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DYNAMIC STUDY OF BAR STRUCTURES USING PRECONDITIONED CONJUGATE GRADIENT AND ELEMENT-BY-ELEMENT

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Abstract. *Nowadays the demand for resolutions of structures and continuous models with high complexity has increased hugely, what implies in large amount of numerical process. The necessity leads to a demand of efficient techniques to solve linear systems faster and to store less information during the computation. This paper deals with dynamic analysis of bar structures modeled by classic FEM (Finite Element Method) and the PCG (Preconditionated Conjugate Gradient) to solve the resultants linear systems. The study is divided in two topics. The first one is the investigation of the influence of the initial guess in PCG algorithm to solve the linear systems obtained with FEM during dynamic analysis. The second is the resolution of the dynamic problem through the EBE (Element-By-Element) technique in combination with PCG, to avoid the assembly of the global matrix composed by mass and stiffness. For the first part, initial guess analysis, it was done a statistic study to compare the results of the different initial guesses observing the time to convergence and number of iterations for an interval of frequency. For the EBE study, it was analyzed the amount of required memory and the time to solve the dynamic problem.*

Keywords: *Element-by-element, Preconditioned Conjugate Gradient, FEM, Dynamic Structure Analysis*

1. INTRODUCTION

The actual conjuncture in the design of structures differ mainly to the past by the complexity that has increased hugely. Problems, nowadays, require an acceptable relation between load support and structure weight, which demands big number of components increasing the computational requirements. Even with the level of computer technology, the complexity of the problems require algorithms that can provide faster calculations and store less information as possible. The motivation of this study concentrate in the application of iterative solvers in dynamic problems as Frequency Response Function (FRF). Problems involving FRF normally demands several operation processing and in a huge repetition process, because the resultant linear system is solved for each discrete frequency in the spectrum. For porous continua, for example, the stiffness and mass matrices are dependent of the frequency, which implies a recalculation of these terms in each step of FRF what can be observed in the work of Panneton and Atalla (1997).

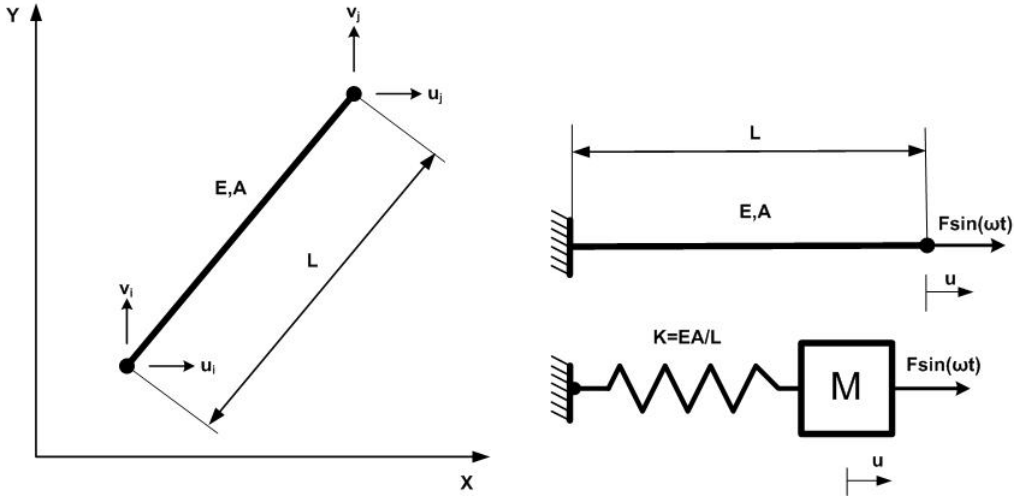
This work is a part of a project that intent to implement simulations involving fluid-structure interactions. The focus of the study is in the iterative solver applied in the FRF dynamic analysis, trying to achieve the responses more rapidly. As a simplification premise, the problem is concerned in the FRF dynamic analysis of bar structures modeled by classic linear finite element method (FEM). Which are classic structure and have simple element matrices (stiffness and mass), concentrating the analysis in the iterative solver. The solving algorithm used in the work is the Preconditionated Conjugate Gradient (PCG) and another analysis is done using Element-By-Element in combination PCG (EBE/PCG) method aiming to verify the performance this choice in FRF problems.

The paper is divided in four parts, the first one explain the classic FEM bar model and the formulation for the FRF analysis, shows the PCG algorithm and the EBE characteristics in the context of PCG. The second part shows the structures used in the simulation of PCG and EBE/PCG and their material properties and geometrical characteristics. The third part presents the proposition of the initial guess choice, and explain the chosen group that was tested. The last part concerns about the results and the conclusion.

2. THE FRF STUDY FOR BAR STRUCTURES MODELED BY CLASSIC LINEAR FEM

The analyzed frames is this paper are composed by a group of elements modeled as a prismatic elastic bar (only axial displacements) structure with a linear behavior, where a node is located at each end of the element. Each node has two Degree of Freedoms (DoF), displacement in the horizontal (u) and in the vertical (v), which can be observed in the Fig. (1a). The Eq. (1) shows the Stiffness Matrix for the bar element present by Cook(1995), where E is the elastic modulus, L is the length and A_0 is the cross-sectional area of the element.

$$[K] = \frac{EA_0}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$



(a) (b)

Figure 1. Dynamics results of the first structure simulation with PCG solver: (a) Parameters and Degree of Freedom of two nodes linear elastic bar finite element. (b) Bar and Mass-Spring Analogy.

Frequency Response Function (FRF) study is made using a bar model presented in the Fig. (1b). The Eq. (2) governs this case, where $[M]$ is the element mass matrix , ω is the vibration frequency , $\{F\}$ is the excitation, $[K]$ is the element stiffness matrix (Eq.1) and $\{u\}$ is the node displacement vector. In the Eq. (3) the bar element mass matrix is presented. The Eq. (2) implies that the external excitation of the bar must be in the form of $\{F\}\sin(\omega t)$. The term ρ present in the Eq. (3) refers to the material density, assumed constant in this study.

$$(-\omega^2[M] + [K])\{u\} = \{F\} \tag{2}$$

$$[M] = \frac{\rho A_0 L}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{3}$$

In the same way, for a truss, the FRF equation that govern the global frame is given by Eq. (4), where the subindex G implies in the global matrices, the assemblage of all elements. As a result of classic FEM formulation the (Eq. (4)) is a sparse symmetric linear definite positive systems.

$$(-\omega^2[M_G] + [K_G])\{u_G\} = \{F_G\} \tag{4}$$

For the coupled fluid-structure case, when the poroelastic BBBBB is modeled using Biot-Allard hypothesis, the stiffness and mass matrices become complex and frequency dependent, resulting in a systems present in the Eq. (5). In this case, the element dynamic equilibrium equation must be evaluated for any frequency analysis.

$$(-\omega^2[M_G(\omega)] + [K_G(\omega)])\{u_G\} = \{F_G\} \tag{5}$$

2.1 Preconditioned conjugate gradient method

The Conjugated Gradient is one of the best known iterative techniques for solving sparse Symmetric Linear Definite linear systems (Saad,1996). The Preconditioning Conjugated Gradient (PCG) differ by adding a preconditioning matrix in the algorithm, which reduces the number of iterations to the solution improving the ill conditioned system problem. The preconditioning matrix transforms the initial linear system in an equivalent system (same solution) with a better rate of convergence (Axelsson and Barker,1984).

The algorithm of PCG to solve the linear system, $[A]\{u\} = \{b\}$, is shown as follows:

1. Initialization
   (a) Initial solution $\{u\}_0$
   (b) $\{r\}_0 = b - [A]\{u\}_0$
   (c) $\{z\}_0$ $[J]$ = $\{r\}_0$

(d) $\beta_0 = \{r\}_0^T \{z\}_0$

(e) $\{p\}_0 = \{z\}_0$

2. Iterative Process

(a) $\{h\}_j = [A]\{p\}_j$

(b) $\gamma_j = \{p\}_j^T \{h\}_j$

(c) $\alpha_j = \beta_j / \gamma_j$

(d) $\{u\}_{j+1} = \{u\}_j + \alpha_i \{p\}_j$

(e) $\{r\}_{j+1} = \{r\}_j - \alpha_i \{h\}_j$

(f) If $\|\{r\}_{j+1}\| \,/\, \|\{r\}_0\| \leq \epsilon$, then assume convergence.

(g) $\{z\}_{j+1}[J] = \{r\}_{j+1}$

(h) $\beta_{j+1} = \{z\}_{j+1}^T \{r\}_{j+1}$

(i) c = β_{j+1} / β_j

(j) $\{p\}_{j+1} = \{z\}_{j+1} + \mathrm{c}\{p\}_j$

(k) j = j + 1

(l) Return to (a)

The matrix $[J]$ (steps (c) in the initialization and (g) in the iterative process) represents the preconditioning matrix. For this study, it was used the Jacobi preconditioning, which implies that the preconditioning matrix is composed by the diagonal of the linear system matrix $[A]$, observed in the Eq. (6). Hughes and Muller (1986) expose and test different precoditioners commonly used in context of structural analysis.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \quad \rightarrow \quad [J] = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} \tag{6}$$

2.2 Element-by-element

After the discretization by FEM there is a linear system in the form of $[A]\{u\} = \{b\}$ to solve. This linear system is assembled by the contribution of each element, which can be exposed mathematically by the Eq. (7), where the index e indicated the contribution in the element level, D is the total number of equations, d is the order of element matrix and Φ is the assemblage operator.

$$\begin{cases} [A]_{d\times d} = \Phi_{e=1}^{N} [A]_{D\times D}^{e} \\ \{b\}_{D\times 1} = \Phi_{e=1}^{N} \{b\}_{d\times 1}^{e} \end{cases} \tag{7}$$

Observing a general matrix position given by a_{ij}, based on Eq. (7), it is possible to assembly through the contribution of each element, what can be observed in the Eq. (8).

$$a_{ij} = \sum_{e=1}^{N} a_{ij}^{e} \tag{8}$$

The following proposition may be expanded to the matrix-vector product as follows (Eq. (9)), where the operation in the global level is $[A]\{b\}$.

$$a_{ij} b_i = \sum_{e=1}^{N} a_{ij}^{e} b_i^{e} = a_{ij}^{1} b_i^{1} + a_{ij}^{2} b_i^{2} + \ldots + a_{ij}^{N} b_i^{N} \tag{9}$$

Thus, the global operation can be composed by a sum of elements operation, doing each step in the element level. This idea can be oriented to do a alternative resolution method, where it is possible to avoid the assembly of the coefficient global system, doing the operation in the element level. This approach received the name of EBE (Element-By-Element) technique and turns possible the computing of huge systems that demand a big quantity of memory. The method was first introduced by Hughes et al. (1983a) in the heat conduction problems and after applied to the static structural mechanics (Hughes et al., 1983b).

2.3 EBE PCG Method

The PCG has several matrix-vector products throughout its algorithm that can be done in the element level following EBE concept. The link between this methods can be observed in the two main steps of PCG: the initialization and the iterative process. In the initialization process, there is a matrix-vector product in the step (b), determination of the term $\{r\}_0$ and in the assembly of the precoditioning matrix $[J]$, given by the Eq. (6). This assembly should be storage in one vector $\{J\}$ containing the diagonal of $[J]$, observed in the Eq. (10), where j_i is the position in the storage vector $\{J\}$.

$$j_i = \sum_{e=1}^{N} a_{ii}^e \tag{10}$$

In the iterative process, term $\{h\}_j$, step (a) of the iterative process, is the result of the matrix-vector product $[A]p_j$ and can be done in the element level, observed in the Eq. (11). Once the computed all of this terms all other operation in PCG are vector operations.

$$\{h\}_j^e = A^e\{p\}_j^e \tag{11}$$

2.4 Analyzed Structures

The first structure simulated is a cantilever truss system observed in the Fig. (2), where one of its extremity is excited by a force $F\sin(\omega t)$ and the other the boundary connections has an encastre support (no translation in the support). This structure was used to test the PCG algorithm solver using different initial guesses. All the components in the structure are composed by the same material, whit Young modulus $E = 2.1 \times 10^{10} MPa$ and density $\rho = 7850 kg/m^3$, and the same cross-sectional area $A_0 = 0.002m^2$. The load $\{F\}\sin(\omega t)$ has module $F = 9MN$ and the length of the components are described in the Fig. (2) .

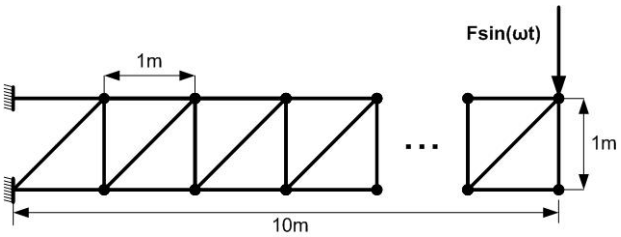


Figure 2. Cantilever truss for PCG analysis

The second structure, used to test the EBE/PCG solver against the PCG, is positioned as showed in the Fig. (3), where the length of the elements and the load position can be observed. The material properties,geometry characteristics and the load are the same.

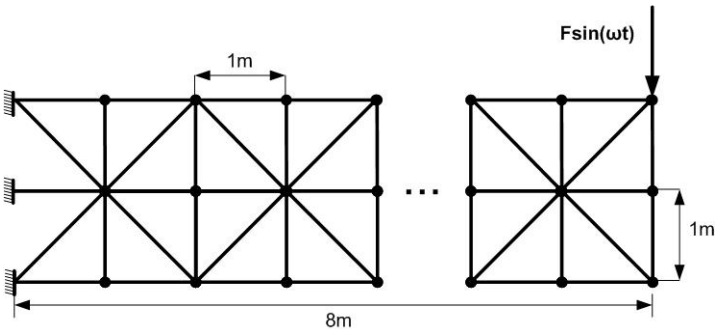


Figure 3. Cantilever truss for EBE/PCG analysis

3. INICIAL GUESS STRATEGY TO PCG

In the PCG study, one of the parameters of the method to solve the linear system is the initial guess. The FRF analysis in FEM, for discrete spectrum implies a resolution of N_ω linear systems presented by Eq. (4), where N_ω is the number interval in the discrete spectrum. Thus, for the discrete frequency ω_i in the spectrum, there is an initial guess $\{u\}_0^{\omega_i}$ that will feed the step (a) in the initialization process in the PCG solver. One of the investigations in this paper is the effect of the choice of this initial guess in the FRF analysis in FEM bar structures, observing the time to convergence and

number of iterations. Assuming N_ω big enough that implies the difference between the amplitude response $\{u\}_n^{\omega_i}$ and $\{u\}_n^{\omega_{i-1}}$ is pretty small, it is suggestive to suppose that the rate of convergence might increase if the $\{u\}_0^{\omega_i}$ is assumed to be $\{u\}_n^{\omega_{i-1}}$. Where the subindex n is the number of interactions to the convergence. Taking this idea, there was proposed three different strategies to the standard vector zero initial guess Eq. (12).

$$\{u\}_0^{\omega_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{12}$$

The first proposed initial guess, following the idea described above, is to choose the last iteration solution for ω_{i-1} and choose it as the initial guess for ω_i, what can be observed in the Eq. (13), where n is the number of iterations to achieve the convergence criteria.

$$\{u\}_0^{\omega_i} = \{u\}_n^{\omega_{i-1}} \tag{13}$$

The others initial guesses are different arbitrary modifications in the first proposed Eq. (13), where the intuit is to verify implication in the rate of convergence. One of the modification represents an normalization in the first vector $\{u\}_n^{\omega_{i-1}}$ based in its the first position, what can be observed in the Eq. (14). The other modification corresponds to an alteration of the first position by a scalar ξ, showed in the Eq. (15).

$$\{u\}_0^{\omega_i} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \rightarrow \{u\}_0^{\omega_i} = \begin{bmatrix} 1 \\ \frac{u_2}{u_1} \\ \vdots \\ \frac{u_n}{u_1} \end{bmatrix} \tag{14}$$

$$\{u\}_0^{\omega_i} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \rightarrow \{u\}_0^{\omega_i} = \begin{bmatrix} \xi \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{15}$$

3.1 Simulation with the different initial guesses

To analyze the efficiency of each initial guess, the structure showed in the Fig. (2) was simulated with FEM bar model and the PCG algorithm. Solving the same problem with the different initial guesses proposed as strategy. To simplify the terminology the following symbols are used to represent each initial guess:

- IG0: Represents the standard vector zero initial guess presented in Eq. (12).
- IG1: Indicates the initial guess as the last solution for the discrete frequency ω_{i-1} as in Eq. (13).
- IGnorm: Relative to the normalized initial guess, Eq. (14).
- IGmod: Illustrates the modified initial guess observed in the Eq. (15).

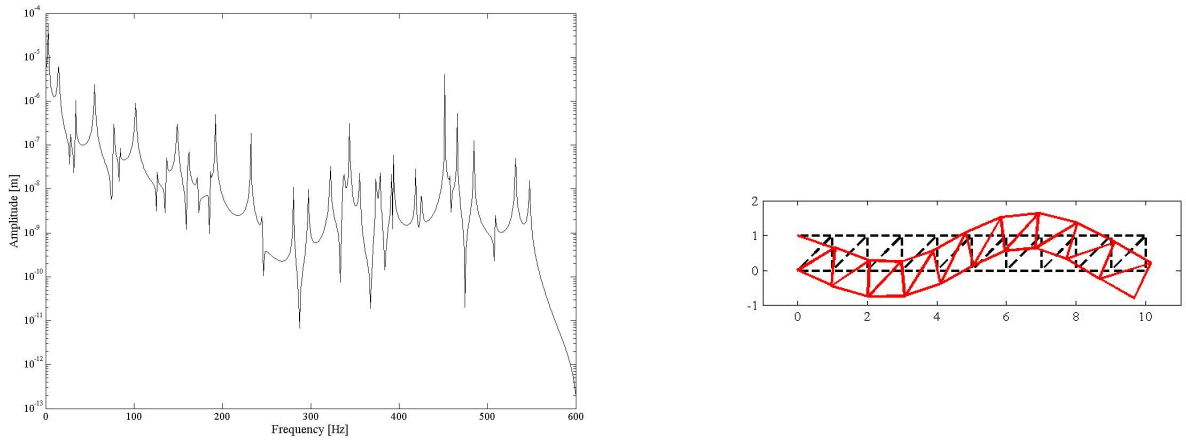
4. RESULTS

The results are divided in two parts, the first one refers to the PCG initial guesses analysis, which uses the first structure presented in the Fig. (2). While the second part is related to the EBE/PCG comparison with PCG and the simulation was done with the structure presented in the Fig. (3). The tolerance in the step (f) of the iterative process was assumed as $\epsilon = 10^{-12}$. All the programming to calculation and plotting was done using the software Matlab©.

4.1 PCG initial guesses analysis

The simulation of the first bar structure was done using only the PCG solver with different initial guesses. As expected, the FRF analysis gave the same response independently of the choice. The Fig. (4a) shows the FRF to the last node (point of load application) on the vertical degree of freedom. It can be observed in the Fig. (4b) the 4th mode of vibration (in red) plotted against the first configuration position (in black). The parameter ξ, in IGmod, was arbitrary assumed as 1000.

The Fig. (5) refers to the number of iterations and the time to convergence in the spectrum (0 to 600 Hz) for each initial guess. It is important to observe that IGnorm and IGmod presented higher values, but the points are distributed in a



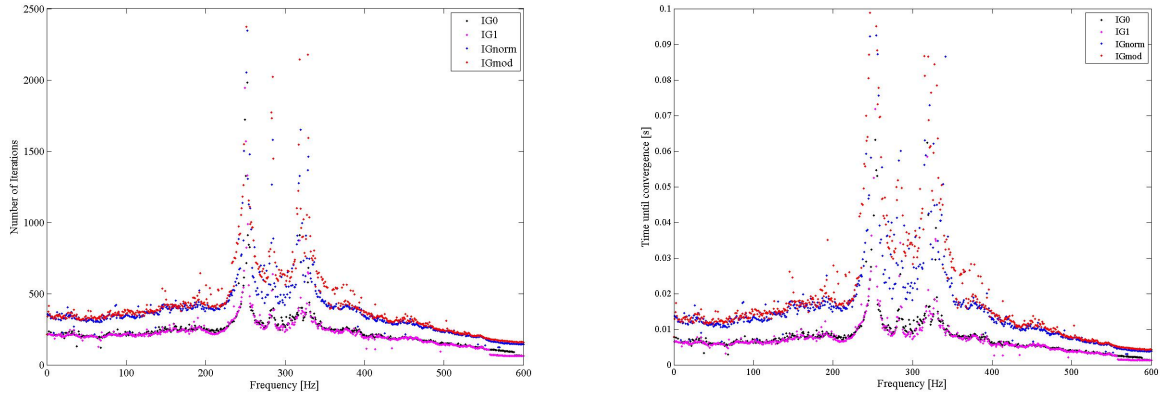
(a) (b)

Figure 4. Dynamics results of the first structure simulation with PCG solver: (a) FRF for a single degree of freedom of PCG analysis structure. (b) 4th Mode of vibration of PCG analysis structure.

pattern, such way that the IG0 and IG1 are in a lower level. To compare responses, an hypothesis test was done to verify if there are equal means between samples, divided in two groups: number of iterations and time to convergence. As a premise of hypothesis test, the samples must be (at least) normally distributed and have the same variance (Sachs, 1984). To verify if the data of the samples fits in a normal distribution the Komolgoroff-Smirnov goodness of fit test (Sachs, 1984, Rohatgi, 1984) was applied and the equality of the variances can be checked by chi-square test .

To compare the initial guess choice, the tests of each sample was separated in two different groups, Number of Iteration and Time to Convergence. According to Komolgoroff-Smirnov goodness of fit test, with an $\alpha = 0.05$ significance level, there is no sample that presents a normal distribution , what turned unnecessary chi-square test to verify the variances equality. Once proved statistically that the samples does not fit in a normal distribution, the hypothesis test can't be used. In this case a nonparametric test must be applied and for this study the Wilcoxon rank-sum test was used (Lehmann and D'Abrera, 1975). All the tests was done in comparison with the standard zero vector initial guess IG0, and was possible to verify that in both groups, Number of Iterations and Time to Convergence, IG1 can be considered equivalent and the others (IGnorm and IGmod) different, with $\alpha = 0.05$ significance level.

It is important to observe Fig. (5a) and (5b) that for an interval of 240 to 320 Hz the number of iterations and the time to convergence has increased hugely. It was verified that this has no relation with bad conditioning of the system matrix, apparently the points of bad conditioning does not correspond with this interval. It seems that the method for some frequencies does not converge in the same pattern.



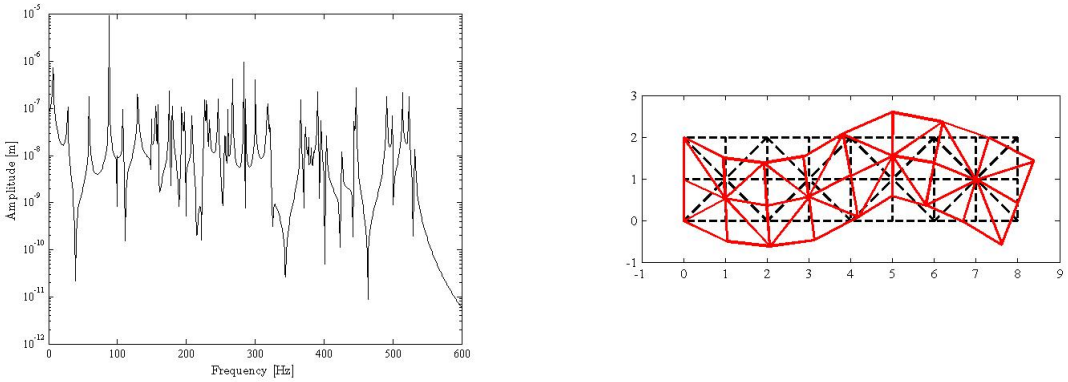
(a) (b)

Figure 5. Results of the first structure simulation with different Initial Guesses in PCG solver:(a) Number of iterations per discrete frequency in the spectrum. (b) Time to convergence per discrete frequency in the spectrum.

4.2 EBE/PCG comparison with PCG

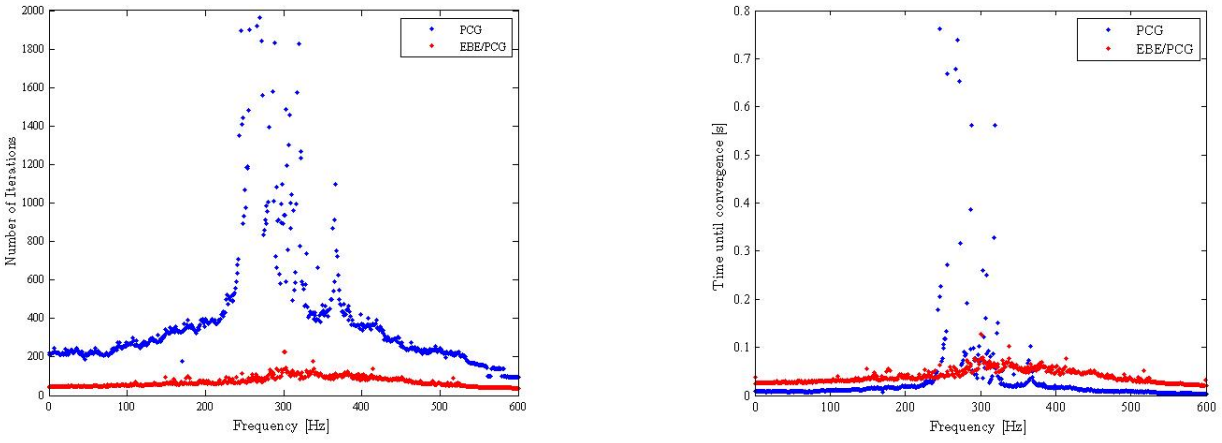
In sequence it was compared the PCG with EBE/PCG. As expected, the FRF analysis gave the same response for both cases. The FRF for the vertical degree o freedom of the upper side node nearest of the encastre what can be observed in Fig. (6a) and the Fig. (6b) shows the 4th mode of vibration (in red) for the structure.

The curve with respect to the number of iterations required to converge and time until convergence for each frequency ω_i of the spectrum is showed in the Fig. (7a) and (7b). In the range of 230 to 370 Hz the same PCG convergence problem can be noticed, something that EBE/PCG did not show. It seems that EBE/PCG can avoid this kind of behavior better than PCG with matrix global assembly. For values outside that critical interval (230 to 370 Hz) the PCG showed faster results, but with higher number of iterations. This can be explain taking account the number of operations, in PCG the local mass and stiffness matrices are calculated once, differently of the EBE/PCG that required this calculation for every step in the process.



(a) (b)

Figure 6. Dynamics results of the second structure simulation with EBE/PCG solver: (a) FRF for a single degree of freedom of EBE/PCG analysis structure. (b) 4th Mode of vibration of EBE/PCG analysis structure.



(a) (b)

Figure 7. Results of the second structure simulation with EBE/PCG solver: (a) Number of iterations per discrete frequency in the spectrum. (b) Time to convergence per discrete frequency in the spectrum.

5. CONCLUSIONS

Observing the results in the PCG analysis it might indicate that for PCG the IG0 initial guess is the better option, Though the IG1 had presented an equal curve in number of iterations and time to convergence, what was proved statistically, IG0 is easier taking a programming view. The others initial guesses can be rejected.

The comparison between PCG and EBE/PCG has present interesting results. EBE did not show any king of critical frequency interval, because the number of iteration and the time to convergence have showed an steady condition. In comparison with the PCG, the time has presented higher levels, what represents, for complex systems, that this parameter

can be majored, turning the resolution non feasible. In the other hand there is safe of memory store, because there is no need to assembly the global load, what turns possible to simulate very huge problems in FEM. Thus, the choice of the resolution approach depends on the problem requirements. There other computer options, as parallelization (Liu, Zhou and Yang, 2007), that can bring the computing time to an acceptable level.

6. ACKNOWLEDGEMENTS

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