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SHAKEDOWN ANALYSIS CONSIDERING LIMITED KINEMATIC HARDENING MATERIALS

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Abstract. *In design or safety assessment of mechanical structures, the use of the Design by Analysis (DBA) route is a modern trend. However, for making possible to apply DBA to structures under variable loads, two basic failure modes considered by ASME or European Standards must be precluded. Those modes are the alternate plasticity and incremental collapse (with instantaneous plastic collapse as a particular case). Shakedown theory is a tool that permit us to assure that those kinds of failures will be avoided. However, in practical applications, very large nonlinear optimization problems are generated. Due to this facts, only in recent years have been possible to obtain algorithms sufficiently accurate, robust and efficient, for dealing with this class of problems. In this paper, one of these shakedown algorithms, developed for dealing with elastic ideally-plastic structures, is enhanced to include limited kinematic hardening, a more realistic material behavior. This is done in the continuous model by using internal thermodynamic variables. A corresponding discrete model is obtained using an axisymmetric mixed finite element with an internal variable. A thick wall sphere, under variable thermal and pressure loads, is used in an example to show the importance of considering the limited kinematic hardening in the shakedown calculations.*

Keywords: *Shakedown, Limited Kinematic Hardening.*

1. INTRODUCTION

The use of the Design by Analysis (DBA) route to demonstrate the structural safety, when designing mechanical components as pipes and pressure vessels subject to variable thermal and mechanical loads, is a modern trend and it makes possible to us a more flexible approach.

When, "in service inspections" detect defects and other deviations from the hypothesis adopted by Design by Formulas (DBF) route, the use of DBA is necessary to proceed a safety assessment. This frequently occurs and it is important when considering life extension of components as for example, in nuclear industry.

In plastic range, to implement the DBA route, is necessary to assure that the structure can outstanding the variable actions without suffer any of the following failure modes considered in the ASME or European Standards: alternating plasticity (AP) and incremental collapse (IC or ratcheting). Plastic collapse (PC) can also be considered as a particular case of incremental collapse, where the statical application of only one of the possible loading distributions may lead to impending unbounded plastic deformation. The analysis that permits us to assure that these types of failure does not occur is the shakedown analysis, including limit analysis as a particular case.

Based on Melan's statical formulation (Koiter, 1960, Polizzotto et al., 1991 and Nguyen, 2000), shakedown analysis is a direct method which needs only the extremum values of the loads and material properties to be performed. To any load program contained in a prescribed range of variable loads, shakedown analysis permit us to assure that the failure modes cited above will be precluded.

In spite of the fact that the shakedown theory has been established for ideal plasticity in the 50th and that, ever since, a number of developments has been made, the implementation of the theory in a fashion capable to deal with real situations, in industrial level problems, only in recent years has been achieved. This occurs because the implementation of the theory, in real cases, result in a very large optimization problem with non-linear constraints. The development and extensions in direct methods in recent years and the development of robust finite elements, efficient optimization methods and the ease access to more powerful computers has become the Design by Analysis route, possible (Zeman, 1996, M.Staat and M.Heitzer, 2003 and Staat, 2005). But, for this to be possible, the integration of those techniques in an efficient, accurate and robust algorithm needs to be done. Zouain and co-workers developed an algorithm of this kind, described in (Zouain et al., 2002 and Zouain, 2004), to perform shakedown analysis of elastic-ideally plastic structures.

However, in the practical use of the theory, the realistic properties of the materials, as for example, limited kinematic hardening, should be considered. Furthermore, in order to represent the ratcheting phenomenon, it is necessary to consider limited kinematic hardening. The extensions of the basic theory to include nonlinear or limited hardening behaviors came only recently(see e.g. Stein et al., 1990, Polizzotto et al., 1991, and Nguyen, 2000). The present study is based on the theory of shakedown with thermodynamic internal variables to represent hardening that can be found in (Nguyen, 2000, Polizzotto et al., 1991 or Stein et al., 1993).

We will use here the constitutive model proposed by E. Stein and coworkers (Stein et al., 1990, 1992 and 1993). Based in that model, Nery (2007) extended the Zouain algorithm to consider limited kinematic hardening and developed a 2D mixed axisymmetric finite element with internal variable to deal with axisymmetric shakedown problems.

2. BASIC ASSUMPTIONS AND NOTATION

Consider a body $\mathcal{B}$ occupying an open bounded region of the euclidean space, with regular boundary. Let $v \in \mathcal{V}$ be a velocity field complying with prescribed boundary conditions. Between the strain rate field $d \in \mathcal{W}$ and v there is a relation:

$$d = \mathcal{D}v \tag{1}$$

where $\mathcal{D}$ is the tangent deformation operator, mapping $\mathcal{V}$ into $\mathcal{W}$. Small deformations are assumed. $\sigma \in \mathcal{W}'$ is the stress field and $L \in \mathcal{F}$, the load systems space. $\mathcal{W}$ and $\mathcal{W}'$ are dual spaces. Between σ and L there is a relation:

$$\sigma = \mathcal{D}'L \tag{2}$$

where $\mathcal{D}'$ is the equilibrium operator. $\mathcal{D}$ and $\mathcal{D}'$ are self-adjoint operators. The virtual power principle states:

$$\langle \sigma, \mathcal{D}v \rangle = \langle L, v \rangle, \quad \forall v \in \mathcal{V} \tag{3}$$

The standard generalized material model (Halphen and Nguyen, 1975) and isothermal processes ($\dot{\Theta} = 0$) are considered here for deriving the constitutive equations. The local states method (Lemaitre and Chaboche, 1990) is used and aiming to consider kinematic hardening, are adopted the following generalized state variables:

$\boldsymbol{\epsilon} = (\varepsilon, 0)$ generalized strain
$\boldsymbol{\epsilon}^e = (\varepsilon^e, \omega)$ generalized reversible strain
$\boldsymbol{\epsilon}^p = (\varepsilon^p, \beta)$ generalized irreversible strain
$\boldsymbol{\sigma} = (\sigma, A)$ generalized stress

where, ε is the observable strain, ε^e is the elastic strain, ω is the reversible internal hardening variable, ε^p is the plastic strain, β is the irreversible internal hardening variable, σ is the Cauchy stress tensor and A is the back stress.

With additive decomposition of strain we have $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p$ and then:

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{4}$$

$$0 = \omega + \beta \tag{5}$$

The state laws are obtained from a quadratic free energy potential, in ε^e and β. Assuming that the elastic and hardening variables are not coupled, the following relations are obtained:

$$\sigma = \mathbb{E}\varepsilon^e \tag{6}$$

and

$$A = -\mathbb{H}\beta \tag{7}$$

with the tensors $\mathbb{E}$ e $\mathbb{H}$ constants.

The evolution laws are derived from a dissipation potential defined by Hill's maximum dissipation principle. As usual we call here $\dot{\varepsilon}^p = d^p$. The flow law is derived from this potential. In the case of Mises criteria and associative flow law, the plastic relations are equivalent to a classical form:

$$(d^p, \dot{\beta}) = \dot{\lambda}\nabla f(\sigma, A) \tag{8}$$

Here $\nabla f(\sigma, A)$ denotes the gradient of f (f is the yield surface in stress space) and $\dot{\lambda}$ is a vector field of plastic multipliers. At any body point, the components of $\dot{\lambda}$ are related to each plastic mode in f by the complementarity conditions:

$$\dot{\lambda} f(\sigma, A) = 0 \quad f(\sigma, A) \leqslant 0 \quad \dot{\lambda} \geqslant 0 \tag{9}$$

(this inequalities hold componentwise).

3. SHAKEDOWN ANALYSIS

3.1 Load domain

The data for shakedown analysis is a prescribed domain Δ^0 in the load space, which contains any feasible load history. However, it is better to consider the correspondent domain, Δ^E, in elastic stress space, to permit us to deal with mechanical and thermal loads in an unified way. Δ^E is assumed here convex and bounded. Any interior point of polyhedron Δ^E is a convex combination of its vertex. If exists a non-linear dependence between the loads, may be necessary to discretize a function that defines the load coupling. To avoid this, it is still better to consider the total uncoupling of loads, defining for each body point a local uncoupled envelope Δ which collect the extremum values of stresses corresponding to the loads, in each body point, independently of the point in the load cycle, to which this stress corresponds. Consider the set of all the local values of elastic stresses associated to any feasible loading, i.e

$$\Delta(x) := \{\sigma^E(x) \quad | \quad \forall \sigma^E \in \Delta^E\}, \quad \forall x \in \mathcal{B} \tag{10}$$

The pointwise envelope of set Δ^E is

$$\Delta := \{\sigma \quad | \quad \sigma(x) \in \Delta(x), \quad \forall x\} \supset \Delta^E \tag{11}$$

3.2 Shakedown and limited kinematical hardening

For ideal plasticity, the theorem due to Bleich-Melan states that any load factor μ^* is safe if there exists a time-independent residual (self-equilibrated) stress field σ^r such that its superposition with any stress belonging to the amplified load domain $\mu^*\Delta$ is plastically admissible. Then, for elastic shakedown, the limit load factor μ is the supremum of all safe factors. This may be translated as an elastic shakedown equilibrium variational principle:

$$\mu := \sup_{(\mu^*, \sigma^r) \in \mathbb{R} \times \mathcal{W}'} \{\mu^* \geqslant 0 \quad | \quad \mu^*\Delta + \sigma^r \subset P, \quad \sigma^r \in S^r\} \tag{12}$$

S^r is a residual stress space i.e. stress fields in equilibrium with null loads.

Shakedown behavior of kinematic hardening material bodies was studied by (Stein et al., 1990, 1992 and 1993) using a 3D overlay-model for the material. The main idea was to approach the behavior of metals by a composite of elastic-ideally plastic micro-elements in a dense spectrum, numbered with a scalar variable $\xi \in [0, 1]$ and deforming together.

Let

$$\Phi(\boldsymbol{\sigma}) := \frac{3}{2}\|\mathbf{S}\|^2 \tag{13}$$

be the Mises yield function. Here the generalized stress deviator is denoted $\mathbf{S} = (S, A)$ where S is the deviator tensor of macroscopic stress and A is an internal thermodynamic stress like variable, named back stress.

Stein's work showed that the theorem of Melan can be stated for materials with hardening, in terms of back stress A as: If exist a load factor $m > 1$, a time independent residual stress field, $\sigma^r(x) \in S^r$ and a time-independent back stress field $A(\mathbf{x}, \xi)$ satisfying

$$\Phi(A(x, 0)) \leqslant [\sigma_Y(x) - \sigma_{Y0}(x)]^2 \tag{14}$$

such as for all possible loads in the load domain, the condition

$$\Phi(m\sigma^E(x, t) + \sigma^r(x) - A(x, 0)) \leqslant [\sigma_{Y0}(x)]^2 \tag{15}$$

is fulfilled for all body points beyond a time t, where $m > 1$ is a safety factor against non adaptation, then the total plastic energy dissipated within an arbitrary load path contained within the load domain is bounded, i.e. the elastic shakedown occurs. The material parameter σ_{Y0} is the initial yield stress and σ_Y is the ultimate stress. Is is important to notice that, the Stein's model does not depend on the hardening curve shape once only σ_{Y0} and σ_Y appears in equations. Because this fact, we could use a linear model for hardening, to simplify the calculations. The correspondent statical principle is:

$$\mu = \sup_{(\mu^*, \sigma^r, A)} \{\mu^* \geqslant 0 \mid \Phi(\mu^*\sigma^E + \sigma^r - A) \leqslant \sigma_{Y0}^2; \quad \Phi(A) \leqslant (\sigma_Y - \sigma_{Y0})^2; \quad \sigma^r \in S^r\} \tag{16}$$

From the statical principle, mixed and kinematic principles can be derived. We chose here to use a mixed principle, but the others can be also used to be discretized aiming to obtain numerical solutions. Introducing the restriction over σ^r into the objective function as a penalty we obtain the mixed principle:

$$\mu = \sup_{(\mu^*, \sigma, A)} \inf_{v} \{\mu^* + \langle \sigma, \mathcal{D}v \rangle, \mid \Phi(\mu^*\sigma^E + \sigma - A) \leqslant \sigma_{Y0}^2; \quad \Phi(A) \leqslant (\sigma_Y - \sigma_{Y0})^2\} \tag{17}$$

The yield functions corresponding to the conditions of Stein's statement are:

$$f_1(\sigma, A) = \frac{3}{2}\|S - A^{dev}\|^2 - (\sigma_{Y0})^2 \tag{18}$$

$$f_2(A) = \frac{3}{2}\|A^{dev}\|^2 - (\sigma_Y - \sigma_{Y0})^2 \tag{19}$$

Stein's model is completed by assuming associated flow rules for both the plastic strain rate d^p and the hardening flux $\dot{\beta}$. Since we have two plastic modes this is written as

$$d^p = \dot{\lambda}_1 \nabla_\sigma f_1 + \dot{\lambda}_2 \nabla_\sigma f_2 \qquad \dot{\beta} = \dot{\lambda}_1 \nabla_A f_1 + \dot{\lambda}_2 \nabla_A f_2 \tag{20}$$

where $\nabla_\sigma f_1$ is the partial gradient of $f_1(\sigma, A)$ with respect to σ, and so on. It follows, by deriving Eq.(18) and Eq.(19), that the evolution equations are

$$d^p = 3\dot{\lambda}_1(S - A) \tag{21}$$

$$\dot{\beta} = -3\dot{\lambda}_1(S - A) + 3\dot{\lambda}_2 A \tag{22}$$

together with the complementarity constraints (see e.g. Pycko and Maier, 1995), for $i = 1, 2$

$$\dot{\lambda}_i f_i(\sigma, A) = 0 \qquad f_i(\sigma, A) \leqslant 0 \qquad \dot{\lambda}_i \geqslant 0 \tag{23}$$

3.3 The discrete problem

The mixed principle presented at Eq.(17) can be discretized to obtain a numerical solution. We used here, mixed axisymmetric triangular finite elements, with internal variable, interpolations. The velocity field is interpolated quadratically, the deviatoric stress field is interpolated linearly and the hydrostatic stress and internal variable A are constant over the element. This element overcome the locking problem in axisymmetric problems. We work over the optimality conditions of the mixed principle. Firstly, introducing the approximation functions in the principle of virtual power, we compute the usual discrete strain-displacement matrix B such that the kinematic compatibility and self-equilibrium equations read now

$$d = Bv \qquad B^T\sigma^r = 0 \tag{24}$$

Next, we consider the whole set of constraints in the mixed principle for the n_{elem} elements mesh. The plastic admissibility has to be imposed in p points in each elements for each basic load n_Δ of the load domain. As the load domain Δ is convex and the stress interpolation is linear, then is necessary to enforce plastic admissibility only at the triangle vertices to assure this condition over the whole element. Thus, there are pn_{elem} points in the mesh where plastic admissibility is explicitly enforced for each basic load. This, results, for the Stein's bimodal yield surface in $m := 2pn_{\text{elem}}n_\Delta$ inequality constraints, that are enumerated using a single index $k = 1 : m$ in correspondence to (ℓ, i, j) with $\ell = 1, n_\Delta$, $i = 1 : 2$ and $j = 1 : p\, n_{\text{elem}}$.

Considering $\sum := \sum_{k=1:m}$, the optimality conditions for limited hardening with internal variables can be stated as follows:

$$B^T\sigma^r = 0 \tag{25}$$

$$\sum d^k = Bv \tag{26}$$

$$\sum \dot{\beta}^k + \dot{\beta}^A = 0 \tag{27}$$

$$\sum \sigma^k \cdot d^k = 1 \tag{28}$$

$$d^k = \dot{\lambda}^k \nabla_\sigma f^k \qquad k = 1 : m \tag{29}$$

$$\dot{\beta}^k = \dot{\lambda}^k \nabla_A f^k \qquad k = 1 : m \tag{30}$$

$$\dot{\beta}^A = \dot{\lambda}^A \nabla_A f^A \tag{31}$$

$$\dot{\lambda}^k f^k = 0 \qquad k = 1 : m \tag{32}$$

$$\dot{\lambda}^A f^A = 0 \tag{33}$$

$$f^k := f_1(\mu\sigma^k + \sigma^r, A) \leqslant 0 \qquad k = 1 : m \tag{34}$$

$$f^A := f_2(A) \leqslant 0 \tag{35}$$

$$\dot{\lambda}^k \geqslant 0 \qquad k = 1 : m \tag{36}$$

$$\dot{\lambda}^A \geqslant 0 \tag{37}$$

To solve the shakedown problems one needs to find:

$$\{v, \sigma^r, A, \mu, \dot\lambda^k, \dot\lambda^A\} \tag{38}$$

Nery (2007) extended the algorithm developed by Zouain et al,(2002) for shakedown analysis with elastic ideally-plastic materials, to dealing also with limited kinematic hardening. The internal variable A was considered together with residual stress in a single vector, but not constrained to be residual. The discrete deformation operator B was constructed to have null elements in the positions corresponding to internal variable components. The new obtained vectors were:

$$\boldsymbol{\sigma}^r = (\sigma^r, A) \quad \boldsymbol{d}^k = (d^k, \dot\beta^k) \quad \boldsymbol{\sigma}^k = (\sigma^k, 0) \quad \dot{\boldsymbol{\lambda}}^k = (\dot\lambda^k, \dot\lambda^A) \tag{39}$$

With this definitions, the optimality conditions are written:

$$B^T \boldsymbol{\sigma}^r = 0 \tag{40}$$

$$\sum \dot{\boldsymbol{\lambda}}^k \nabla_{\boldsymbol{\sigma}} f^k = Bv \tag{41}$$

$$\sum \boldsymbol{\sigma}^k \cdot \dot{\boldsymbol{\lambda}}^k \nabla_{\boldsymbol{\sigma}} f^k = 1 \tag{42}$$

$$\dot{\boldsymbol{\lambda}}^k f^k = 0 \qquad k = 1:m \tag{43}$$

$$f^k := f_{S1}(\mu \boldsymbol{\sigma}^k + \boldsymbol{\sigma^r}) \leqslant 0 \qquad k = 1:m \tag{44}$$

$$f^A := f_{S2}(A) \leqslant 0 \tag{45}$$

$$\dot{\boldsymbol{\lambda}}^k \geqslant 0 \qquad k = 1:m \tag{46}$$

The above system of nonlinear equations and inequalities is solved then by using the algorithm described in Zouain et al. (2002) and the numerical solution is obtained.

4. APPLICATION-THICK WALL SPHERE UNDER VARIABLE PRESSURE AND THERMAL LOADS

Let us to consider a thick wall sphere subjected to variable internal pressure and thermal loads. This problem was considered by Yan (1999) in his doctorate thesis for an elastic ideally-plastic material. There is shakedown analytical solution for this problem. Will be considered a numerical application of the algorithm developed in the previous sections to validate it in the case of perfect plasticity and after the analysis will be extend to evaluate the effect of to consider the limited kinematic hardening.

Due to central symmetry, two stress components (in the two meridional planes) are equal and the Tresca yield condition used in Yan's work and the Mises yield condition (used here) coincide as showed by Lubliner (1990), p.194. The sphere material have the Young modulus $E = 210000MPa$, the Poisson coefficient $\nu = 0.3$ and the thermal expansion coefficient $\alpha = 1 \times 10^{-5}/C^o$.

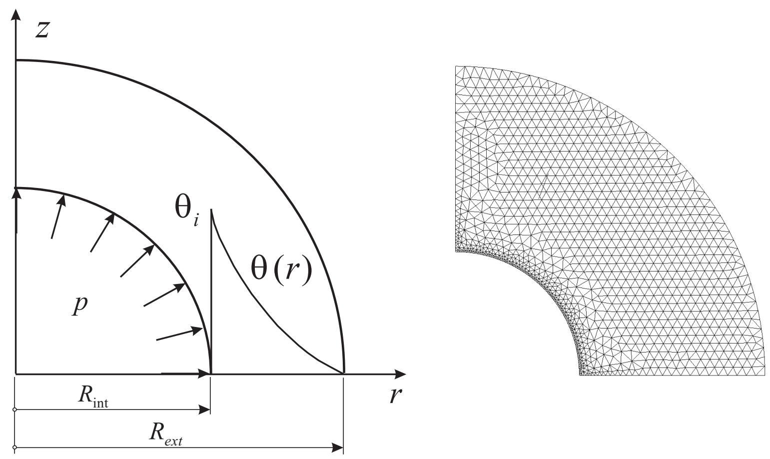

Figure 1. *Temperature distribution through the sphere wall and the 2195 axisymmetric finite elements mesh.*

The following geometric non-dimensional parameter is defined:

$$\ell := \frac{R_{ext}}{R_{int}} \tag{47}$$

Following Yan's work, we will consider here a thick wall sphere with $R_{ext} = 1.0$ m and $R_{int} = 0.4$ m. Noting θ_{int} as the temperature in the internal sphere wall face and $\theta_{ext} = 0$ as the temperature in the external wall face, the temperature profile to be considered through the sphere wall is:

$$\theta(r) = \theta_i \frac{(R_{ext}/r) - 1}{(\ell - 1)} \tag{48}$$

For the sphere, the following reference loading parameters are defined:

$$p_f := \frac{4\sigma_Y}{3}\left(1 - \frac{1}{\ell^3}\right), \tag{49}$$

$$\theta_f := \frac{4(1-\nu)\sigma_Y}{E\alpha_\theta}\frac{\ell^2 + \ell + 1}{\ell + 2\ell^2} \tag{50}$$

Variable loads are defined as $p = \alpha\overline{p}$ and $\theta = \alpha\overline{\theta}$, with $0 \leq \alpha \leq 1$, where $\overline{p}$ and $\overline{\theta}$ are the extremum values that can be reached by pressure p and temperature θ respectively. In the Yan (1999) work, two load cases were considered:

4.1 Constant internal pressure and variable temperature

In this case, the load domain in the pressure-temperature space will have two vertices:

$$V(1) = (\overline{p}, 0) \qquad and \qquad V(2) = (\overline{p}, \overline{\theta}) \tag{51}$$

4.2 Variable internal pressure and temperature loads

The load domain in this case have four vertices:

$$V(1) = (0,0), \qquad V(2) = (\overline{p}, 0), \qquad V(3) = (\overline{p}, \overline{\theta}), \qquad V(4) = (0, \overline{\theta}) \tag{52}$$

4.3 Results

The figure 2 shows the comparison among the numerical values obtained here, the numerical values obtained by Yan (1999) and the analytical values. When p varies independently of q, one can notice a marked reduction in the shakedown domain.

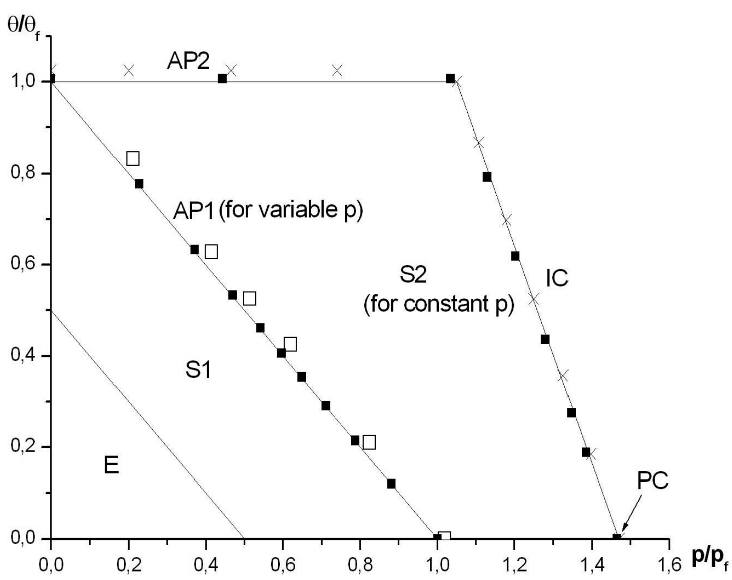

Figure 2. *Comparison among numerical values obtained in this work (black squares) with those obtained by Yan (noted by X and by white squares) and with the analytical solutions (straight lines) for constant and variable* p*, for elastic ideally-plastic material and* $\ell = 2.5$*. The substantial reduction in the shakedown domain for variable* p *can be observed.*

The numerical results obtained by Yan showed good agreement with the analytical ones. Our numerical results, presents good agreement with both, except along the straight line which divide the shakedown domain for constant p and the incremental collapse domain.

The interaction diagram can be subdivided in sub domains displaying the different collapse mechanisms and the shakedown domain. The following sub-domains can be distinguished:

a) Entirely elastic behavior region noted by E.
b) Shakedown occurs at the region S1 if the internal pressure varies independently of the temperature variation.
c) Shakedown occurs at the region S2 if the internal pressure stays constant when the temperature varies.
d) Varying the temperature, the alternate plasticity occurs at the region AP2 if p is constant or AP1 if p varies.
e) Incremental collapse occurs at the region IC.
f) The plastic collapse occurs at the point noted by PC.

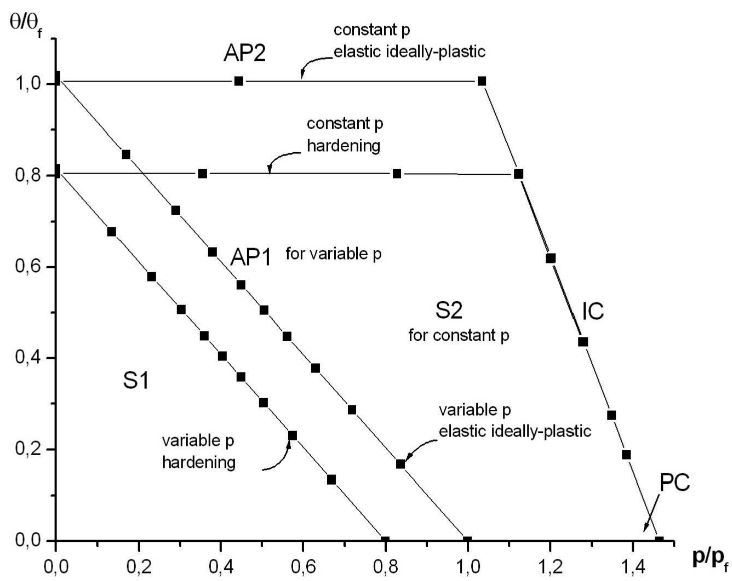

Figure 3. *Hardening effect displayed in the sphere interaction diagram for* $\ell = 2.5$. *The numerical values are indicated by black squares. The lines are not analytical solutions but interpolation of the numerical values, plotted only to delimit the different domains. The elastic domain is not showed.*

In Fig. 3 with the same notation of the Fig. 2, the reduction of the shakedown domains due to limited kinematic hardening is showed, for both cases, p constant and p variable, when temperature varies. One can notice that the plastic collapse load is not affected by the hardening existence. This indicate that, using an elastic ideally-plastic material model, the yield stress to be considered should be σ_Y and not σ_{Y0}. But, in spite of this being the appropriate consideration for elastic ideally-plastic model, it doesn't guarantee the safety related to the alternate plasticity. That fact shows the importance of considering hardening, instead to use an elastic ideally-plastic material.

5. CONCLUDING REMARKS

In this paper, a numerical procedure for shakedown analysis for materials presenting limited kinematic hardening was considered. The algorithm developed by (Zouain et al., 2002, Zouain, 2004) to deal with the shakedown analysis of an elastic ideally-plastic structures was adapted to be used jointly with the overlay-model developed by (Stein et al., 1990, 1992 and 1993) to consider limited kinematic hardening. The results obtained by the use of this extended algorithm, exhibited a good matching with previous analytical and numerical results obtained for elastic ideally-plastic materials by Yan. The results for limited kinematic hardening materials was obtained and plotted together to show the influence of hardening in this axisymmetric example. Considering the unicity of collapse load, independently of the material be elastic ideally-plastic or with limited kinematic hardening, the importance of alternate plasticity that reduces the secure domain, was shown.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

Halphen, B. and Nguyen, Q. S., 1975, "Sur Les Matériaux Standard Généralisés", Journal de Méchanique Théorique et Appliquée, Vol. 14, pp. 1-37.

Koiter, W. T., 1960, "General Theorems for Elastic-Plastic Solids",In I. N. Sneddon and R. Hill (Eds.), Progress in Solid Mechanics, Vol. 1, pp. 165-221, North-Holland, Amsterdan.

Lemaitre, J. and Chaboche, J. L., 1990, "Mechanics of Solid Materials", Cambridge University Press.

Lubliner, J. , 1990, "Plasticity Theory", Macmillan, New York.

Nery, D. E. S., 2007, "Estados Limites de Componentes Mecânicos Considerando Encruamento Cinemático Limitado", D.Sc. Thesis, PEM/COPPE/UFRJ, Rio de Janeiro, Brazil.

Nguyen, Q. S., 2000, Stability and Nonlinear Solid Mechanics, Willey.

Polizzotto, C., Borino, G., Caddemi, S. and Fuschi, P., 1991, "Shakedown Problems for Material Models with Internal Variables", European Journal of Mechanics A/Solids, Vol.10, No. 6, pp. 621-639.

Pycko, S. and Maier, G., 1995, "Shakedown Theorems for Some Classes of Nonassociative Hardening Elastic-Plastic Material Models", International Journal of Plasticity, Vol.11, No. 4, pp. 367-395.

Staat, M., 2005, "Local and Global Collapse Pressure of Longitudinal Flawed Pipes and Cylindrical Vessels", International Journal of Pressure Vessel and Piping, Vol.82, pp. 217-225.

Staat, M., Schwartz, M. and Heitzer, M., 2003, "Design by Analysis of Pressure Components by Nonlinear Optimization", In J. L. Zeman (Eds.), Proceedings of the Pressure Vessel Technology 2003- icpvt10, july, pp.59-65.

Stein, E., Zhang, G. and Huang, Y., 1993, "Modeling and Computation of Shakedown Problems for Nonlinear Hardening Materials", Computer Methods in Applied Mechanics and Engineering, Vol.103, pp. 247-272.

Stein, E., Zhang,G. and König, J. A., 1990, "Micromechanical Modelling and Computation of Shakedown with Nonlinear Kinematic Hardening Including Examples for 2-D Problems", In D. Axelrad and W. Muschik (Eds), Recent Developments of Micromechanics.

Stein, E., Zhang, G. and König, J. A., 1992, "Shakedown with Nonlinear Strain-hardening Including Structural Computation Using Finite Element Method", International Journal of Plasticity, Vol.8, pp. 1-31.

Yan, A.M., 1999,"Contribution to the Direct Limit State Analysis of Plastified and Cracked Structures", Thèse de Doctorat, Faculté de Sciences Apliqées, Université de Liège, Belgique.

Zeman, J. L. , 1996, "Some Aspects of the Work of the European Working Groups Relating to Basic Pressure Vessel Design", International Journal of Pressure Vessel and Piping, Vol. 70, No. 1, pp. 3-10.

Zouain, N., 2004, "Shakedown and Safety Assessment", In E. Stein, R. de Borst and T. J. R. Hughes (Eds.), Encyclopedia of Computational Mechanics, John Willey, pp.291-334.

Zouain, N., Borges L. A. and Silveira, J. L., 2002, "An Algorithm for Shakedown Analysis with Nonlinear Yield Functions", Computer Methods in Applied Mechanics and Engineering,Vol.191, No. 23-24,pp. 2463-2481.

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