ASSESSMENT OF FRACTIONAL-ORDER CONTROLLERS FOR ACTIVE VIBRATION CONTROL

Ávila, Edson Borges, edsonborgesmec@yahoo.com.br
Sales, Thiago de Paula, thiago.mecanica@gmail.com
Rade, Domingos Alves, domingos@ufu.br
Lacerda, Helder Barbieri, helder@mecanica.ufu.br
Federal University of Uberlândia – School of Mechanical Engineering – Prof. José Eduardo Tannús Reis Structure Mechanics Laboratory, Santa Mônica Campus – Building 1O – P.O. Box 593 – Zip Code 38400-902 – Uberlândia, MG – Brazil.

Abstract. *Differential models of non-integer orders, or fractional models, have been increasingly used in many fields of science and engineering, such as rheology (modeling of viscoelastic behavior), electrochemical processes, dielectric polarization, heat transfer phenomena, and chaos, among others. In spite of the more involved theoretical and numerical procedures, the use of such models is justified by a more accurate modeling, as has been demonstrated by many authors. The application of fractional models in the area of active control of dynamical systems deserves particular mention due to the increasing number of publications concerning the theme in the last years. The present paper address the generalization of the traditional PID controller by the use of fractional operators, which are briefly reviewed. One demonstrates the efficiency of fractional controllers, as compared to traditional integer-order controllers, explaining the influence of the fractional order over the closed-loop system response, by considering the implementation to an undamped dynamical system. It is concluded that the utilization of the Fractional Calculus as associated to system control is well justified as an intereresting alternative to traditional design of controlers, as such technique provides extended modeling flexibility and improved accuracy.*

Keywords: *Fractional operators; Active vibration control; PID controller.*

1. INTRODUCTION

Fractional Calculus is an old mathematical analysis field dating from 300 years ago that has attracted the attention from several famous mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, Laurent, Lacroix, Leibniz, Grunwald e Letnikov and Weyl. It has been employed in the last three decades in modern applications of differential and integral equations to the modeling of several types of problems of Science and Engineering, such as: signal processing (Barbosa *et al*., 2006; Bultheel and Martinez-Sulbaran, 2007); electrical systems (Yifei *et al*., 2005); fluid mechanics (Amaral, 2003); viscoelasticity (Bagley and Torvik, 1983; Glockle and Nonnenmacher, 1991; Maia, 1998; Adolfsson *et al*., 2005; Bagley, 2007; Jia *et al*., 2007); mathematical biology (Cole, 1933; Anastasio, 1994); electrochemics (Oldham, 1972; Goto and Ishii, 1975); rheology (Cavazos *et al*., 2007); heat transfer (Agrawal, 2004); economy (Meerschaert, 2006); electromagnetism (Engheta, 1996; Machado *et al*., 2006); diffusion problems (Pedron, 2003; Gonçalves *et al*., 2005; Pedron and Mendes, 2005; Andrade, 2006; Gonçalves *et al*., 2006).

The fundamental motivation for the practical use of the Fractional Calculus is the possibility of a more accurate modeling of physical phenomena, at the expense of a higher analytical and numerical complexity in comparison with the traditional Calculus tools.

The relevance of such theme in Science and Engineering is proved by the expressive and growing number of publications that exists in the form of books and scientific articles, besides the existence of international conferences dedicated to the topic.

In this context, the present work has the aim of reviewing the fundamentals of Fractional Calculus and to illustrate its application to the field of active control of dynamical systems. This is done by presenting the generalization of traditional largely known active controllers and by performing an implementation of a fractional controller to an undamped mechanical vibrating system. Also, another major objective of this work is to investigate the influence of the controller parameters over the closed-loop system response. Some reference works within this area are the ones of Hartley and Lorenzo (2002), Valério and Costa (2006), Podlubny (1994), Dorcak (1994) and Podlubny *et al*. (1997).

2. FUNDAMENTALS OF FRACTIONAL CALCULUS

The works by Kilbas *et al*. (2006), Miller and Ross (1993) and Sabatier *et al*. (2007) bring comprehensive presentations of the fundamentals of Fractional Calculus, which are briefly reviewed in this section.

The Riemann-Liouville fractional integration of order α is defined as:

$$J^{\alpha} f(t) = D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \,,\ \alpha > 0 \,, \tag{1}$$

in which $\Gamma(\bullet)$ denotes the gamma function, defined as:

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt \tag{2}$$

with $z>0$ (or $\mathrm{Re}(z)>0$, if z is a complex number).

Two distinct definitions for fractional derivatives are commonly utilized in solving engineering problems that involves fractional differential equations:

− The Caputo Fractional Derivative:

$$D_C^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}d\tau \tag{3}$$

− The Riemann-Liouville Fractional Derivative:

$$D^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \tag{4}$$

in which n is the least integer value greater than or equal to α. For null initial conditions, both definitions become equivalent.

One can demonstrate that the Laplace transform of a Riemann-Liouville fractional derivative is given by (Kilbas *et al*., 2006):

$$\mathfrak{L}\left\{D^{\alpha} f(t)\right\}=s^{\alpha}\mathfrak{L}\left\{f(t)\right\}-\sum_{k=0}^{n-1} s^k D^{\alpha-1-k} f(0),\ n-1<\alpha<n. \tag{5}$$

When $0<\alpha<1$, Eq. (5) reduces to:

$$\mathfrak{L}\left\{D^{\alpha} f(t)\right\}=s^{\alpha}\mathfrak{L}\left\{f(t)\right\}. \tag{6}$$

For obtaining the solution of fractional differential equations, two wide classes of numerical methods are commonly used. One of them utilizes direct approximation of fractional derivatives in the time domain. Methods that belong to this class are called direct methods. The second class, named indirect methods, utilize different approximations of the fractional derivative operator.

An efficient indirect method developed for the simulation of complex fractional systems involves the approximation of the fractional integration operator in the frequency domain with a state space representation (Poinot and Trigeassou, 2003). Such approximation utilizes an ideal integrator $1/s$ acting together with a conventional phase filter proposed by Oustaloup (1995):

$$A_v(j\omega)=\prod_{i=1}^{N}\frac{1+j(\omega/\omega_i')}{1+j(\omega/\omega_i)} \tag{7}$$

where ω_1' is the lowest and ω_N is the largest frequencies that define the frequency band in which the problem analysis will be performed, and:

$$\omega_i=\alpha\,\omega_i',\ \omega_{i-1}'=\eta\,\omega_i,\ n=1-\frac{\log\alpha}{\log\alpha\eta},\ \eta=\left(\frac{\omega_N}{\omega_1'}\right)^{\frac{n}{N-n}},\ \alpha=\eta^{\frac{1-n}{n}},\ G_n=\left|\prod_{i=1}^{N}\frac{1+j/\omega_i'}{1+j/\omega_i}\right|^{-1}, \tag{8}$$

in which $n>0$ is the fractional integration order and $j=\sqrt{-1}$ is the imaginary unit.

The approximation for the fractional integrator is then given by:

$$J^n(s)=D^{-n}(s)=\frac{G_n}{s}\prod_{i=1}^{N}\frac{1+s/\omega_i'}{1+s/\omega_i}, \tag{9}$$

whose block diagram is presented in Fig. 1.

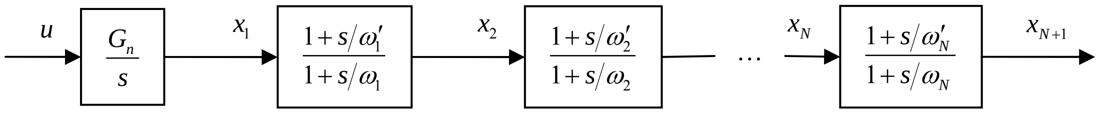

Figure 1. Block diagram of the fractional integrator approximation, $D^{-n}(s)$.

The state space representation of the operator is given by (Poinot and Trigeassou, 2003):

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha & 1 & & & \vdots \\ 0 & -\alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{N+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \omega_1 & -\omega_1 & & & \vdots \\ 0 & \omega_2 & -\omega_2 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_N & -\omega_N \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{N+1} \end{bmatrix} + \begin{bmatrix} G_n \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \cdot u . \tag{10}$$

3. GENERALIZATION OF THE TRADITIONAL PID CONTROLLER

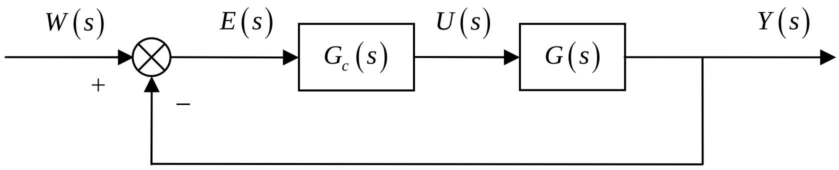

Figure 2. Control system with unity feedback.

The active control of dynamical systems is typically performed through the known PID (Proportional-Integral-Derivative) controllers and their variants, from which PI and PD controllers are two examples. One considers controlled processes with unity feedback as shown in Fig. 2. In this figure, $G(s)$ designates the transfer function of the process to be controlled, $G_c(s)$ designates the controller's transfer function, $W(s)$ designates an input, $E(s)$ designates an error, $U(s)$ designates the controller's output and $Y(s)$ designates the system's output.

In the Laplace domain, the transfer function associated to a PID controller is found to be:

$$G_c(s) = \frac{U(s)}{E(s)} = k_1 + \frac{k_2}{s} + k_3 s = k_c\left(1 + \frac{1}{T_i s} + T_d s\right), \tag{11}$$

in which $k_1 = k_c$, $k_2 = k_c/T_i$ and $k_3 = k_c T_d$ are the controller's parameters to be chosen in such a way that the closed-loop system response satisfies some design criteria. In the time domain the controller is represented by the differential equation that relates the time signals of the error $e(t)$ to the controller's output $u(t)$:

$$u(t) = k_1 e(t) + k_2 \int e(t)\,dt + k_3 \frac{d}{dt} e(t) = k_1 e(t) + k_2 D^{-1} e(t) + k_3 D^{1} e(t) . \tag{12}$$

If fractional order elements are available, an immediate generalization resulting in a fractional PID controller can be made through the inclusion of fractional integration and differentiation into the traditional integer-order PID controller. From Eq. (11), one has that the fractional PID controller transfer function is given by:

$$G_c(s) = \frac{U(s)}{E(s)} = k_1 + \frac{k_2}{s^{\lambda}} + k_3 s^{\mu} = k_c\left(1 + \frac{1}{T_i s^{\lambda}} + T_d s^{\mu}\right), \ \lambda > 0,\ \mu > 0 . \tag{13}$$

The following fractional differential equation in the time domain is associated to this controller:

$$u(t) = k_1 e(t) + k_2 D^{-\lambda} e(t) + k_3 D^{\mu} e(t) . \tag{14}$$

It should be pointed-out that, as the control law given by Eq. (14) involves fractional integration of order λ and fractional differentiation of order μ, the fractional PID controller is generally referred to as a PI$^{\lambda}$D$^{\mu}$ controller.

One can demonstrate that the closed-loop system transfer function for the control system with unity feedback as presented in Fig. 2 is given by:

$$T(s)=\frac{Y(s)}{W(s)}=\frac{G_c(s)G(s)}{1+G_c(s)G(s)} \tag{15}$$

with the following associated characteristic equation:

$$1+G_c(s)G(s)=0\,. \tag{16}$$

Then, for the traditional PID controller, one has that the associated characteristic equation is:

$$1+k_c\left(1+\frac{1}{T_i s}+T_d s\right)G(s)=0\,, \tag{17}$$

while that for a $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller one would have the following characteristic equation:

$$1+k_c\left(1+\frac{1}{T_i s^{\lambda}}+T_d s^{\mu}\right)G(s)=0 \tag{18}$$

As one has as fundamental objective in control design the correct allocation of the closed-loop system transfer function poles, it is verified by direct comparison between Eqs. (17) and (18) that the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller provides enhanced freedom and improved design capability that those ones provided by the traditional integer-order PID controller.

4. IMPLEMENTATION OF A FRACTIONAL CONTROLLER TO AN UNDAMPED SYSTEM

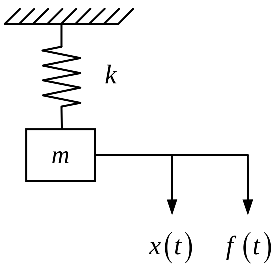

Figure 3. One-degree-of-freedom vibrating system to be controlled.

Consider the undamped mechanical system presented in Fig. 3 and described by the following differential equation of motion:

$$m\ddot{x}(t)+k\,x(t)=f(t)\,, \tag{19}$$

in which m represents the system mass, k represents the stiffness, $x(t)$ represents the displacement, $f(t)$ represents an external force applied to the mass and the superposed two dots denotes a second order time differentiation. One desires to active control the vibrating motion of the mass by means of a control force that will act on the system and that is given by $f_c(t)=g\,D^{\alpha}x(t)$, $0<\alpha<2$. A block diagram of the system acted upon by the control force is presented in Fig. 4.

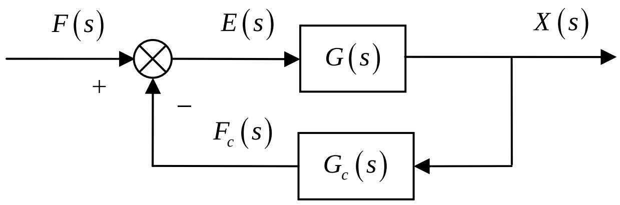

Figure 4. Block diagram of the control system.

When the control force is acting over the system presented in Fig. 3, the following differential relationships hold in the time domain:

$$m\ddot{x}(t)+k\,x(t)=e(t)\,; \tag{20}$$

$$e(t)=f(t)-f_c(t)\,. \tag{21}$$

Applying the Laplace transform to the Eq. (20), one has the transfer function of the process to be controlled:

$$G(s)=\frac{X(s)}{E(s)}=\frac{1}{m\,s^2+k}\,. \tag{22}$$

The transfer function of the controller to be utilized is:

$$G_c(s)=\frac{F_c(s)}{X(s)}=g\,s^{\alpha}\,. \tag{23}$$

Noticing that from Eq. (21) one haves:

$$E(s)=F(s)-F_c(s)=F(s)-g\,s^{\alpha}X(s) \tag{24}$$

and from Eq. (22) one has:

$$E(s)=X(s)/G(s)\,, \tag{25}$$

one can write the closed-loop system transfer function as follows:

$$T(s)=\frac{X(s)}{F(s)}=\frac{1}{m\,s^2+g\,s^{\alpha}+k}\,, \tag{26}$$

whose block diagram is given in Fig. 5.

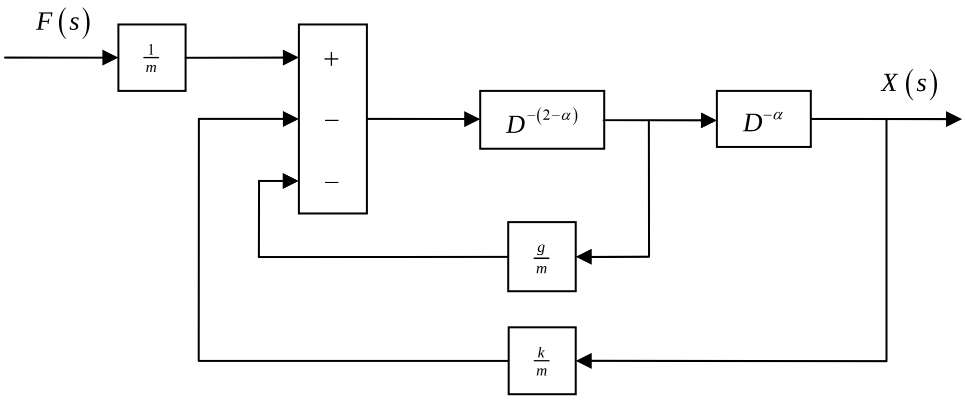

Figure 5. Closed-loop system transfer function block diagram.

One can, then, by means of an indirect method of numerical approximation of the fractional derivative operator, as the one described in Section 2, calculate the system response to unit step inputs, that is, $f(t)=1\ \text{N}$ for $t\geq 0$ and $f(t)=0\ \text{N}$ for $t<0$. The system mass and stiffness take the values 0.5 kg and 500 N/m, respectively. Several values of g and α were utilized to evaluate the influence of these parameters over the system response.

5. SIMULATION RESULTS AND ANALYSIS

The simulation results are presented as plots of the system's displacement responses $x(t)$. Figures 6 to 15 present the behavior of the system response for several values of g with α being kept constant, while the Figs. 16 to 22 present the behavior of the system response for several values of α, with g being kept constant.

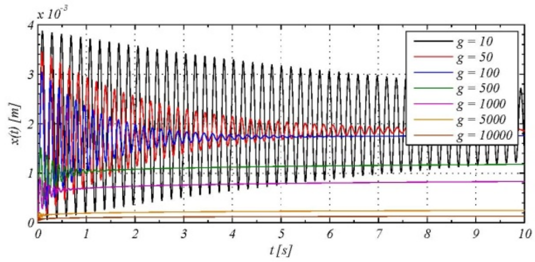

Figure 6. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 0.125$.

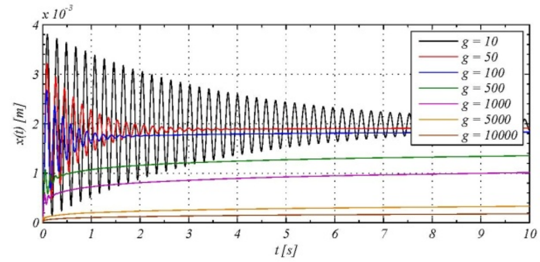

Figure 7. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 0.25$.

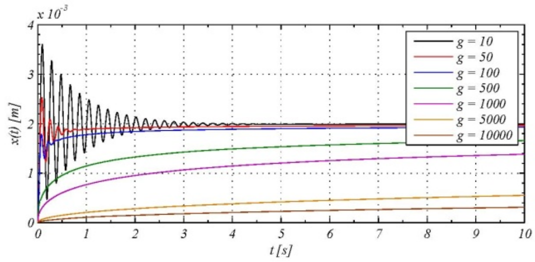

Figure 8. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 0.5$.

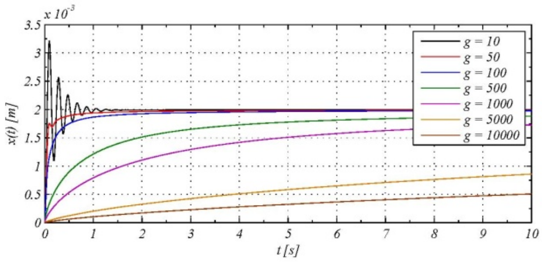

Figure 9. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 0.75$.

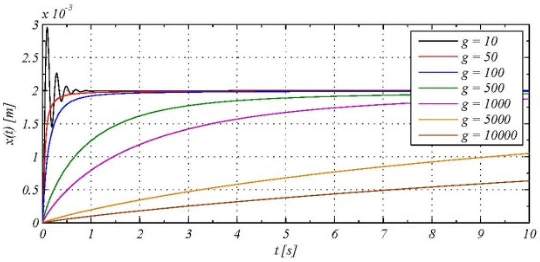

Figure 10. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 0.875$.

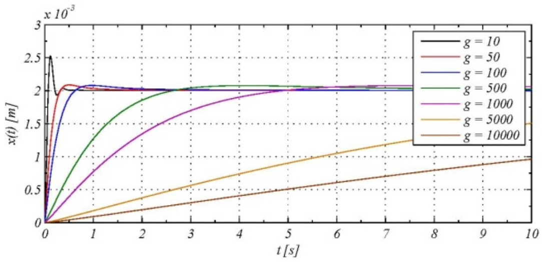

Figure 11. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 1.125$.

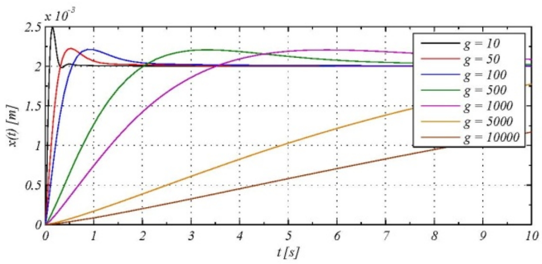

Figure 12. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 1.25$.

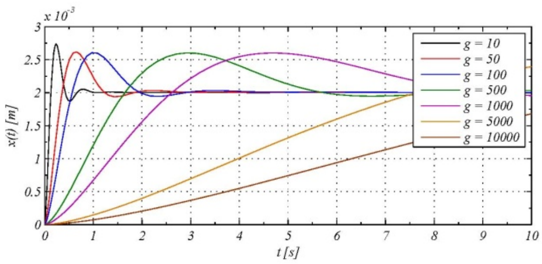

Figure 13. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 1.5$.

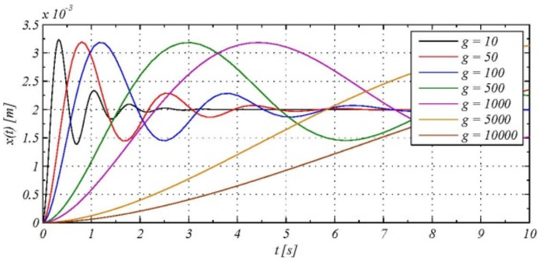

Figure 14. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 1.75$.

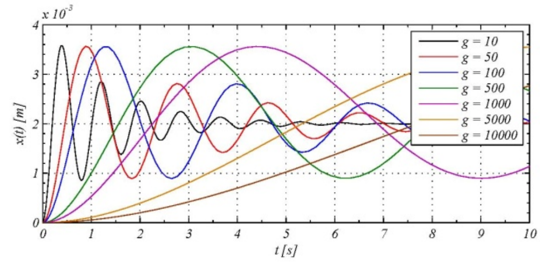

Figure 15. Dynamical response of $x(t)$ to an unit step excitation input for $\alpha = 1.875$.

From an analysis of the plots presented in Figs. 6 to 15, one can highlight that an increasing value of g causes: a reduction or an increase in the number of oscillations (for the cases in which the system presents an oscillatory response) for $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively; a reduction in the maximum overshoot; an increase in the rise time; and an increase in the steady state error (when present). The augmentation in the value of g can be interpreted as an augmentation in the system damping, implying in modification of the system characteristics: small values of g leads to an underdamped system behavior, while larger values of g leads to an overdamped system behavior.

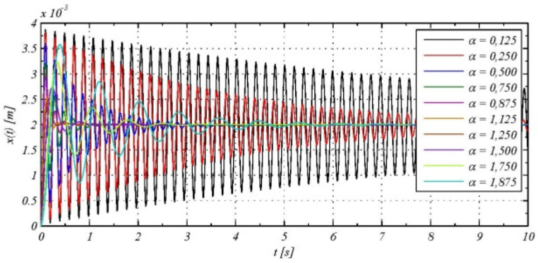

Figure 16. Dynamical response of $x(t)$ to an unit step excitation input for $g = 10$.

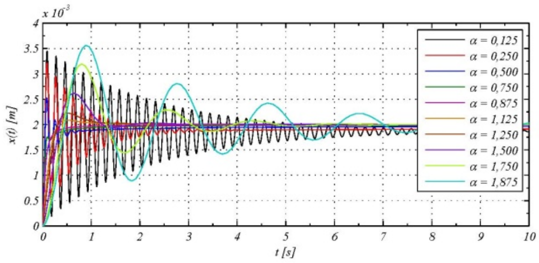

Figure 17. Dynamical response of $x(t)$ to an unit step excitation input for $g = 50$.

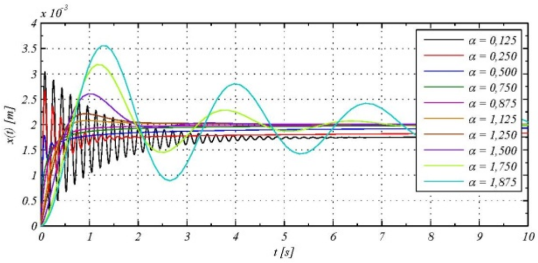

Figure 18. Dynamical response of $x(t)$ to an unit step excitation input for $g = 100$.

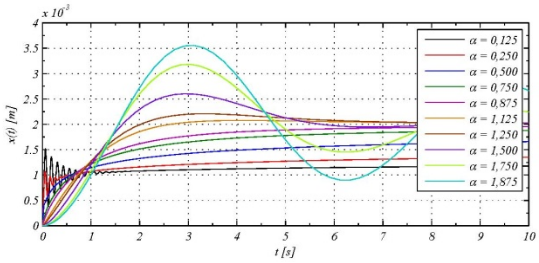

Figure 19. Dynamical response of $x(t)$ to an unit step excitation input for $g = 500$.

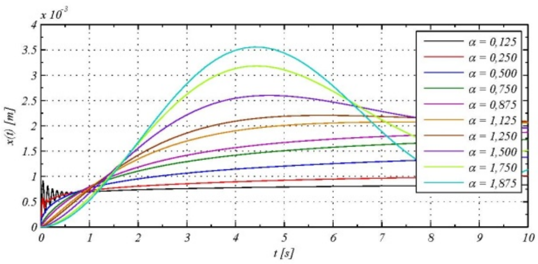

Figure 20. Dynamical response of $x(t)$ to an unit step excitation input for $g = 1000$.

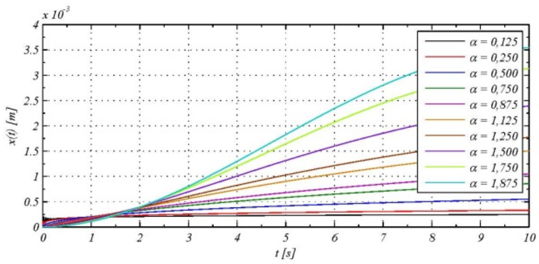

Figure 21. Dynamical response of $x(t)$ to an unit step excitation input for $g = 5000$.

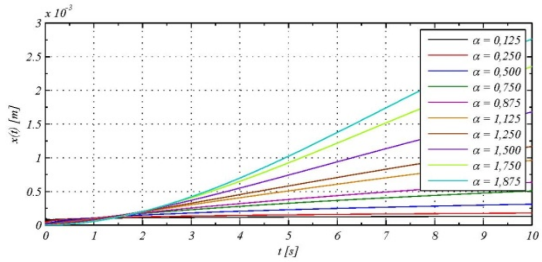

Figure 22. Dynamical response of $x(t)$ to an unit step excitation input for $g = 10000$.

By making an analysis of the plots presented in the Figs. 16 to 22 when g is kept constant, one can see that there is a reduction in the number of oscillations (when the system presents an oscillatory response) as α increases up to 1 and that there is an increase in the number of oscillations when α approaches 2. This fact can be explained by the nature of the control force: a value of α near 0 induces a control force similar to that generated by a spring; an α value near 1 induces a control force similar to that generated by a viscous damper; and an α value near 2 induces a control force similar to a system inertia force. Besides that, intermediate values of α between these conditions would imply a superposition of the effects induced by those forces corresponding to an integer value of α. As examples, for $\alpha = 0,5$, the control force would have both characteristics of a spring force and of a viscous damping force; for $\alpha = 1,5$, the control force would have both characteristics of a viscous damping force and of an inertia force. Associated to the increase in the α value, are also related an initial reduction and a later increase in the maximum overshoot, an increase in the rise time and a reduction in the steady state error (when present).

6. CONCLUSION

A fractional controller was implemented to an one degree of freedom undamped system. The system response to unit step inputs was simulated and allowed to evaluate the influence of controller parameters on the closed-loop response. The simulation results enables to conclude that utilization of the Fractional Calculus to system control is well justified and deserves attention by control designers, since fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers are associated to an enhanced design flexibility.

7. ACKNOWLEDGEMENTS

D. A. Rade acknowledges the Brazilian Research Council – CNPq and Minas Gerais State Agency FAPEMIG for the continued support to his research work.

T. P. Sales acknowledges the CNPq for the financial support, through project number 503605/2007-7. The third author also thanks CNPq for his research grant.

8. REFERENCES

Adolfsson, K., Enelund, M., Olsson, P., 2005, “On the Fractional Order Model of Viscoelasticity”, Mechanics of Time-Dependent Materials, Vol. 9, pp. 15-34.

Agrawal, O. P., 2004, “Application of Fractional Derivatives in Thermal Analysis of Disk Brakes”, Nonlinear Dynamics, Vol. 38, pp. 191-206.

Amaral, B. D., 2003, “Solução da Equação de Transporte Multidimensional em Geometria Cartesiana e Meio Infinito Usando Derivada Fracionária”, 104 pp., Dissertação de Mestrado, Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil.

Anastasio, T. J., 1994, “The Fractional-Order Dynamics of Brainstem Vestibulo-Oculomotor Neurons”, Biol. Cybernet., Vol. 72, Nº 1, pp. 69-79.

Bagley, R. L., Torvik, P. J., 1983, “A Theoretical Basic for the Application of Fractional Calculus to Viscoelasticity”, Journal of Rheology, Vol. 27, Nº 3, pp. 201-210.

Andrade, M. F., 2006, “Equações de Difusão Fracionárias e Não-Lineares: Soluções e Difusão Anômala”, 64 pp., Dissertação de Mestrado, Universidade Estadual de Maringá, Maringá, Brasil.

Bagley, R., 2007, “On the Equivalence of the Riemann-Liouville and the Caputo Fractional Order Derivatives in Modeling of Linear Viscoelastic Materials”, An International Journal for Theory and Applications, Vol. 10, Nº 2, pp. 123-126.

Barbosa, R. S., Machado, J. A. T., Silva, M. F., 2006, “Time Domain Design of Fractional Differintegrators Using Least-Squares”, Signal Processing, Vol. 86, pp. 2567-2581.

Bultheel, A., Martínez-Sulbaran, H., 2004, “Recent Developments in the Theory of the Fractional Fourier and Linear Canonical Transforms”, Bulletin of the Belgian Mathematical Society - Simon Stevin, Vol. 13, Nº 5, pp. 971-1005.

Cavazos, F. R. G., Melo, M. E. R., González, V. A. G., Salazar, C. A. G., Loera, A. G., 2007, “Aplicación del Cálculo Fraccional a la Reología de Materiales Poliméricos”, Ingenierías, Vol. X, Nº 35, pp. 42-47.
Cole, K. S., 1933, “Electic Conductance of Biological Systems”, Proc. Cold Spring Harbor Symp. Wuant. Biol., New York, pp. 107-116.
Dorcak, L., 1994, “Numerical models for the simulation of the fractional-order control systems”, Slovak Academy of Sciences, Institute of Experimental Physics, Technical University of Kosice, pp. 261-264.
Engheta, N., 1996, “On Fractional Calculus and Fractional Multipoles in Electromagnetism”, IEEE Trans. Antennas and Propagation, Vol. 44, Nº 4, pp. 554-566.
Glockle, W. G. and Nonnenmacher, T. F., 1991, “Fractional Integral Operators and Fox Functions in the Theory of Viscoelasticity”, Macromolecules, Vol. 24, pp. 6426-6434.
Gonçalves, G., Lenzi, M. K., M., Lenzi, E. K., Antonio, F. J., Schot, A., 2006, “Equação da Difusão Fracionária Não-Linear: Solução Exata”, Acta Sci. Technol, Vol. 28, Nº 1, pp. 47-53.
Gonçalves, G., Lenzi, M. K., Moraes, L. S., Lenzi, E. K., Andrade, M. F., 2005, “Difusão Anômola e Equações de Difusão”, Acta Sci. Technol, Vol. 27, Nº 2, pp. 123-131.
Goto, M., and Ishii, D., 1975, “Semidifferential Electroanalysis”, J. Electroanal. Chem. And Interfacial Electrochem., Vol. 61, pp. 361-365.
Hartley, T. T., Lorenzo, C. F., 2002, “Dynamics and control of initialized fractional-order systems”, Nonlinear Dynamics, Vol. 29, pp. 201-233.
Jia, J-H., Shen, X-Y., Hua, H-X., 2007, “Viscoelastic Behavior Analysis and Application of the Fractional Derivative Maxwell Model”, Journal of Vibration and Control, Vol. 13, Nº 4, pp. 385-401.
Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., 2006, “Theory and Applications of Fractional Differential Equations”, Amsterdam, Elsevier, 541 pp..
Machado, J. A. T., Jesus, I. S., Galhano, A., Cunha, J. B., 2006, “Fractional Order Electromagnetics”, Signal Processing, Vol. 86, pp. 2637-2644.
Maia, N. M. M., Silva, J. M. M. and Ribeiro, A. M. R., 1998, “On a General Model for Damping, Journal of Sound and Vibration”, Vol. 218, Nº 5, pp. 749-767.
Meerschaert, M. M., 2006, “Fractional Calculus Models in Finance”, Department of Mathemetics & Statistics, University of Otago.
Miller, K. S., Ross, B., 1993, “An Introduction to the Fractional Calculus and Fractional Differential Equations”, Canada, Wiley-Interscience Publication, 376 pp..
Oldham, K. B., 1972, “A Signal-Independent Electroanalytical Method”, Anal. Chem., Vol. 44, Nº 1, pp. 196-198.
Oustaloup, A., 1995, “La dérivation non entière: théorie, synthèse et applications” Hermès.
Pedron, I. T., 2003, “Estudos em Difusão Anômala”, 97 pp., Tese de Doutorado, Departamento de Física, Universidade Estadual de Maringá, Brasil.
Pedron, I. T., Mendes, R. S., 2005, “Difusão Anômala e Equações Generalizadas de Difusão”, Revista Brasileira de Ensino de Física, Vol. 27, Nº 2, pp. 251-258.
Podlubny, I., 1994, “Fractional-order systems and fractional-order controllers”, Slovak Academy of Sciences Institute of Experimental Physics.
Podlubny, I.; Dorcak, L., Kostial, I., 1997, “On fractional derivatives, fractional-order dynamic systems and $PI^{\lambda}D^{\mu}$ controllers”, Proceeedings of the 36th IEEE Conference on Decision and Control, December 10–12, San Diego, pp. 4985–4990. ISBN 0780341902.
Poinot, T., Trigeassou, J. C., 2003, “A method for modelling and simulation of fractional systems”, Signal Processing, Vol. 83, pp. 2319-2333.
Sabatier, J., Agrawal, O. P., Machado, J. A. T., 2007, “Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering”, Netherlands, Springer, 568 pp..
Yifei, P., Xiao, Y., Ke, L., Jiliu, Z., Ni, Z., Yi, Z., Xiaoxian, P., 2005, “Structuring Analog Fractance Circuit for ½ Order Fractional Calculus”, IEEE, p. 1039-1042.

9. RESPONSABILITY NOTICE

The authors are the only responsible for the printed material included in this paper.