comparing constrained and unconstrained results for the fluid fraction as a function of the time, while case (3) is considered in Figure 3, which presents the pressure $p$ as a function of the ratio $x/t$ for both constrained and unconstrained results.

Figure 1 presents the results for cases (1) to (3) in the plane $x$–$t$ considering an unitary value for the first derivative of the pressure with respect to the fluid fraction – namely $c^2$=1. Case (1) presents the same results for constrained and unconstrained solutions (**1C** and **1U**) – since the initial values have been conveniently chosen, assuring $\phi < \varepsilon$ everywhere, but obviously the transition unsaturated$\rightarrow$saturated flow is never observed.

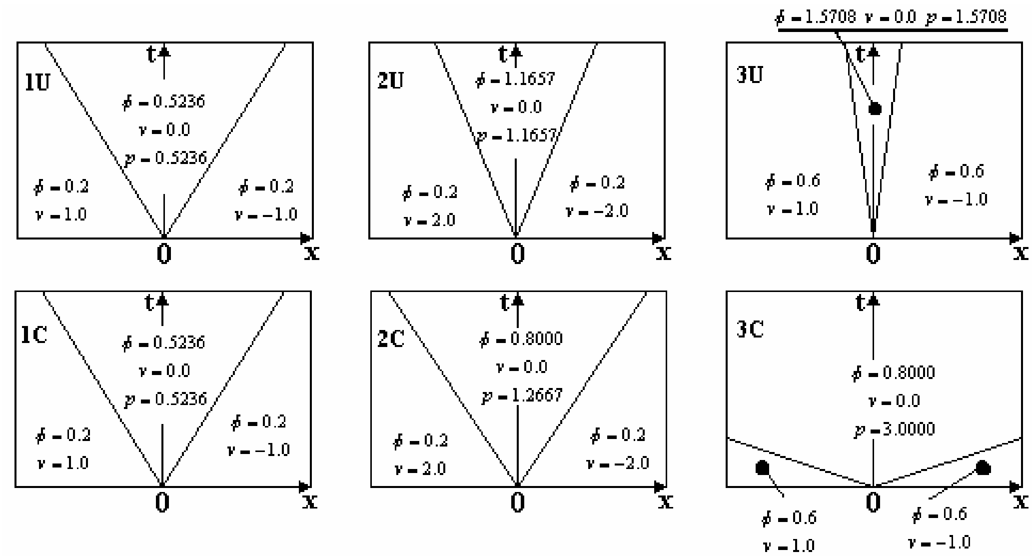

Figure 1. Solutions presented in the plane $x-t$ for cases (1) to (3), unconstrained (**U**) and constrained (**C**), assuming $c$=1.

The transition unsaturated$\rightarrow$saturated flow may be observed in the other two cases for constrained results – namely **2C** and **3C**. The region in the plane $x$–$t$ in which $\phi$=$\varepsilon$ =0.8 corresponds to a saturated flow. This phenomenon cannot be observed for unconstrained results, **2U** and **3U**, since the physical meaning has been lost (because regions where $\phi > \varepsilon$ are allowed). It is interesting to observe the difference between the shock speeds and between the maximum pressures in case 3. (Considering $c$=1, for the case **3C**: $p_*$=$s_2$=-$s_1$=3 and for the case **3U**: $p_*$=1.5708 and $s_2$=-$s_1$=0.1459.)

Figure 2 depicts constrained and unconstrained results for case (2) considering the evolution of the fluid fraction $\phi$ at $x$=1. Up to $t$=1.5, both results **2C** and **2U** are coincident at $x$=1. At this point transition unsaturated$\rightarrow$saturated flow takes place for case **2C**, while case **2U** remains unsaturated up to $t$=2.4, when it suddenly becomes physically unrealistic, with $\phi > \varepsilon$, illustrating the relevance of the procedure introduced in this work.

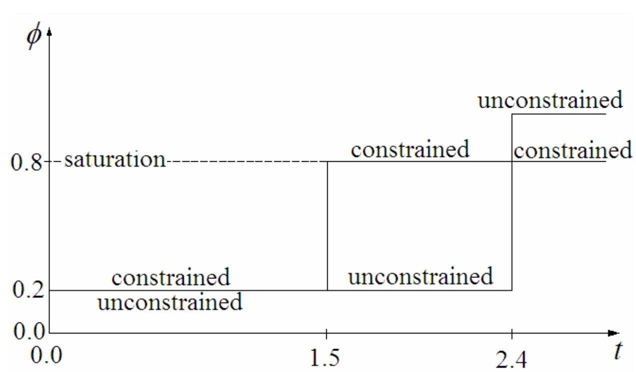


Figure 2. Fluid fraction $\phi$ as a function of the time $t$ at the position $x$=1, assuming $c$=1, for case (2).

The pressure variation with $x/t$ is shown in figure 3, for constrained and unconstrained results, in which not only the maximum pressures are distinct (for the case **3C**: $p_*$ =3 and for the case **3U**: $p_*$=1.5708), but also the extension of the regions of the ratio $x/t$ subjected to maximum pressures, again emphasizing the applicability of the solution methodology proposed in this work.

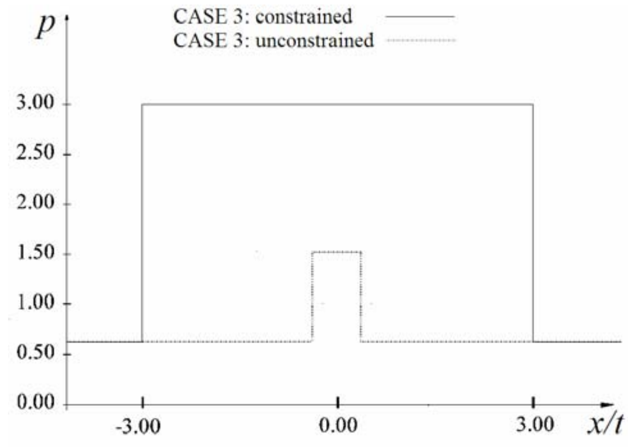


Figure 3. Pressure as a function of the ratio *x*/*t* for case (3), assuming *c*=1. The dashed line corresponds to the unconstrained description (without physical meaning) while the continuous line corresponds to the constrained description.

## 7. FINAL REMARKS

In this work a mathematical model for flows through unsaturated porous media, identifying the transition unsaturated/saturated flow, was proposed by including a constraint that must be satisfied to build physically realistic generalized solutions for any initial data. The complete solution of a constrained nonlinear hyperbolic problem with shock waves – an associated Riemann problem containing a restriction (an upper bound for the fluid fraction, represented by the porosity), was presented as well as its application to flows through porous media. Some cases were simulated in order to show the differences between the constrained (always physically consistent) and the unconstrained (sometimes without physical sense – depending on the previously chosen initial data) descriptions.

## 8. ACKNOWLEDGEMENTS

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