# THE LIMIT ANALYSIS MODEL FOR ORTHOGONAL CUTTING

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***Abstract.*** *The main objective of this work is to present a limit analysis formulation to model orthogonal cutting processes in steady state condition. The mathematical characterization of the problem is based on the concepts of equilibrium, kinematic and constitutive relations for elastic ideally plastic materials. Solving the problem involves the Finite Elements Method coupled with techniques of mathematical programming. A velocity-stress principle for limit analysis is considered as primitive for the finite element discretization. The proposed strategy includes the determination of the flow configuration. Therefore, cutting forces and others parameters are computed for different configurations, three of them proposed by others authors and a new one suggested by ourselves in the present paper. In order to validate the model, the results are compared with that ones obtained from the classical literature.*

***Keywords****: Limit Analysis, OrthogonalCutting, Numerical Methods*

## 1. INTRODUCTION

Many efforts have been made to model analytically the progression of wear and cutting forces in the progress of various machining processes. However, the complex mechanisms related to the friction between the tool and chip, and the large number of variables that can influence the process, complicate considerably this kind of analysis.

The orthogonal cutting model based on direct methods provide important information to control the process without the need to study the transient phase, in other words, a model based on the analysis of stresses and deformation rates in the setting of permanent regime is sufficient to adequately describe the relevant mechanical phenomena. However, this description requires the prior knowledge of the configuration in steady flow. This configuration is not only defined by a matrix of shaping such as extrusion processes. Other parameters such as length of the chip and the length of the tool contact with the chip, which for the same condition of machining provide differents configurations of flow.

Thus, the direct methods used in the study of orthogonal cutting on a permanent basis are characterized as problems of free boundary, requiring the search or the definition of the configuration system during the process of solution. This search for the configuration of flow does not imply in the elastoplastic solution of the model, but the analysis of instantaneous mechanical flow conditions to determine the settings that determinate a geometric configuration feasible for steady flow.

In a cutting process, the piece moves at a constant speed on the tool, so steady state flow is maintained to achieve this condition. Incremental analysis breaks down a large amount of data intermediaries of minor relevance to the process. The limit analysis of plasticity approach the limit load and the correct flow of plastic materials directly, which are derived from a end principle. Many authors have proposed for this setting, being the most classic the Merchant model (Armarego and Brown, 1969), currently many publications have criticized the simplicity of this proposal by introducing new parameters to control the description of this setting steady flow.

Many researchers in the machining of metals have made efforts to develop a model of analysis in cut cases to provide a clear understanding of the forces, stresses and deformations involved in the process, which allows predict important parameters for cutting, complementing the experimental data obtained. Often a theorical analysis is essential to the assembly of devices of the experiment. Therefore, we can use this approach in estimating the life of the tool, calculation of the cut power, and much more ambitious in implementing the control of the process in real time.

With the development of computational techniques, the use of numerical methods to study these processes have become even more frequent. The analysis by finite elements method has been widely used in these studies mainly in complex geometries, where the theoretical treatment becomes very complicated. In this work we used the finite elements method combined with a limit analysis formulation. For problems with metal cutting, an incremental analysis should be conducted until the limit load condition for which the piece begins to flow under constant load condition.

In analytical approaches, in a plane strain, there is a clear conflict on the nature of the deformed zone in the process of cutting metals. Several researchers, such as Piispanen, Merchant, Kobayashi and Thomsen(Armarego and Brown, 1969)., has favored the model of the shear line (thin-plane), and others like Palmer and Oxley Okushima and Hitomi(Armarego and Brown, 1969), based on analysis of the thick deformation region (thick-plane). To cutting process into revolution bodies with symmetry few theoretical results are available in literature. Most of the solutions mentioned in the literature are based on analytical solutions of the kinematic theorem of limit analysis (Lubliner, 1990).

Obviously, the tool-chip interface friction is one of the most important parameters in this process, however, some characteristics of the global process behavior can a priori be analysed whithout this parameter consideration. For instance, the nature of the deformed zone, the stress behavior along the shear plane and the tool-chip contact lenght.

In this work delopment, the main goal is to construct a limit analysis model with unilateral contact conditions for machining under orthogonal cutting conditions without consideration of the wear, to determinate the magnitude of the machining forces, associated with the finite element method, exploring an adaptive mesh refinement strategy to localize plastic regions (Borges *et al*, 2001).In this study will be used the settings of the flow defined in the literature, isolated or combined with each other (Armarego and Brown, 1969; Molinari, 2008; Tyan and Yang, 1992). From this analysis it is intended in future to obtain the configuration of flow automatically during the solution process without the need for semi-analytical definitions or heuristics.

In the limit analysis the ultimate goal is to obtain the necessary power to keep the process under the permanent plastic flow. The plastic flow is characterized by the development of speed, kinematics permitted, associated with a field of stresses static and plastically admissible and balanced with a external load. This model is also quite widespread in the literature as a classic application of the theory of plasticity (Lubliner, 1990) and in publications of specific conformation of metals.

## 2. LIMIT ANALYSIS

The load acting under the workpiece that assures a steady-state velocity field is related to a static and plastic admissible stress field. These flow conditions state the development of a plastic flow under constant loading and are the same that characterize the incipient plastic collapse phenomenon experimented for elastic-ideally plastic materials.

Under the assumption of propotional loading, the limit analysis problem consists in finding a load factor $\alpha$ such that the body undergoes plastic collapse when subject to the reference load $F$ uniformly amplified by $\alpha$. In turn, a system of loads produces plastic collapse if there exists a stress field in equilibrium with these loads, wich is plastically admissible and related by the constitutive equations to a plastic strain rate field being kinematically admissible. Thus the limit analysis problem consists in finding $\alpha \in \Re$, a stress field $T \in W'$, a plastic strain rate field $\mathrm{D^p} \in W$ and a velocity field $v \in V$ such that,

$$\mathrm{D}=Dv \qquad v \in V \qquad D \in W \tag{1}$$

$$T \in S(\alpha f) \tag{2}$$

$$T \in \partial\chi(D^p) \Leftrightarrow D^p \in C_p(T) \tag{3}$$

The equation (1) imposes that the collapse plastic strains rate is related to a kinematically admissible velocity field $v$ by means of the tangent deformation operator $D$, and W means the space of the regular deformation rates.

The unilateral kinematic conditions require the imposition of this boundary mixed conditions (Wriggers and Panagiotopoulos, 1999 ; Naccarato, 2006), where $\Gamma_c$ means the unilateral velocity boundary,

$$v_N \leq 0, \qquad r_N \leq 0, \qquad r_N v_N = 0 \qquad \text{in } \Gamma_c \tag{4}$$

$$v_N \in V, \qquad r_N(v_N^* - v_N) \geq 0, \qquad \forall v_N^* \in V \tag{5}$$

The term $S(\alpha f)$ present in equation (2) denotes the set of all stress field in equilibrium with the givem system of forces $\alpha f$, that satisfying the Principle of Virtual Power. The internal power for any par $\mathrm{T} \in W'$ and $D \in W$ can be defined by the dual product,

$$\langle \mathrm{T}, D \rangle = \int_B \mathrm{T} \cdot D dB \tag{6}$$

In order, $V'$ is the space of loads and external dissipated power by a load system $f \in V'$ in a velocity field $v \in V$, and is given by the dual product,

$$\langle f, v \rangle = \int_B b \cdot v dB + \int_{\Gamma_\tau} a \cdot v d\Gamma \tag{7}$$

where $b$ and $a$ are the body and surface loads respectively.

The equilibrium conditions, linking a tension field and a system of proportional loading by an $F \in V'$ prescribed, is imposed by the virtual potencial principle,

$$\left\langle T, Dv^* \right\rangle + \int_{\Gamma_c} b_c(-v^*, r)\, d\Gamma \geq \left\langle \alpha f, v^* \right\rangle \qquad \forall v^* \in V \tag{8}$$

where $\alpha_c \in \Re$ is the proportionality factor or load factor. The previous equation can be expressed in a compact form as $T \in S_\alpha$ where $S_\alpha$ represents the set of all tensions fiel in equilibrium with a fixed system of forces $\alpha f$ ,

$$S(\alpha f) = \left\{ T \in W \,\middle|\, \left\langle T, Dv \right\rangle + \left\langle b_c(v,r) \right\rangle_{\Gamma_c} \geq \alpha \left\langle f, v \right\rangle, \forall v \in V \right\} \tag{9}$$

Thus, the unilateral contact problem can be understood and solved by a classical control problem as,

$$\begin{cases} div(T) + \alpha b = 0 \quad on \quad B, \\ T \cdot n = \alpha a \quad on \quad \Gamma_\tau \\ v = 0 \quad on \quad \Gamma_v \\ v_n \leq 0, \begin{cases} if \quad v_n < 0 \Rightarrow r_n = 0 \quad (no\, contact) \\ if \quad v_n = 0 \Rightarrow r_n \leq 0, \quad then \\ \begin{cases} if \quad \|r_T\| - \mu |r_n| < 0, \quad then \quad v_T = 0 \\ or \quad if \quad \|r_T\| - \mu |r_n| = 0, \quad then \quad v_{Ti} = -\lambda \dfrac{r_{Ti}}{\|r_{Ti}\|} \end{cases} \end{cases} \end{cases} \tag{10}$$

The constitutive relation discribing an elastic ideally-plastic material is writen in (3). The symbol $\partial\chi(D^p)$ denotes the subdifferential of the plastic dissipation function $\chi$ , that is, the set of all stress field such that (Borges *et al*,2001),

$$\chi(D^{p*}) - \chi(D^p) \geq \left\langle T, D^{p*} - D^p \right\rangle \forall D^{p*} \in W \tag{11}$$

For these materials the dissipation function is related to the set $P$ of plastic admissible stress field by,

$$\chi(D^p) = \sup_{T^* \in P} \left\langle T^*, D^p \right\rangle \tag{12}$$

Frequently the set $P$ is defined as,

$$P = \left\{ T \in W \,\middle|\, f(T) \leq 0 \in B \right\} \tag{13}$$

where $f(t)$ is a flow function, and the inequality above is then understood as imposing that each component $f_k$ , wich is a regular convex function of $T$ , is non-positive. Then, at any point of $B$ , equation (3) is equivalent to the normality rule $D^p = \nabla f(T)\dot{\lambda}$ , where $\nabla f(T)$ denotes the gradient of $f$ and $\dot{\lambda}$ is the m-vector field of plastic multipliers. At any point of , the components of $\dot{\lambda}$ are related to each plastic mode in $f$ by the complementarity condition $\dot{\lambda}_j \geq 0$, $f_j \leq 0$ and $f_j \cdot \dot{\lambda}_j = 0$.

The classical extremum principles of limit analysis, that is the kinematical, statical and mixed formulations, can be derived from the optimality conditions (1-3). The discretized versions of these formulations lead to a single type of finite dimensional problem, which can be cast in four strictly equivalent forms, namely the statical, mixed and kinematical dicrete formulations, and set of discrete optimality conditions (Borges *et al*, 2001)

### 2.1. Discret Model

The discret model of the limit analysis of the mixed formulation dicribed in Naccarato(2006) can be dualized and written in three other equivalent form, said discrete static formulation, kinematic and the discrete set of conditions for optimality. In particular, the algoritm used in this paper and originally proposed in Naccarato(2006) uses as base a form of the problem described by the optimality condition (Borges *et al*, 1995).

The discrete limit analysis problem consists in finding a load factor $\alpha \in \Re$, a stress vector $T \in \Re^q$, a velocity vector $v \in \Re^n$ and a plastic multipliers vector $\dot{\lambda} \in \Re^m$, such that the system represented by a deformation matrix $B : \Re^n \to \Re^q$ and a convex function $f(t) \in \Re^m$, undergoes by a deformation matrix load being proportional to a given force vector $F \in \Re^n$. It is assumed that all rigid motions are ruled out by the kinematical constraints, so that the kernel of matrix $B$ only contains the null velocity vector

The discretized version of the limit analysis formulation leads to a finite dimensional problem that can be seen as a discrete version of the equations (1-3), that is,

$$Bv - \nabla f(T)\dot{\lambda} = 0 \tag{14}$$

$$B^T T - \alpha F = 0 \tag{15}$$

$$F \cdot v = 1 \tag{16}$$

$$f_j(T)\dot{\lambda}_j = 0; \qquad f(T) \leq 0; \qquad \dot{\lambda} \geq 0 \qquad j = 1,\dots,m \tag{17}$$

together with the restrictions of unilateral contact and Coulomb friction, where,

$$v_n \leq 0, \qquad r_n \leq 0, \qquad r_n v_n = 0 \tag{18}$$

and

$$v_n \leq 0, \begin{cases} if \quad v_n < 0 \Rightarrow r_n = 0 \quad (no\, contact) \\ if \quad v_n = 0 \Rightarrow r_n \leq 0, \qquad then \\ \quad \begin{cases} if \quad \|r_T\| - \mu |r_n| < 0, \quad then \quad v_T = 0 \\ or \quad if \quad \|r_T\| - \mu |r_n| = 0, \quad then \quad v_{Ti} = -\lambda \dfrac{r_{Ti}}{\|r_{Ti}\|} \end{cases} \end{cases} \tag{19}$$

A strategy for solving this discrete problem is dicribed by Borges *et al* (2001), and is not discussed here. In the same way, no particular emphasis is placed on the adaptive strategy used. Details about the procedure are presented in Borges *et al* (2001).

## 3. ORTHOGONAL CUTTING MODEL

Using an Eulerian reference coordinate to describe the steady state motion of the work-piece relative to a stationary cutting tool, an orthogonal metal cutting process for a controlled contact toll is shown in Figure 1.

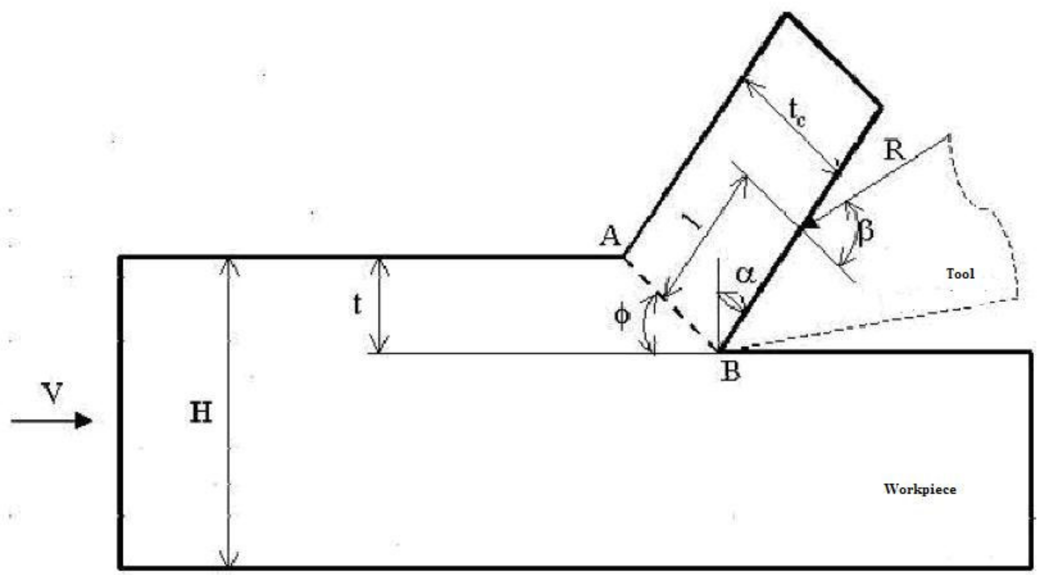



Figure 1. Orthogonal Cutting scheme

The model basically consists in a workpiece of thickness *H* moving towards a stationary tool at a constant speed while a non-deformed chip thickness *t* (the cutting depth) is being cut away; in the same way, a deformed chip thickness $t_c$ is machined. A layer of large shear deformation occurs along the plane *AB* (shear plane) inclined at an angle $\phi$ (shear angle) to the horizontal line, $\alpha$ is the tool rake angle, $\beta$ is the friction angle between the resultant force *R* and the normal to the rake face. The width of the chip is assumed to be large as compared with the cutting depth *t* and the chip thickness $t_c$. This assures the two dimensional plane strain model. The controled contact tool model is adapted by many researchers (Armarego and Brown, 1969 and Tyan and Yang, 1992) and is adopted herein. In that case the tool chip contact lenght is previously settled and this lenght is named $l$. Since a controlled contact tool is used, a full contact of chip with the rake surface is assumed and for the frictionless model the stress shear is zero in the region.

The geometry of the chip is described by the chip stream angle $\eta$ and the chip thickness $t_c$. For the general case, those parameters are variables of the problem. However, in the limit analysis by finit elements, an a priori knowledge of geometry of the chip is required. For problems with friction more attention needs to be done to this caracteristic of the problem, but for the frctionless study, based on Tyan and Yang (1992), it is assumed that the chip stream angle is equal to the rake angle and the chip thickness is equal to the cutting depth. The deformed zone position, defined by the shear angle, assumed know approach, is obtained by a limit analysis procedure in association with an adaptive mesh strategy.

In the model of this paper the effects of strain rate and temperature are not consideres; the tool is assumed rigid and the workpiece is modelled to be a infinitely ductile. The last consideration is adequate with the continuous chip formation model adopted. Form most metals the herdening rate falls to small values for larga strain and so it reachs a near constant saturation stress. The high strain rates that accompany the machinin operation are said to raise the yeld strenght of the material and make it approximate the idealized plastic material. So a Von Mises plastic workpiece in the sense of assymptotic yeld behavior is assumed.

## 4. NUMERICAL APPLICATIONS

All models of orthogonal cutting in the literature has basically the same principle, described previously. In this paper are analyzed four different geometries that describe the process of orthogonal cutting, three of them existing in the literature and a new one proposal developed in this work, figure 2 shows the different geometries that were simulated.

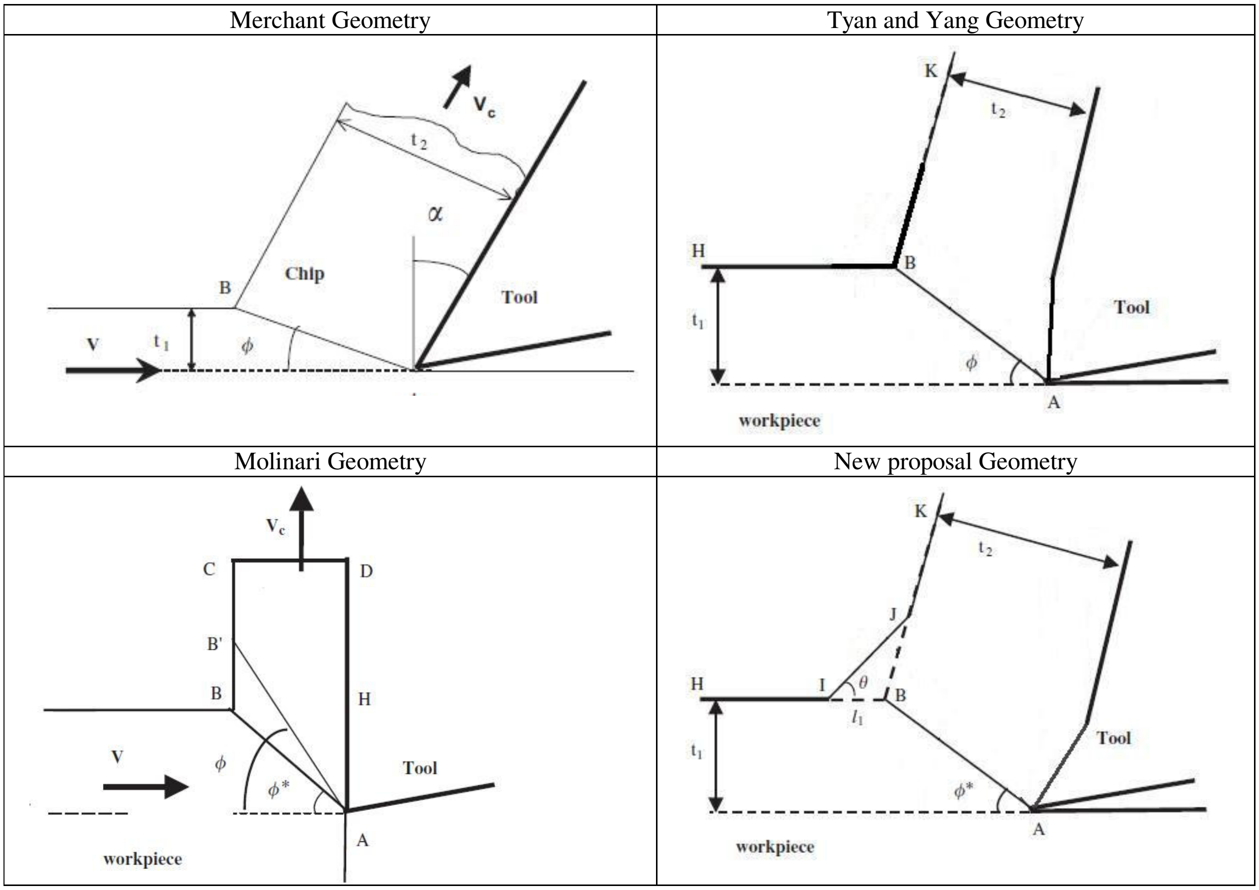



Figure 2. Geometric models of orthogonal cutting

In modeling, a parameter that is a difficult choice is the region of tool-chip contact. In the model of controlled contact, which is used by several researchers (Armarego and Brown, 1969), the length of tool-chip contact is first established, for example, Tyan and Yang (1992), suggest a typical lenght for the contact as $l = 0,2H$ . In this work after the study of a series of cutting parameters and analysis of some results presented later was considered that the better value for the lenght og the chip would be $l = 0,25H$ , being this value fixed throughout the modeling. Thus it is guaranteed that the contact in all cases will occurs only in this lenght, this is the normal stresses exists only in this region.

After considered all the above examined four different geometries for the orthogonal cutting are studied to simulate the most severe cases of cut (Tyang and Yang, 1992). One of the parameters differentiating the analysis was the rake angle, for which figures were adopted 10, 20, 30 and 80 degrees in some cases. For each of these angles three different depths of cut were analyzed: $t_1 = 0,1H$ , $t_1 = 0,2H$ , $t_1 = 0,3H$ . As the lenght of contact was defined previously as $l = 0,25H$ , can be express the following relationships: $t_1/l = 0,4$ ; $t_1/l = 0,8$ ; $t_1/l = 1,2$ . These parameters are approximate to the typical already adopted by some researchers, which enables the comparison of results with other existing literature (Tyan and Yang, 1992) and Nacaratto (2006).

The finite element mesh adopted were obtained through a process of adaptive refinement of meshes for limit analysis (Borges et al., 2001). With this strategy was possible to capture the region of localized plastic deformation, allowing estimation of the shear plane. This is a important parameter in the analysis because, depending on the combination of the value of the depth of cut and rake angle, the plastic region can vary substantially, then it could be a perfectly designed plan or can extend for a greater region, as shown in figure 3, where we present examples for $\eta = \alpha - 8^\circ$ , $\theta = 45^\circ$ and $\mu = 0$ , not so with a plan.

Some authors call these different regions of plastification as thin or thick plane of shear (Armarego and Brown, 1969). In general, frame the problem in one of the two cases is not a trivial task, but the choice of meshes suitable for each of these mechanisms is crucial to have good results. Thus, the procedure of adapted mesh is part of the solution and ignored by many authors who adopt methods of analytical solution.

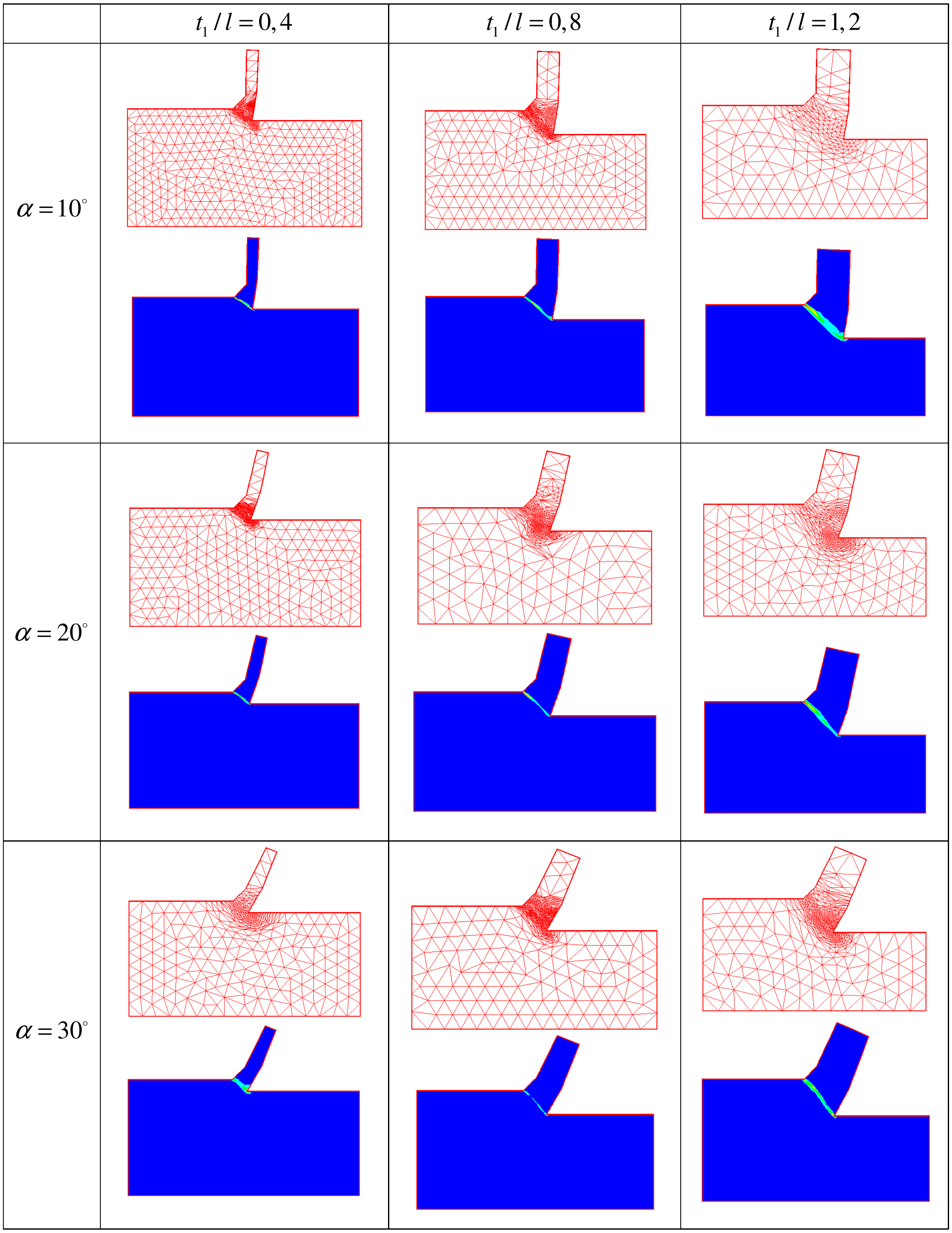



Figure 3 – Plastic deformation region (shear plane) and adaptive mesh

Another important consideration to be mentioned is the issue of friction. We know that the model with friction is essential. However, as a first experience and structuring of the model will not be assessed by comparing the influence of the theories that important parameter. However, the best laws of friction for this model are not the conventional laws, making it necessary to advance the study of appropiate laws of friction.

After that consideration for the friction model were compared with the different theories proposed to obtain the forces of machining and the shear angle. Initially, to validate the results, we simulated the different theories depending on the rake angle and the cut-off switch $\frac{t_1}{l}$, so you can compare the values obtained with the existing literature and validate the model.

To compare the results obtainde with the analytical solutions with those of Finite Element Methods was analyzed the Figure 4, in which was showed that analytical modeling is a more conservative method then the finite elements used.

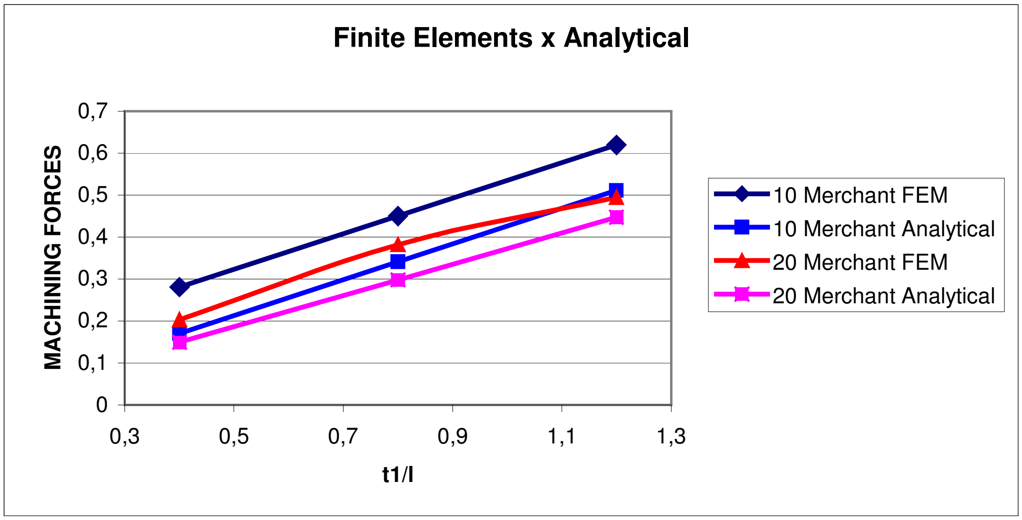



Figure 4 – Comparation with Finite Elements Method versus Analytical solution

Tyan and Yang (1992) found that the machining forces, an important parameter of the project, varies depending on the cutting parameters such as: the rake angle, the friction coefficient, depht of cut. To simulate this behavior in the different theories proposed has varied the depth of cut( $t_1 = 0{,}1H$ , $t_1 = 0{,}2H$ , $t_1 = 0{,}3H$ ) with the same coefficient of friction( $\mu = 0$ ), also vary the rake angle $\alpha = 10^\circ$ , $\alpha = 20^\circ$ , $\alpha = 30^\circ$, and only for the Merchant geometry $\alpha = 80^\circ$. The figures (5-8) shows the values obtained for the machining forces admensionalized according to the proposed cut-off switch. Please note that the models of Merchant (Armarego and Brown, 1969) and Molinari (2008) which are of analytical solution have been modeled to compare the results of finite element method (FEM).

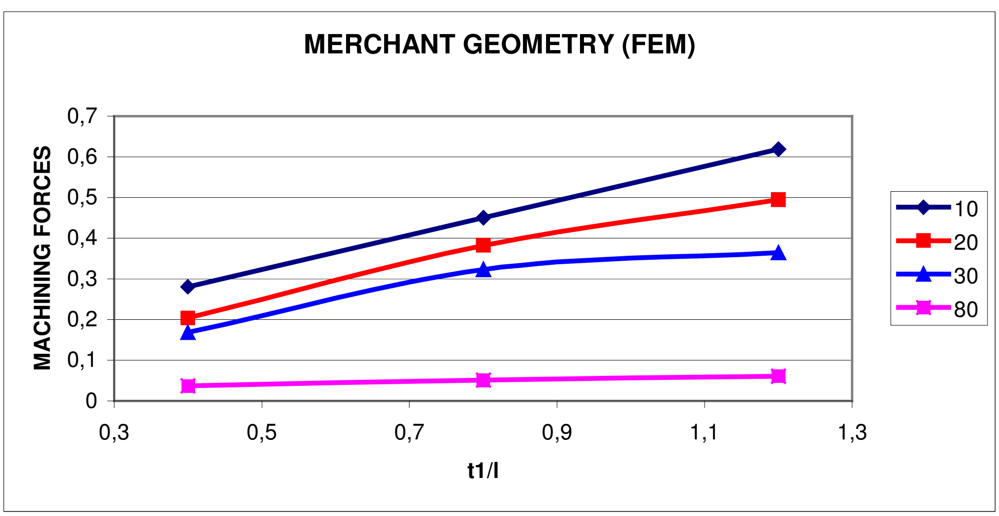



Figure 5. Machining forces for Merchant geometry in function of ranke angle

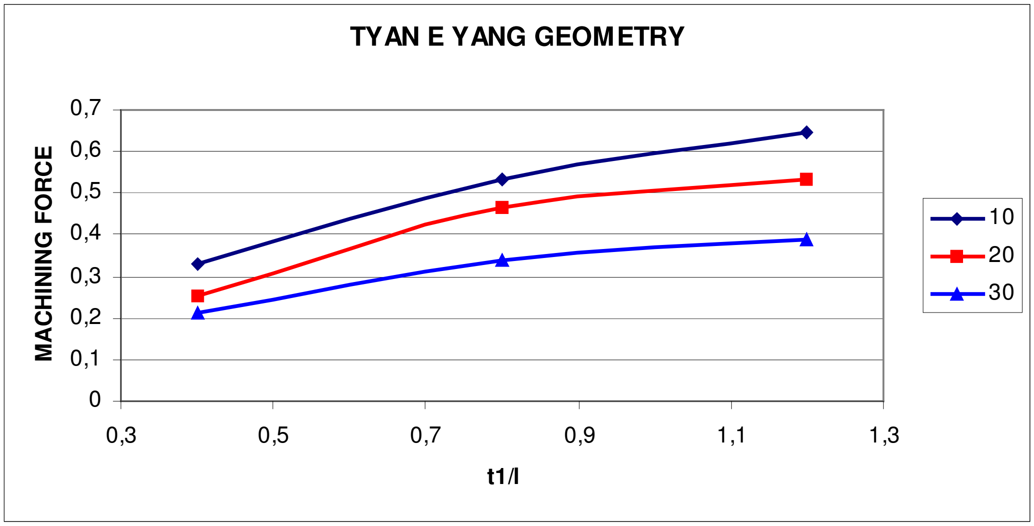



Figure 6. Machining forces for Tyan anda Yang geometry in function of ranke angle

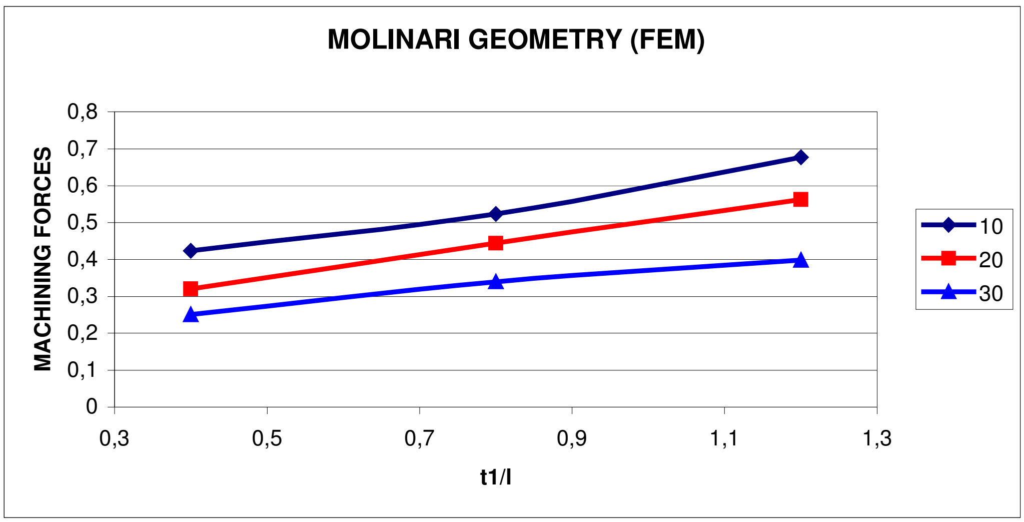



Figure 7. Machining forces for Molinari geometry in function of ranke angle

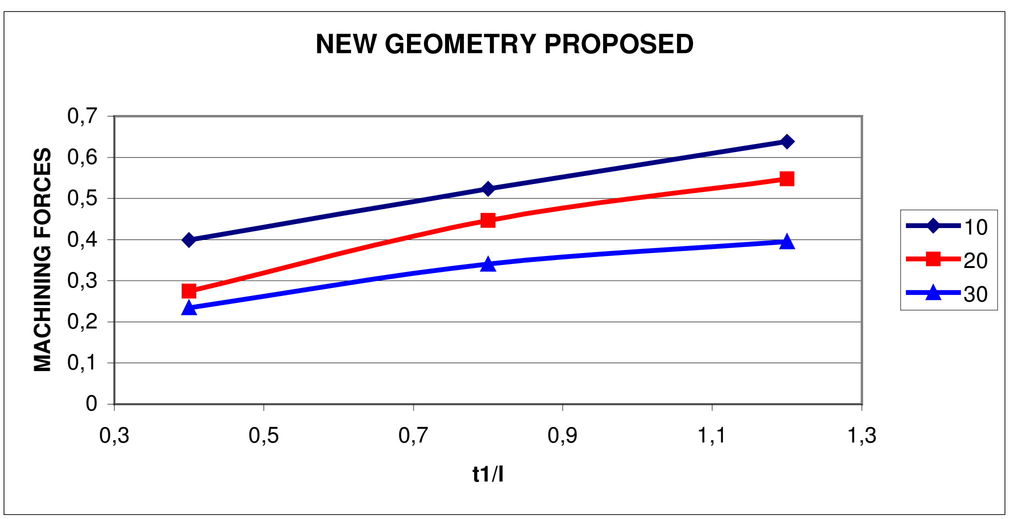



Figure 8. Machining forces for the geometry proposed in the paper in function of ranke angle

In all the theories proposed the line velocity field have to leave in the same direction as the chip, indicating the tendency of its movement after the chip-tool contact. Of all the models evaluated that provided the best results of the analysis was the geometry proposed here (mixed). Figure below are the fields of speeds for the different models for the parameters: $\alpha = 30^{\circ}$, $t_1 = 0,3H$, $\theta = 45^{\circ}$, $\eta = \alpha - 8^{\circ}$ (only for Tyan and Yang model) and for friction $\mu = 0$.

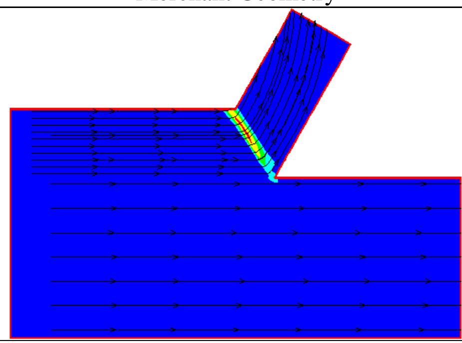
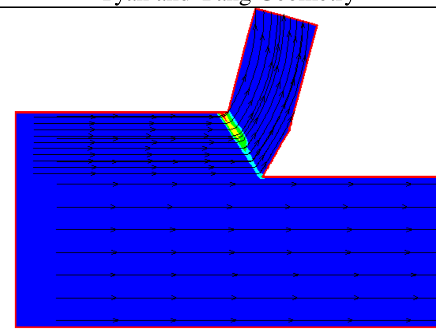
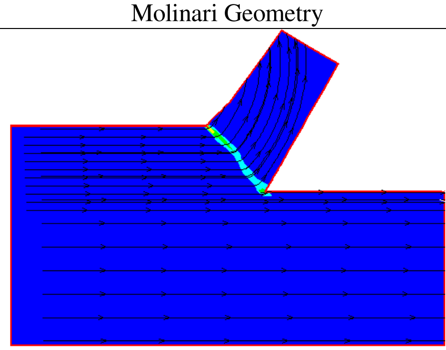
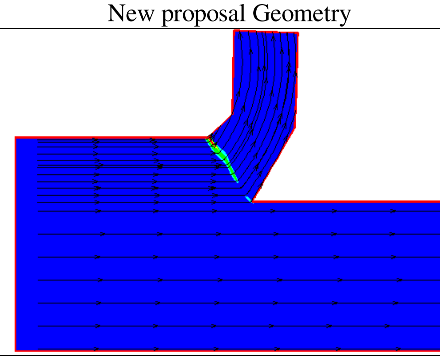
| Merchant Geometry | Tyan and Yang Geometry |
|---|---|
|  |  |
| Molinari Geometry | New proposal Geometry |
|  |  |

Figure 9. Velocity fields for diferents geometries

## 5. FINAL REMARKS

A numerical method was introduced to study an orthogonal cutting process. The friction less model is not a real practical condition but make possible to indentify the way that the model can be improved. In the next works the friction and an unilateral boundary contact condition to the tool-chip contact interface will be incorporated.

## 6. ACKNOWLEDGEMENTS

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