ANALYTICAL SOLUTION OF THE POINT KINETICS EQUATIONS IN SUBCRITICAL SYSTEMS FOR LINEAR REACTIVITY VARIATIONS AND EXTERNAL SOURCES

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Abstract. *The analytical solution of point kinetics equations with a group of delayed neutrons is useful in predicting the variation in neutron density during the operation of a nuclear reactor in subcritical systems. With the conception of a new generation of nuclear reactors, guided by ppaper accelerators (ADS), it is necessary to have a fast and accurate prediction of power and reactivity transients due to the variation of external sources. In this paper, an analytical solution for point kinetics equations in subcritical systems, which differ substantially from the conventional equations for critical systems is presented. The results obtained proved to be precise when applied to a subcritical nuclear reactor through the linear increase of the external neutron source.*

***Keywords**: Point kinetics; Subcritical systems; ADS*

1. INTRODUCTION

The analytical solution of point kinetics equations with a group of delayed neutrons is useful to predict neutron density variation during the start-up of a nuclear reactor (Stacey, 2001). With the conception of a new generation of accelerator driven system nuclear reactors (ADSs) (Schikorr, 2001), it becomes necessary to rapidly and accurately predict power and reactivity transients in the event of a possible variation in the intensity of external sources.

Considering the time-dependent neutron transport equation,

$$V^{-1}\frac{\partial}{\partial t}\left|\Psi\right\rangle = -A\left|\Psi\right\rangle + \left(1-\beta\right)\frac{\chi_p}{4\pi}F\left|\Psi\right\rangle + \sum_{i=1}^{I}\frac{\chi_{d,i}}{4\pi}\lambda_i\left|c_i\right\rangle + \left|q_{ext}(t)\right\rangle , \tag{1}$$

and an importance function associated to equation

$$A_o^{\dagger}\left|n_o^{+}\right\rangle + F_o\frac{F_o^{\dagger}}{4\pi}\left|n_o^{+}\right\rangle + \frac{\gamma}{W_o}\left|\Sigma_f^{(o)}\right\rangle = \left|0\right\rangle . \tag{2}$$

This importance function is the same proposed by Gandini and Salvatores (2002), albeit with a slightly different notation and taking the angular dependency into account.

Weighing Eq. (1) with vector $\left\langle n_o^{+}\right|$ results in:

$$\frac{\partial}{\partial t}\left\langle n_o^{+}\left|V^{-1}\right|\Psi\right\rangle = \left\langle n_o^{+}\left|-A\right|\Psi\right\rangle + \left\langle n_o^{+}\left|\left(1-\beta\right)\frac{\chi_p}{4\pi}F\right|\Psi\right\rangle + \sum_{i=1}^{I}\lambda_i\left\langle n_o^{+}\left|\frac{\chi_{d,i}}{4\pi}\right|c_i\right\rangle + \left\langle n_o^{+}\middle|q_{ext}\left(t\right)\right\rangle . \tag{3}$$

Suppose now the system os disturbed as follows:

$$-A \rightarrow -A_o + \delta A \tag{4}$$

$$F \rightarrow F_o + \delta F \tag{5}$$

$$q_{ext}\left(t\right) \rightarrow q_{ext}^{(o)} + \delta q_{ext}\left(t\right) , \tag{6}$$

where $-A_o$, F_o and $q_{ext}^{(o)}$ correspond to the non-disturbed stationary states.

Replacing these disturbances in Eq. (3) and adding and subtracting

$$\sum_{i=1}^{I}\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi}\beta_i\left(F_o+\delta F\right)\right| \Psi \right\rangle , \tag{7}$$

On the right side of the resulting equation one has

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle n_o^+ \left| V^{-1} \right| \Psi \right\rangle = \left\langle n_o^+ \left| \delta A \right| \Psi \right\rangle + \left\langle n_o^+ \left\| \left[\left(1-\beta\right)\frac{\chi_p}{4\pi} + \sum_{i=1}^{I}\frac{\chi_{d,i}}{4\pi}\beta_i \right]\delta F \right| \Psi \right\rangle + \sum_{i=1}^{I}\lambda_i \left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \right| c_i \right\rangle - \\ \sum_{i=1}^{I}\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi}\beta_i F \right| \Psi \right\rangle + 1 + \left\langle n_o^+ \left| -A_o \right| \Psi \right\rangle + \left\langle n_o^+ \left| F_o \left[\left(1-\beta\right)\frac{\chi_p}{4\pi} + \sum_{i=1}^{I}\frac{\chi_{d,i}}{4\pi}\beta_i \right] \right\| \Psi \right\rangle + \left\langle n_o^+ \left| \delta q_{ext}\left(t\right) \right\rangle\right. \end{aligned} . \tag{8}$$

where in the calculations above the relation of source reciprocity $\left\langle n_o^+ \left| q_{ext}^{(o)} \right\rangle\right. = 1$ was used.

Weighing Eq. (2) with vector $\left\langle \Psi \right|$ and adequately manipulating it, taking into account the properties of the linear operators, one has:

$$\left\langle n_o^+ \left| -A_o \right| \Psi \right\rangle + \left\langle n_o^+ \left| F_o \left[\left(1-\beta\right)\frac{\chi_p}{4\pi} + \sum_{i=1}^{I}\frac{\chi_{d,i}}{4\pi}\beta_i \right] \right\| \Psi \right\rangle = -\frac{\gamma}{W_o}\left\langle \Sigma_f^{(o)} \left| \Psi \right\rangle\right. . \tag{9}$$

Replacing the right side of Eq. (9) in Eq. (8), one finds

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle n_o^+ \left| V^{-1} \right| \Psi \right\rangle = \left\langle n_o^+ \left| \delta A + \frac{\chi}{4\pi}\delta F \right| \Psi \right\rangle - \sum_{i=1}^{I}\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi}\beta_i F \right| \Psi \right\rangle + \sum_{i=1}^{I}\lambda_i \left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \right| c_i \right\rangle + 1 \\ -\frac{\gamma}{W_o}\left\langle \Sigma_f^{(o)} \left| \Psi \right\rangle\right. + \left\langle n_o^+ \left| \delta q_{ext}\left(t\right) \right\rangle\right. \end{aligned} \tag{10}$$

where,

$$F_o^{\dagger} = \chi = \left(1-\beta\right)\chi_p + \sum_{i=1}^{I}\chi_{d,i}\beta_i \tag{11}$$

Considering the following factorization,

$$\Psi \approx n(t)\cdot\phi_0 \tag{12}$$

and replacing Eq. (12) in Eq. (10) one finds that

$$\begin{aligned}\left\langle n_o^+ \left| V^{-1} \right| \phi_o \right\rangle \frac{dn\left(t\right)}{dt} = \left\langle n_o^+ \left| \delta A + \frac{\chi}{4\pi} \right| \phi_o \right\rangle n(t) - \sum_{i=1}^{I}\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi}\beta_i F \right| \phi_o \right\rangle n(t) + \sum_{i=1}^{I}\lambda_i \left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \right| c_i \right\rangle + \left[1-n(t)\right] \\ + \left\langle n_o^+ \left| \delta q_{ext}\left(t\right) \right\rangle\right. \end{aligned} , \tag{13}$$

where we considered identity,

$$\frac{\gamma}{W_o}\left\langle \Sigma_f^{(o)} \middle| \phi_o \right\rangle \equiv 1 \; . \tag{14}$$

Dividing all the member of Eq. (13) by a normalization factor $I=\left\langle n_o^+ \middle| \left(\frac{\chi}{4\pi}\right) F \middle| \phi_o \right\rangle$, one has

$$\frac{\left\langle n_o^+ \middle| V^{-1} \middle| \phi_o \right\rangle}{I}\frac{dn(t)}{dt}=\frac{\left\langle n_o^+ \middle| \delta A+\frac{\chi}{4\pi}\delta F \middle| \phi_o \right\rangle}{I}n(t)-\sum_{i=1}^{I}\frac{\left\langle n_o^+ \middle| \frac{\chi_{d,i}}{4\pi}\beta_i F \middle| \phi_o \right\rangle}{I}n(t)+\sum_{i=1}^{I}\lambda_i\frac{\left\langle n_o^+ \middle| \frac{\chi_{d,i}}{4\pi} \middle| c_i \right\rangle}{I}+ . \tag{15}$$
$$\frac{1}{I}\left[1-n(t)\right]+\frac{\left\langle n_o^+ \middle| \delta q_{ext}(t) \right\rangle}{I}$$

Applying the following definitions of the integral parameters

$$l_{eff}=\frac{\left\langle n_o^+ \middle| V^{-1} \middle| \phi_o \right\rangle}{I} \tag{16}$$

$$\rho(t)=\frac{\left\langle n_o^+ \middle| \delta A+(\chi/4\pi)\delta F \middle| \phi_o \right\rangle}{I} \tag{17}$$

$$\beta_{eff}=\sum_{i=1}^{I}\frac{\left\langle n_o^+ \middle| (\chi_{d,i}/4\pi)\beta_i F \middle| \phi_o \right\rangle}{I} \tag{18}$$

$$\xi_i=\frac{\left\langle n_o^+ \middle| (\chi_{d,i}/4\pi) \middle| c_i \right\rangle}{I} \tag{19}$$

$$\frac{1}{I}=\zeta \tag{20}$$

$$q(t)=\frac{\left\langle n_o^+ \middle| \delta q_{ext}(t) \right\rangle}{I} \quad , \tag{21}$$

Eq. (15) can be written thus:

$$l_{eff}\frac{dn(t)}{dt}=\left[\rho(t)-\beta_{eff}\right]n(t)+\sum_{i=1}^{I}\lambda_i\xi_i(t)+\zeta\left[1-n(t)\right]+q(t) \; . \tag{22}$$

The equation for the delayed neutron precursors is:

$$\frac{\partial}{\partial t}\left| c_i \right\rangle=\beta_i F\left| \Psi \right\rangle-\lambda_i\left| c_i \right\rangle \; . \tag{23}$$

Multiplying Eq. (23) by $\left\langle n_o^+ \middle| (\chi_{d,i}/4\pi) \right.$, applying the factorization as represented by Eq. (12) and dividing by the normalization factor I, one has

$$\frac{d}{dt}\frac{\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \right| c_i \right\rangle}{I} = \frac{\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \beta_i F \right| \phi_o \right\rangle}{I} n(t) - \lambda_i \frac{\left\langle n_o^+ \left| \frac{\chi_{d,i}}{4\pi} \right| c_i \right\rangle}{I} \quad , \tag{24}$$

where, given the definitions for the integral parameters, one has the following equation:

$$\frac{d}{dt}\xi_i(t) = \beta_{i,eff} n(t) - \lambda_i \xi_i(t) \ . \tag{25}$$

The point kinetics equation system formed by Eq. (22) and (25) is similar to the equation system defined by Gandini and Salvatores, being different, however, in the integral parameter α which, albeit present in the reference equation system (2002), is not found in the equation system (22) and (25) presented here (Palma et. al., 2009).

The set of Eq. (22) and (25) is subjected to the initial conditions:

$$n(0) = n_0 \tag{26}$$

$$\xi_i(0) = \frac{\beta_i}{\lambda_i} n_0 \tag{27}$$

The goal of this paper is to study the variations in neutron density in subcritical systems in the presence of an external neutron source that varies in a linear manner in time according to the expression:

$$q(t) = q_0 + r_q t, \tag{28}$$

where q_0 is the initial intensity of the source of neutrons inserted in the system and r_q is the linear insertion rate for external neutrons.

Let us also consider an insertion of reactivity in the system that linearly varies according to the expression:

$$\rho(t) = \rho_0 + r_\rho t, \tag{29}$$

where ρ_0 is the initial reactivity in the system and r_ρ is the linear reactivity insertion rate.

2. MATHEMATICAL FORMULATION

Deriving Eq. (22) in relation to time one can write:

$$l_{eff}\frac{d^2 n(t)}{dt^2} = \frac{d\rho(t)}{dt} n(t) + \left[\rho(t) - \beta_{eff}\right]\frac{dn(t)}{dt} + \sum_{i=1}^{I} \lambda_i \frac{d\xi_i(t)}{dt} + \varsigma\left[1 - \frac{dn(t)}{dt}\right] + \frac{dq(t)}{dt} . \tag{30}$$

Considering a group of precursors and replacing Eq. (25) in Eq. (22) one obtains, following the elimination of the dependency of precursor concentration, the following expression:

$$l_{eff}\frac{d^2 n(t)}{dt^2} = \left[\frac{d\rho(t)}{dt} - \lambda\varsigma + \lambda\rho(t)\right] n(t) + \left[\rho(t) - \beta_{eff} - \varsigma - \lambda l_{eff}\right]\frac{dn(t)}{dt} + \left[\frac{dq(t)}{dt} + q(t) + \lambda\varsigma\right], \tag{31}$$

subjected to the initial conditions:

$$n(0) = n_0 \tag{32}$$

$$\frac{dn(0)}{dt} = 0 \tag{33}$$

Disregarding term $l_{eff}\,\frac{d^2n(t)}{dt^2}$ in relation to all the other in Eq. (31) and replacing Eq (28) and (29) as well as its derivates in Eq. (31) one obtains the following differential equation:

$$\frac{dn(t)}{dt} + \lambda\left(\frac{k_1 - t}{\Delta - t}\right)n(t) = k_3\left(\frac{k_2 + t}{\Delta - t}\right), \tag{34}$$

where these are defined:

$$\Delta = \frac{\beta_{eff} + \varsigma + \lambda l_{eff} - \rho_0}{r_\rho} \tag{35}$$

$$k_1 = \frac{\lambda\varsigma - \lambda q_0 - r_\rho}{\lambda r_\rho} \tag{36}$$

$$k_2 = \frac{r_q + \lambda\rho_0 + \lambda\varsigma}{\lambda r_q} \tag{37}$$

$$k_3 = \frac{\lambda r_q}{r_\rho}\,. \tag{38}$$

The differential equation that rules the neutron density in the system can be written thus:

$$\frac{dn(t)}{dt} + f(t)n(t) = g(t), \tag{39}$$

and may be solved with the use of the integrating factor method (Arfken, 2001), providing solutions represented by:

$$n(t) = e^{-\int f(t)dt}\left[\int e^{\int f(t)dt} g(t)\,dt + \Omega\right], \tag{40}$$

where the integration constant Ω can be determined from the initial condition as expressed by Eq. (32).

Identifying functions $f(t)$ and $g(t)$ in Eq. (34):

$$f(t) = \lambda\left(\frac{k_1 - t}{\Delta - t}\right) \tag{41}$$

$$g(t) = k_3\left(\frac{k_2 + t}{\Delta - t}\right), \tag{42}$$

and replacing it in Eq. (40) one can write the following expression for neutron density:

$$n(t) = e^{-\int \lambda\left(\frac{k_1 - t}{\Delta - t}\right)dt}\left[k_3\int e^{\int \lambda\left(\frac{k_1 - t}{\Delta - t}\right)dt}\left(\frac{k_2 + t}{\Delta - t}\right)dt + \Omega\right], \tag{43}$$

In replacing variables $u = \Delta - t$ and defining $\theta = \lambda\left(\Delta - k_1\right)$ one can re-write Eq. (43) thus:

$$n\left(t\right) = \frac{e^{\lambda u}}{u^{\theta}}\left\{k_3\left[\left(k_2 + \Delta\right)\int e^{-\lambda u}u^{\theta-1}du - \int e^{-\lambda}u^{\theta}du\right] + \Omega\right\}, \tag{44}$$

Both integrals in Eq. (44) are tabled (Gradshteyn and Ryzhik, 2007) and produce the following expression for function $n\left(t\right)$:

$$n\left(t\right) = \frac{e^{\lambda u}}{u^{\theta}}\left\{k_3\left[-\lambda\left(k_2 + \Delta\right)\Gamma\left(\theta, \lambda u\right) + \Gamma\left(\theta + 1, \lambda u\right)\right] + \Omega\right\}, \tag{45}$$

where in Eq. (45) one can find the incomplete gamma functions.

Imposing the initial condition $n\left(0\right) = n_0$ and grouping the terms in a convenient manner, one obtains the following expression for neutron density:

$$n\left(t\right) = A_1\frac{e^{-\lambda t}}{\left(\Delta - t\right)^{\theta}}\left\{A_2\Gamma\left(\theta, \lambda\left(\Delta - t\right)\right) - \Gamma\left(\theta + 1, \lambda\left(\Delta - t\right)\right) + A_3\right\}, \tag{46}$$

where constants A_1, A_2 and A_3 are defined thus:

$$A_1 = \frac{k_3 e^{\lambda\Delta}}{\lambda^{\theta+1}} \tag{47}$$

$$A_2 = \lambda\left(k_2 + \Delta\right) \tag{48}$$

$$A_3 = \frac{n_0\left(\Delta\right)^{\theta}e^{\lambda\Delta}}{k_3\lambda^{-\left(\theta+1\right)}} + \Gamma\left(\theta + 1, \lambda\Delta\right) - A_2\Gamma\left(\theta, \lambda\Delta\right), \tag{49}$$

The results obtained from Eq. (46) are presented in the next section.

3. RESULTS

As a reference in the validation of the analytical approximation for neutron density obtained in this paper, the method of finite differences will be used for the numerical solution of point kinetics equations in sub-critical systems considering a group of delayed neutron precursors, Eq (22) and (25). The following expressions were used to implement the implicit temporal integration method (Hashimoto et. al., 2000):

$$l_{eff}\frac{n^{i+1} - n^{i}}{\Delta t} = \left(\rho^{i+1} - \beta - \varsigma\right)n^{i+1} + \lambda\xi^{i+1} + q^{i+1} \tag{50}$$

$$\frac{\xi^{i+1} - \xi^{i}}{\Delta t} = \beta n^{i+1} - \lambda\xi^{i+1}, \tag{51}$$

The following nuclear parameters were used in the numerical simulations carried out in this paper: $\lambda = 0.00127$, $\varsigma = 0.0493$, $\beta = 0.00009$, $l_{eff} = 6.25\times10^{-5}$, $\rho\left(t\right) = -0.06 + 0.001t$ and $q\left(t\right) = 0.5 + 0.1t$. Figure 1 shows the neutron density obtained from the analytical approximation proposed in this paper, Eq. (46), and from the numerical reference method.

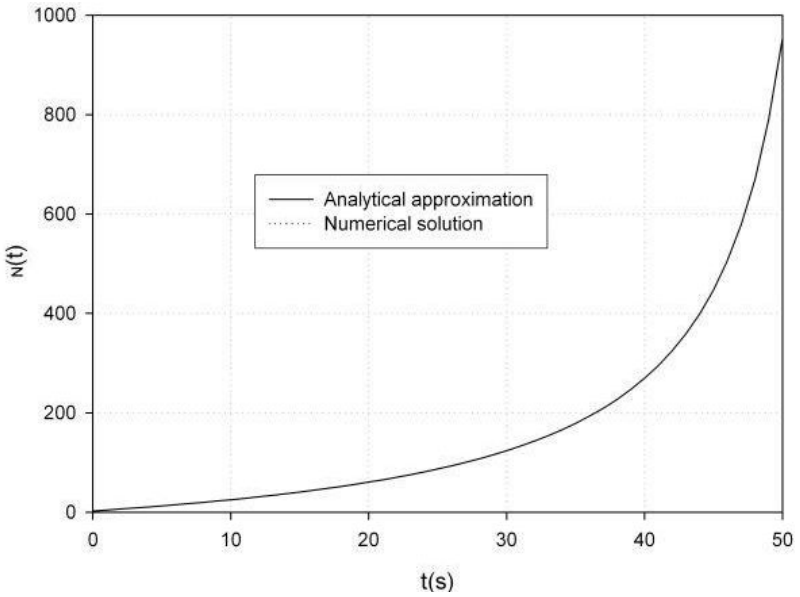

Figure 1 - Comparison between neutron density as obtained from Eq. (46) with the numerical reference method.

Graph 1 allows the conclusion that the approximation proposed overlaps the numerical reference method. This behaviour can be justified as the typical values of parameter l_{eff} for ADS reactors are smaller than for PWR type reactors, which indicates that disregarding the second derivate is a good approximation.

Figure 2 shows that it is possible to visualize the behaviour of neutron density $n(t)$, strongly influenced by the variation of the external neutron source $q(t)$.

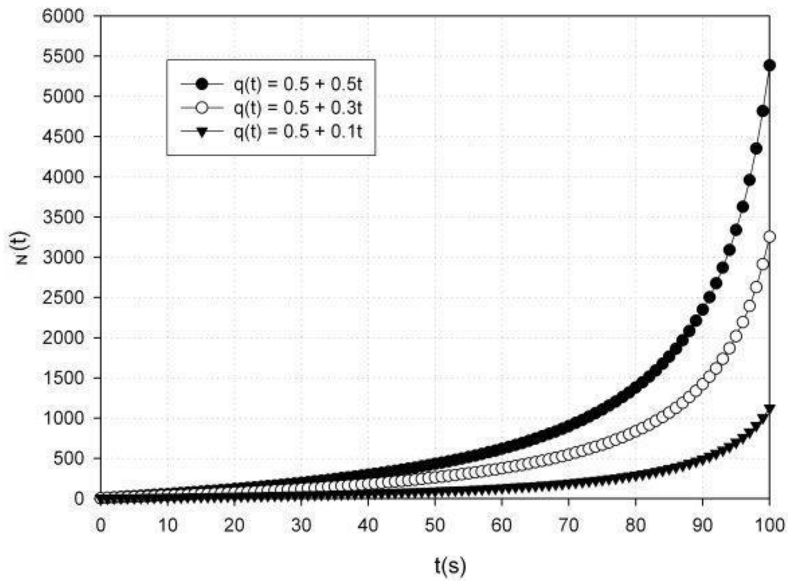

Figure 2 - Neutron density behaviour as obtained from Eq. (46) for different external sources q(t).

4. CONCLUSIONS

An analytical approximation has been developed in this paper, seeking to predict neutron density $n(t)$ in sub-critical systems with linear reactivity insertions and source external to the system. The formulation proposed consists of the solution of the point kinetics equations for sub-critical systems with a group of precursors. The results obtained have shown small percentage deviations in relation to those obtained with the reference method, which was the numerical solution of the point kinetics equations for sub-critical systems.

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7. RESPONSIBILITY NOTICE