COMPLEX FORMULATION IN THE ANALYSIS OF ROTOR-BEARING SYSTEM

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Abstrac: *In the rotor dynamics, the traditional formulation has been using, so much for mathematical modelling, as well as to study the dynamic behavior of the* **Rotor-Bearing** *mechanical systems. However, the complex formulation has lately appeare, as an appropriate tool for the dynamic analysis of rotors. In that sense, the present work makes a comparative study between the complex formulation and the traditional analysis, exploring the advantages and disadvantages of each one of the techniques, when applied to isotropic and anisotropic mechanical systems. It is used, The finite element method is used for the modelling of the components of the Rotor-Bearing system considered in this work.*

Keywords*:* *complex formulation, rotor dynamics, finite element method.*

1. INTRODUCTION

According to the ISO definition, a rotor is a body whose simple form is a shaft supported by bearings. Sometimes the rotors can be subject to excessive vibrations. Therefore, the understanding of rotor dynamics is reached through analyses of their mathematical models. The simplest model is the rotor called of Laval, also knowing as Jeffcott rotor.

The mathematical models became indispensable tools in the study of the dynamic behavior of mechanical systems, because they allow the analysis of the dynamic behavior in critical and stable situations through simulations. Lately, the complex formulation was applied in the analysis of the dynamic behavior of rotating mechanical systems by Lee (1993); Kessler (1999) and Souto (2000), as a more efficient form, when compared with the traditional modeling. Therefore, this paper brings a comparative study between the complex and the traditional formulation, applying both methods for a Rotor-Bearing mechanical system.

The mechanical Rotor-Bearing system considered in this work is showing in Fig. 1. It is composed by a flexible shaft, a rigid disk (called the rotor of the system), and two bearings. The stationary reference system XYZ, is used to describe the motion equations for each components of the system. A transverse section of the rotor in bending state, is defined in relation to the system XYZ through the translations $u(\mathrm{Y},t)$, $v(\mathrm{Y},t)$ in the X and Z directions, supplying the position of the center of the transverse section at the instant t. The orientation of the section is given by the small rotations $\alpha(\mathrm{Y},t)$, $\beta(\mathrm{Y},t)$ around the axes X and Z, respectively.

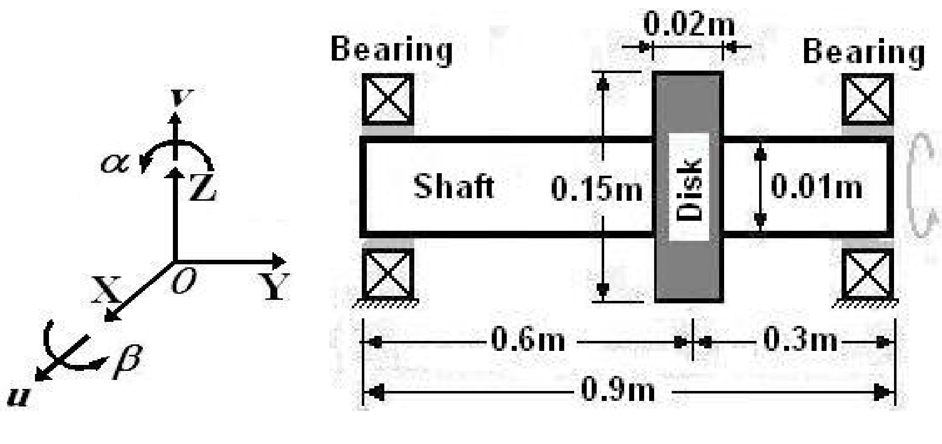

Figure 1. Mechanical Rotor-Bearing System.

2. FINITE ELEMENTS METHOD: TRADITIONAL FORMULATION

The system considerated in this work is reduced to a problem of dynamic modeling of a flexible system composed by rotor, shaft and bearings; which typical configuration is showing in Fig. 1. The finite elements method produces satisfactory results in the study of structural and mechanical problems.

The modeling of the components of the mechanical system was based in the works presented by Tapia (2003), where the shaft is modeled through flexible beam elements of continuous mass, the disk is modeled as rigid disc of concentrated mass, and the bearings through equivalent stiffness and damping coefficients, neglecting the bending moments and the inertia of the bearings. The graphic representation of the components models is showing in Fig. 2.

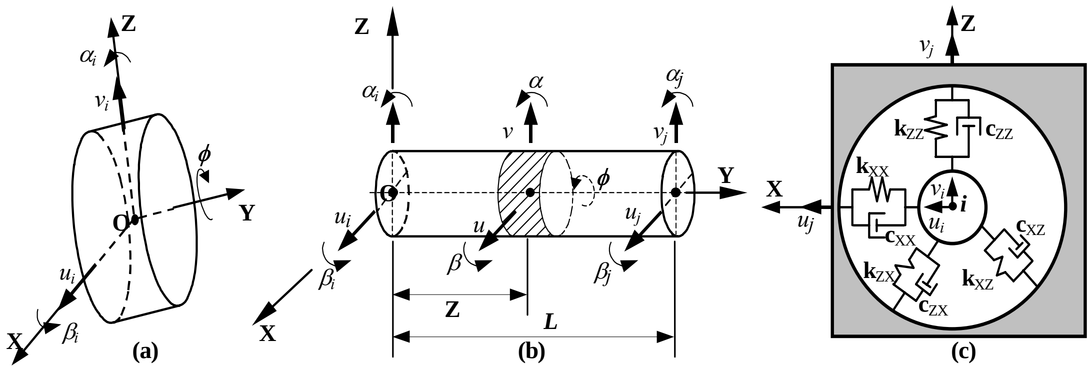

Figure 2- Models of the components of the system: (a) Disc, (b) Shaft, (c) Bearing.

Considering the bending vibrations of the system, the equation of motion of the **Rotor-Bearing** system is written in the matrix form as:

$$\lfloor M_g \rfloor\{\ddot{q}_g\} + \lfloor \Omega \lfloor G_g \rfloor + \lfloor C_g \rfloor \rfloor \{\dot{q}_g\} + \lfloor K_g \rfloor \{q_g\} = \{F_{ex}\}. \tag{1}$$

Where Ω is the rotational speed of the system; $\lfloor M_g \rfloor, \lfloor G_g \rfloor, \lfloor C_g \rfloor, \lfloor K_g \rfloor$ are the global matrices of mass, gyroscopic, damping and stiffness respectively; and $\{F_{ex}\}$ is the vector of the external forces that act on the **Rotor-Bearing** system. Finally, the $\{\ddot{q}_g\}, \{\dot{q}_g\}, \{q_g\}$, are the global vectors of acceleration, velocity and of displacement of the system in generalized coordinates, that for any node i in the mechanical system is defined as:

$$\{\ddot{q}_i\} = \{\ddot{u}_i \quad \ddot{v}_i \quad \ddot{\alpha}_i \quad \ddot{\beta}_i\}, \{\dot{q}_i\} = \{\dot{u}_i \quad \dot{v}_i \quad \dot{\alpha}_i \quad \dot{\beta}_i\}, \{q_i\} = \{u_i \quad v_i \quad \alpha_i \quad \beta_i\}$$

2.1. Free vibration

The objective of the analysis of free vibration in a rotating mechanical system, is to determine: the natural frequencies, the Campbell diagram and the vibration modes, for each value of rotational speed Ω. For the analysis of free vibration, the homogeneous form of Eq. (1), is written as:

$$\lfloor M_g \rfloor\{\ddot{q}_g\} + \lfloor \Omega \lfloor G_g \rfloor + \lfloor C_g \rfloor \rfloor \{\dot{q}_g\} + \lfloor K_g \rfloor \{q_g\} = \{0\} \tag{2}$$

An exponential function can be assumed as solution of Eq. (2), written as: $\{q_g\} = \{q\}e^{\lambda t}$, and substituted in Eq. (2), determining a eigenvalues problem for the equation:

$$\left[\lambda^2 [M_g] + \lambda\left(\Omega [G_g] + [C_g]\right) + [K_g]\right]\{q\} = \{0\}, \tag{3}$$

where: $\{q\}$ is the right eigenvector of system and λ is the eigenvalue. The solution of Eq. (3), being the order of the matrices of nxn, produces n eigenvalues and n eigenvectors. Generally, for the solution of problems of second order, it is convenient to reduce the order of the equation, for first order, through the definition of a state vector $\{\chi\}$ and matrices $[A], [B]$ as:

$$\{\chi\} = \begin{Bmatrix} \{\dot{q}_g\} \\ \{q_g\} \end{Bmatrix},\ [A] = \begin{bmatrix} [0] & [M_g] \\ [M_g] & \Omega[G_g] + [C_g] \end{bmatrix},\ [B] = \begin{bmatrix} -[M_g] & [0] \\ [0] & [K_g] \end{bmatrix}.$$

Using theses mathematics relations, the Eq. (2) can be rewritten as:

$$[A]\{\dot{\chi}\} + [B]\{\chi\} = \{0\}. \tag{4}$$

Therefore a linear homogeneous equation of first order is obtained, which solution can be a exponential function as $\{\chi\} = \{y\}e^{\lambda t}$, when substituted in Eq. (4), supplies two problems of eigenvalues of first order:

$$(\lambda[B] + [A])\{X_{od}\} = \{0\} \text{ ou } \left(\lambda[B]^t + [A]^t\right)\{X_{oe}\} = \{0\}. \tag{5}$$

Where $\{X_{od}\}, \{X_{oe}\}$ are the right and left eigenvectors.

The solution of Eq. (5) has $2n$ eigenvalues and $2n$ eigenvectors, since the matrices [A] and [B] are real matrices of order $2nx2n$. The eigenvalues are defined as $\lambda_i = \sigma_i \pm j \cdot \omega_i$, that correspond to the ith eigenvalue. The imaginary part ω_i of eigenvalue is the ith natural frequency of the rotating system, and the real part σ_i is the damping coefficient for the ith vibrate mode.

2.2. Forced vibration

The analysis of the forced vibration of mechanical systems is made with the purpose of determining the frequency response function of the system. The equation used in that analysis is the non homogeneous form of Eq. (1). As forced vibrations of the systems are usually due to the unbalance, misalignment among shafts, etc., those can eventually appear.

The transfer function matrices can be determined through the inversion of the mechanical impedance matrix. However, in that case, some problems can occur due to the process of the matrices inversion.

Being Eq. (1), the equation of motion of the system, an external sinusoidal excitation force $\{F_{ex}\} = \{F_o\}e^{j\eta t}$ is assumed, and the system response is $\{q\} = \{q_o\}e^{j\eta t}$, being η the frequency of the excitation force, that is substituted in the Eq. (1), giving:

$$\left[-\eta^2[M_g] + j\eta[\Omega[G_g] + [C_g]] + [K_g]\right]\{q_o\}e^{j\eta t} = \{F_o\}e^{j\eta t}. \tag{6}$$

The coefficient vector of the system vibration response is:

$$\{q_o\} = \left[-\eta^2[M_g] + j\eta[\Omega[G_g] + [C_g]] + [K_g]\right]^{-1}\{F_o\}. \tag{7}$$

Finally, the matrix of the transfer functions $([H(\eta)])$ can be defined as:

$$[H(\omega)] = \left[-\eta^2[M_g] + j\eta[\Omega[G_g] + [C_g]] + [K_g]\right]^{-1}. \tag{8}$$

Therefore, for an excitation applied in a certain degree-of-freedom of the system, the response of the system can be taken in all others degree-of-freedom of the system.

3. COMPLEX FORMULATION

The complex formulation is a modeling method for rotating structures analysis, and it is based on the use of the complex coordinates to describe the dynamic behavior of the rotor, through the decomposition of each mode of the system in two sub-modes, a forward and a backward modes.

Considering the description of the traditional formulation, the complex coordinates (p_i , ϕ_i) for a node i of system are defined according to the following relations:

$$p_i = u_i + j * v_i \text{ and } \overline{p}_i = u_i - j * v_i\ ;\ \phi_i = \alpha_i - j * \beta_i \text{ and } \overline{\phi}_i = \alpha_i + j * \beta_i. \tag{9}$$

Being, j the complex number ($\sqrt{-1}$), and $\bar{p}_i, \bar{\phi}_i$ the conjugated complex coordinates, used to describe the motion of any node of the mechanical system. However, in the first studies: Lee (1993); Kessler (1999), e Souto (2000), only the first relation is presented, i.e., $p_i, \bar{p}_i$. The second relation considering the angular degrees of freedom was presented in the study of Dias Jr et al. (2002), i.e., $\phi_i, \bar{\phi}_i$.

Considering the relations of Eq. (9), the transformation matrix between complex and real coordinates is given by the expression:

$$\begin{Bmatrix} u_i \\ v_i \\ \alpha_i \\ \beta_i \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ -j & 0 & j & 0 \\ 0 & 1 & 0 & 1 \\ 0 & j & 0 & -j \end{bmatrix}\begin{Bmatrix} p_i \\ \phi_i \\ \bar{p}_i \\ \bar{\phi}_i \end{Bmatrix}, \quad [T] = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ -j & 0 & j & 0 \\ 0 & 1 & 0 & 1 \\ 0 & j & 0 & -j \end{bmatrix}, \tag{10}$$

where $[T]$ is a transformation matrix.

The equation of motion of the mechanical **Rotor-Bearing** system, in terms of the complex coordinates, can be written by the equation:

$$[M_C]\begin{Bmatrix} \ddot{p}_i \\ \ddot{\bar{p}}_i \end{Bmatrix} + (\Omega[G_C] + [C_C])\begin{Bmatrix} \dot{p}_i \\ \dot{\bar{p}}_i \end{Bmatrix} + [K_C]\begin{Bmatrix} p_i \\ \bar{p}_i \end{Bmatrix} = \{F_C\}, \tag{11}$$

where $[M_C],[G_C],[C_C],[K_C]$ are the mass, gyroscopic, damping and stiffness global matrices of the system, with elements that are complex numbers. Each one of those matrices are defined for the mathematical relations:

$$[M_C] = [T]^{-1}[M_g][T], [G_C] = [T]^{-1}[G_g][T], [C_C] = [T]^{-1}[C_g][T], [K_C] = [T]^{-1}[K_g][T]. \tag{12}$$

Besides, the vectors: $\begin{Bmatrix} \ddot{p}_i \\ \ddot{\bar{p}}_i \end{Bmatrix}, \begin{Bmatrix} \dot{p}_i \\ \dot{\bar{p}}_i \end{Bmatrix}, \begin{Bmatrix} p_i \\ \bar{p}_i \end{Bmatrix}$ are the acceleration, velocity and displacement vectors in complex coordinates; and $\{F_C\}$ is the vector of complex excitation force.

3.1. Free Vibration.

The objective of the free vibration analysis of the complex formulation is to determine: natural frequencies, Campbell diagram, and vibration modes, for each value of the rotational speed Ω. For the analysis of free vibration in the complex formulation, the homogeneous form of Eq. (11) is considered. It is expressed in the following equation:

$$[M_C]\begin{Bmatrix} \ddot{p}_i \\ \ddot{\bar{p}}_i \end{Bmatrix} + (\Omega[G_C] + [C_C])\begin{Bmatrix} \dot{p}_i \\ \dot{\bar{p}}_i \end{Bmatrix} + [K_C]\begin{Bmatrix} p_i \\ \bar{p}_i \end{Bmatrix} = \{0\} \tag{13}$$

An exponential function can be assumed as solution of Eq. (12), $p = P_f \cdot e^{st} + P_b \cdot e^{\bar{s}t}$, where P_f and P_b are complex numbers, and s is also complex, and defined as $\sigma + j \cdot \omega$, being $\bar{s}$ its conjugated complex. Being substituted in Eq. (13), Eq. (12) produces two eigenvalue problems, carrying the same information, given for:

$$\begin{gathered} \left[s^2[M_C] + s(\Omega[G_C] + [C_C]) + [K_C]\right]\begin{Bmatrix} P_f \\ \bar{P}_b \end{Bmatrix} = \{0\}, \\ \left[\bar{s}^2[M_C] + \bar{s}(\Omega[G_C] + [C_C]) + [K_C]\right]\begin{Bmatrix} P_b \\ \bar{P}_f \end{Bmatrix} = \{0\} \end{gathered} \tag{14}$$

In general, the eigenvalues are defined in the form $s_i = \sigma_i + j\omega_i$. As much the eigenvalue as the eigenvector are dependents of the rotational speed of the rotor. Also, ω_i and σ_i are the natural frequency and damping coefficients for the ith eigenvalue of rotating system.

3.2. Forced vibration.

The analysis of forced vibrations of the system is made with the purpose of determining the frequency response function of the system. In that case, it is called of directional frequency response function. The equation used in that analysis is the non homogeneous form of Eq. (11).

The matrix of the transfer functions can be determined through the inversion of mechanical impedance matrix. Being the Eq. (11) the motion equation, an external sinusoidal excitation force is assumed in the form $\{F_C\} = \{F_f e^{j\eta t} + F_b e^{-j\eta t}\}$, and the response of system is $\{p\} = \{P_f e^{j\eta t} + P_b e^{-j\eta t}\}$, being η the frequency of the excitation force. Those relations, when substituted in the motion equation, give the following relations:

$$\begin{aligned} &\left[-\eta^2[M_C] + j\eta[\Omega[G_C] + [C_C]] + [K_C]\right]\{P_f\}e^{j\eta t} + \\ &\left[-\eta^2[M_C] - j\eta[\Omega[G_C] + [C_C]] + [K_C]\right]\{P_b\}e^{-j\eta t} = \{F_f\}e^{j\eta t} - \{F_b\}e^{-j\eta t} \end{aligned} \quad (15)$$

Finally, the matrix of the directional transfer functions can be defined as:

$$[H(\eta)] = \left[-\eta^2[M_C] + j\eta[\Omega[G_C] + [C_C]] + [K_C]\right]^{-1}. \quad (16)$$

Therefore, for an excitation in a degree of freedom pi of the system, the response of the system in all others degrees of freedom p_j of the system can be obtained, the same that it is given by an element of the matrix of the directional frequency response functions, given by Eq. (16). However, H_{ij}, that corresponds to the degree-of-freedom p_i, gives the frequency response function corresponding to the forward mode, and the H_{ij}, that corresponds to the degree-of-freedom $\bar{p}_i$, gives a frequency corresponding to the backward mode, for the same degree of freedom.

4. ANALYZED EXAMPLE

The **Rotor-Bearing** mechanical system considered in the analysis is shown in Fig. 1. Table 1 presents the physical properties and geometric dimensions of each components of the mechanical system.

Table 1. Dimensions and physical properties of the mechanical system components

Shaft	External diameter: 0.010 m. Length: 0.900 m Elasticity Modulus: $2.0*10^{11}$N/m^2 Specify Density: 7850.0 kg/m^3 Poisson Coefficient: 0.3 Shear factor: 0.85	**Disc, Mass**	External diameter: 0.15 m Internal diameter: 0.01 m Thickness: 0.02 m Specify Density : 7850.0kg/m^3

The mathematical model of the mechanical system, shown in Fig. 1, is obtained through the finite elements method. For this purpose, two shaft elements, a disc element and two bearings elements are taken into account, according to Fig.3.

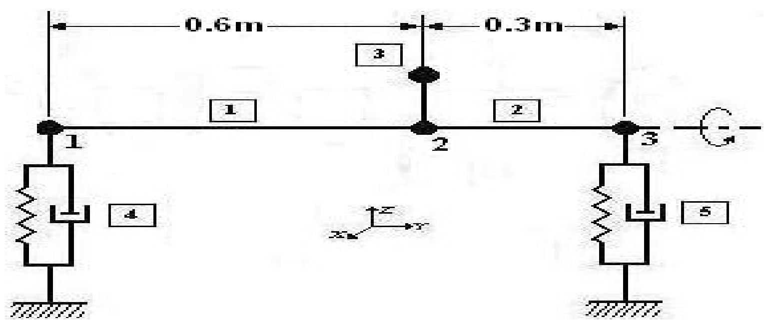

Figure 3. Finite elements model of the **Rotor-Bearing** system for Fig. 1.

The **Rotor-Bearing** mechanical system can be an isotropic or anisotropic system, depending on the stiffness values of the bearings, which are established in Table 2.

Table 2. Parameters of stiffness and damping for the bearings of the **Rotor-Bearing** system.

Bearing 1	Bearing 2	Rotation
k_{xx} =0.18*10^5 N/m , k_{xz} = 0.0 N/m k_{zz} = 0.18*10^5 N/m , k_{zx} = 0.0 N/m c_{xx} = 1.0 Ns/m , c_{xz} = 0.0 Ns/m c_{zz} = 1.0 Ns/m , c_{zx} = 0.0 Ns/m	k_{xx} =0.18*10^5 N/m , k_{xz} = 0.0 N/m k_{zz} = 0.18*10^5 N/m , k_{zx} = 0.0 N/m c_{xx} = 1.0 Ns/m , c_{xz} = 0.0 Ns/m c_{zz} = 1.0 Ns/m , c_{zx} = 0.0 Ns/m	523.60 rad/s

4.1. Isotropic System

To turn the mechanical system of Fig. 1 into an isotropic system, it is necessary that the stiffness values of the two bearings, so much in the horizontal direction (k_{xx}), as in the vertical (k_{zz}), should have the same values, i.e., (k_{xx}=k_{zz}). However, it doesn't implicate that the stiffness values of the two bearings have the same dynamic coefficients.

With the established considerations, the stiffness values of the second bearing are changed in relation to those in Table 2, but always maintaining the condition of isotropy. Fig. 4 presents the traditional frequency response functions (FRF) and the directional FRFs (forward precession), for the different values of stiffness of the bearing 2, staying the values of stiffness of the bearing 1 constant.

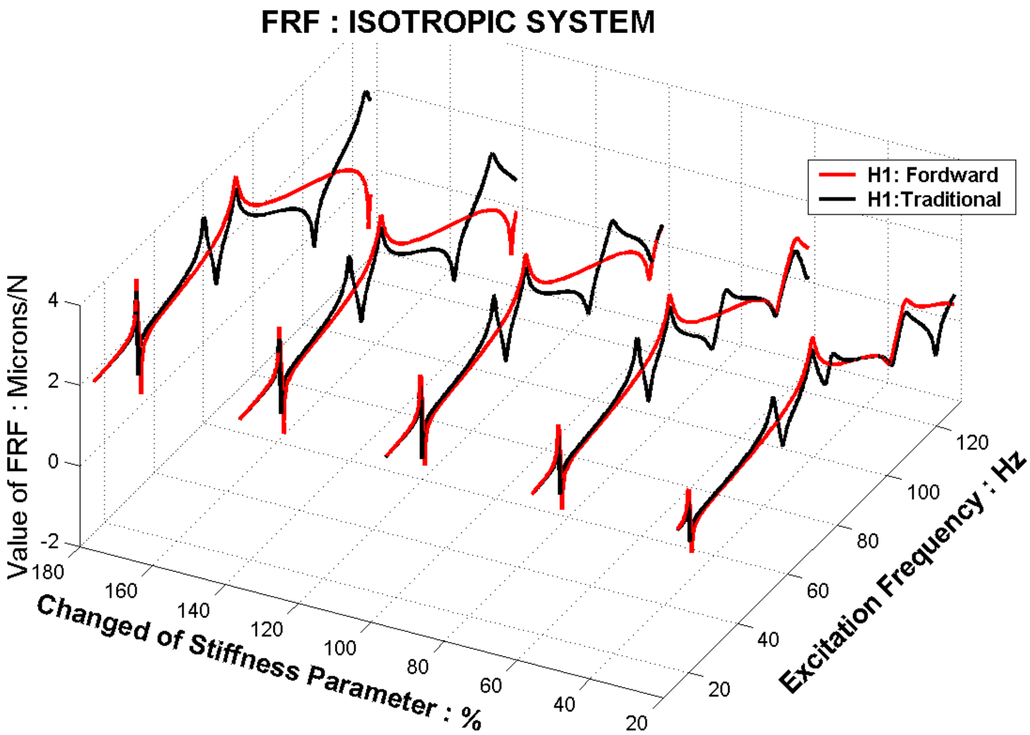

Figure 4. Traditional and Directional FRFs (Forward) for an isotropic system.

Fig. 5 also presents the frequency response functions, for the same isotropic system, and this time, it is considered the backward precession of the directional frequency response function, for the different values of stiffness of the bearing 2.

It is observed in Fig. 4 and Fig. 5, that a separation of the forward precession modes and of the backward precession mode can be obtained through the directional frequency response functions. The same modes appear mixed in the traditional frequency response functions. Similar results are also observed when the stiffness of bearing 1 are changed, and the stiffness of bearing 2 is remains constant. Besides, there are some cases in which some modes, forward or backward, don't appear due to the presence of significant damping, depending on the sensitivity of the modes to the variation of the stiffness parameters of a certain bearing or both bearings. However, the frequency range was properly selected to have a good separation among corresponding forward and backward modes.

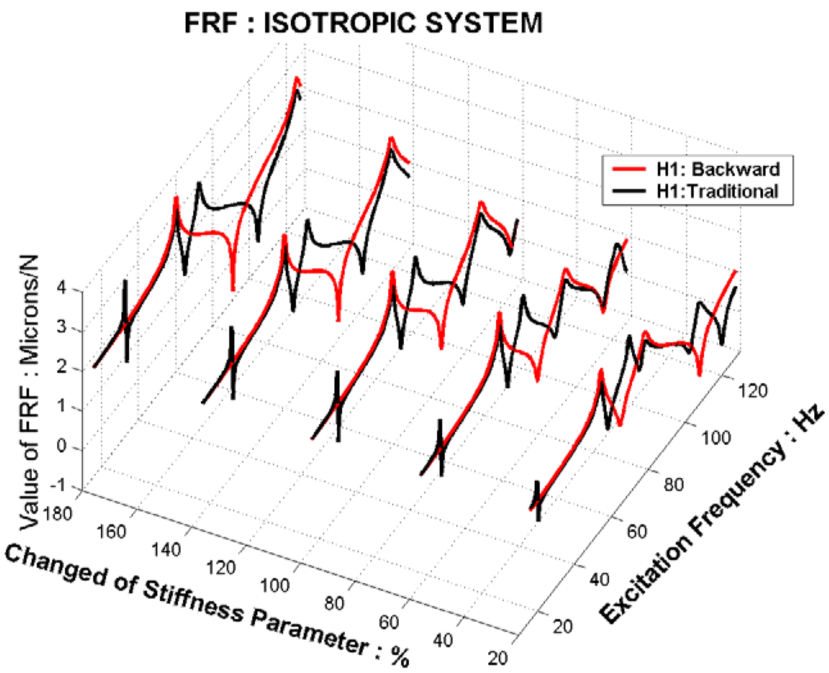

Figure 5. Traditional and Directional FRFs (Backward) for isotropic system.

4.2. Anisotropic system

To turn the mechanical system of Fig. 1 into an anisotropic system, the values of the stiffness of the bearings in both directions, horizontal and vertical, should have different values.

With the established considerations, the stiffness values of the bearing were changed in relation to Table 2, i.e., the stiffness in one direction is changed while in the other it is maintained constant, always maintaining the condition of anisotropy of the system. Fig. 6 presents the traditional and directional frequency response functions (forward precession), for the different values of vertical stiffness of bearings l and 2.

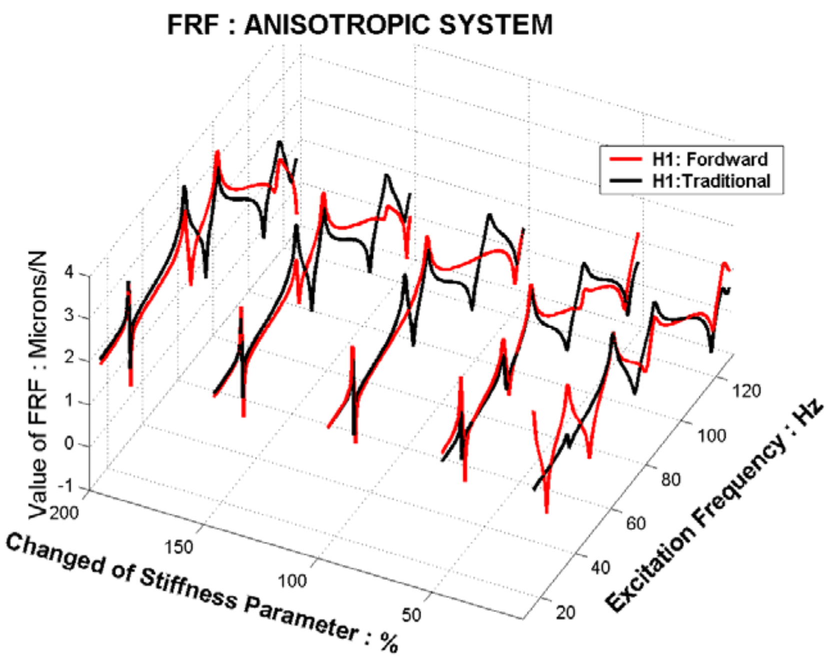

Figure 6. Traditional and Directional FRFs (Forward) for the anisotropic system.

Fig. 7 also presents the frequency response functions, for the same anisotropic systems, taking into account the backward precession of the directional frequency response function, for the different values of vertical stiffness of bearings l and 2.

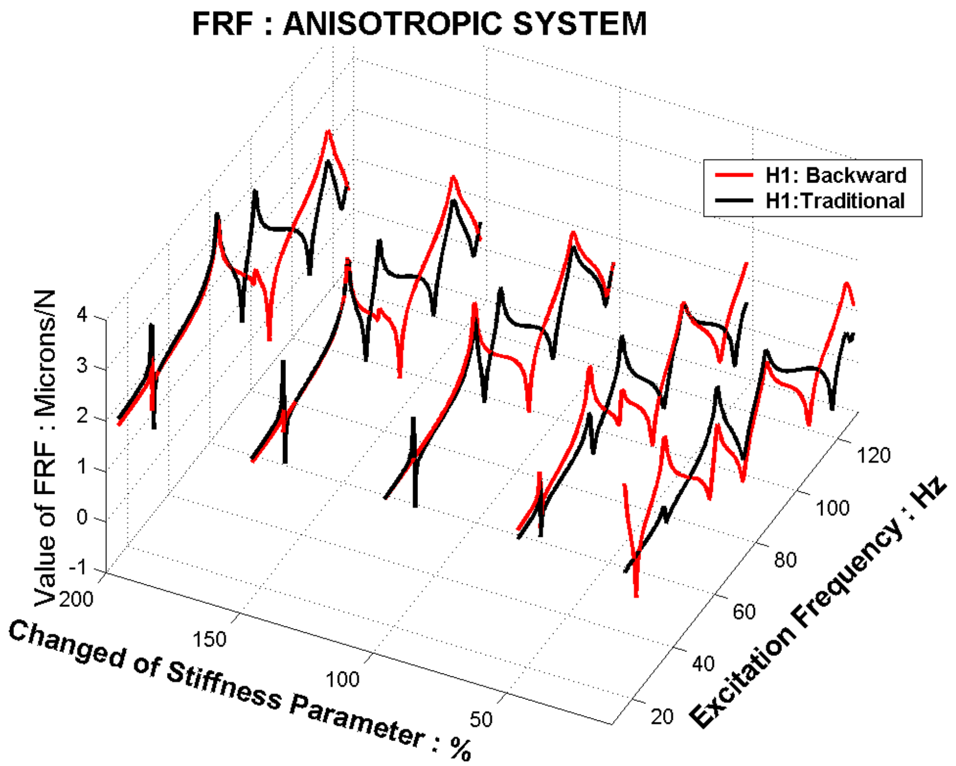

Figure 7. Traditional and Directional FRFs (Backward) for the anisotropic system.

It is observed in Fig. 6 and Fig. 7 that a total separation for the forward precession modes or the of backward precession modes can not be obtained, even by the directional frequency response functions, because some backward precession modes appear in the frequency response functions that correspond to the forward modes. The same effect happens with the frequency response functions that correspond to the backward modes.

Also, a similar behavior is observed when the value of the horizontal stiffness of the bearings is changed, staying constant for the vertical stiffness of the bearings. However, depending on the anisotropy degree of the bearings, this disturbance can be accentuated.

5. CONCLUSIONS

The conclusions in the present work can be summarized as:

In relation to the free vibrations, the order of the global matrices of the mechanical system doesn't decrease as it is mentioned in some works, when the complex formulation is used.

The advantage of the complex formulation in relation to the traditional one occurs for isotropic mechanical system, because a decrease of the spectral density is achieved, with the directional frequency response functions of the system.

In anisotropic mechanical systems, the method of the complex formulation sometimes is not able to take a clear separation of the forward/backward precession modes, and depending on the anisotropy degree, as well as of the number of components that induce the anisotropy, the effectiveness of the directional frequency response functions decreases in its purpose of separating the modes.

6. ACKNOWLEDGEMENTS

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