VIBRATION OF AN EULER-BERNOULLI STEPPED BEAM ON ELASTIC END SUPPORTS

José Juliano de Lima Junior, limajr@unifei.edu.br
Janaína da Cunha Vaz, jana_cvaz@yahoo.com.br
Mechanical Engineering Institute, Federal University of Itajubá, Itajubá, MG, 37500-903, Brazil

Abstract. *A analysis of stepped beam on elastic end supports has been investigated by several authors due to its importance in the structural engineering fields, such as active structures, structural elements with integrated piezoelectric materials, components of shaft-disc systems, turbines and fans blades and many others structural configurations. In This paper a mathematical modeling of a stepped beam on elastic end supports based on the Euler-Bernoulli beam theory is proposed. Compared to the bibliography on the transverse vibration of Euler-Bernoulli beams with one step change in cross-section, publications on beams with more than one step changes is not extensive. The natural frequencies and mode shapes of a stepped beam are discussed and also compared to each other. Combinations of the classical clamped, pinned, sliding, and free types of elastic end supports are considered. The first three frequency parameters of beams with two step changes in cross-section are tabulated for selected sets of system parameters and types of end supports. The method proposed may be extended to beams with any number of step changes in cross-section.*

Keywords: *: vibration, stepped beam, natural frequency, mode shape*

1. INTRODUCTION

Brief reviews of selected publications on transverse vibration beams with changes in cross-sections follow. Taleb and Suppiger (1961) and Levinson (1976) derived the frequency equation for a simply supported stepped beam. Heidebrecht (1967) has shown numerical method to calculated the first natural frequency of the simply-supported beams. Jang and Bert (1989a) and Jang and Bert (1989b) were the first to derive the frequency equations as fourth order determinants equated to zero, for combinations of the classical clamped, pinned and free end supports. Vibration analysis of the stepped beam with one step cross-section change subject to the constraining effect of rotational and translational springs at both ends was presented by Maurizi and Bellés (1993). De Rosa (1994) studied the vibration of a beam with one step change in cross-section with elastic supports at the ends. The frequency equations of Euler-Bernoulli beam with up to three step changes in cross-section and on classical and/or elastic supports were expressed as fourth order determinants equated to zero by Neguleswaran (2002). They tabulated the first three frequency of three types of beams.

Dong (2005) presented a scheme to calculate the laminated composite beam's flexural rigidity and transverse shearing rigidity based on first order shear deformation theory. A stepped beam model was then developed using Timoshenko beam theory to predict analytically the natural frequencies and mode shapes of a stepped laminated composite beam. Modal analysis with piezoelectric materials bonded on beam surface, i. e., stepped piezoelectric beams, was validated by Maurini et al. (2006). They have used Euler-Bernoulli model from finite element analysis and experimental procedures validated the results.

In the present paper the transverse vibration of an beam with changes in cross-section was studied. firstly an beam by Timoshenko beam theory is modeled without to consider changes in cross-sections. After considerations the Euler-Bernoulli model from Timoshenko beam model is obtained. Now is possible to consider changes in beam cross-section using Euler-Bernoulli model. Natural frequencies and mode shapes of a stepped beam are discussed and also compared to each other. Combinations of the classical clamped, pinned, sliding, and free types of elastic end supports are considered. The first three frequency parameters of beams with two step changes in cross-section are tabulated for selected sets of system parameters and types of end supports. The method proposed may be extended to beams with any number of step changes in cross-section.

2. TRANSVERSE VIBRATION BEAM

The modeling of a transverse vibration beam includes the effects of shear distortion and bending moment, knowledged as Timoshenko beam model is derived (Benaroya, 2004).

2.1 Derivation of the beam's equation

The equation governing the transverse vibration of a beam of length L is derived, with the following properties at section x: $A(x)$ is the cross-sectional area, $I(x)$ is the moment of inertia, and $\rho(x)$ is the mass per unit volume. Assume small deflection $w(x,t)$ and rotation $\partial w/\partial x$, and include the bending $M(x,t)$ and shear $Q(x,t)$ effects.

Consider a free body of a section of length dx as shown in Fig. 1. Its slope is due to a bending component ϕ and to a

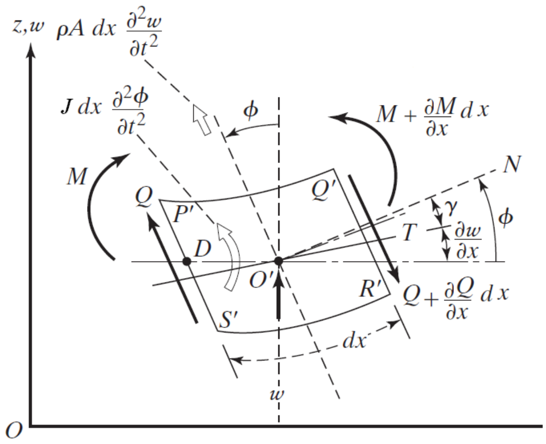

Figure 1. Timoshenko Beam Element.

shear distortion component γ

$$\frac{\partial w}{\partial x} = \phi(x,t) + \gamma(x,t) \tag{1}$$

From elementary beam theory, the bending moment is related to the slope by

$$M = EI\frac{\partial \phi}{\partial x} \tag{2}$$

and the shear is related to the slope by

$$Q = \gamma s A(x) G \tag{3}$$

where $sA(x)$ is called the reduced section. For exemple $s = 5/6$ for a plane rectangular cross-section.

Hamilton's principle is used to derive the boundary value problem and proceed to derive the kinetic and potencial energies, and their variations. The kinetic energy due to translation and rotation for the whole beam is given by

$$T(t) = \frac{1}{2}\int_0^L \rho(x)A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L J(x)\left(\frac{\partial \phi}{\partial t}\right)^2 dx \tag{4}$$

where $J(x)$ is the mass moment of inertia per unit length about the neutral bending axis.

Using the chain rule, the variation in kinetic energy δT is

$$\delta T(t) = \int_0^L \rho(x)A(x)\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) dx + \int_0^L J(x)\frac{\partial \phi}{\partial t}\delta\left(\frac{\partial \phi}{\partial t}\right) dx \tag{5}$$

The partition the work into a conservative parte that is equal to the change in potential and strain energy, and a non-conservative parte that includes the work done by external forces $f(x,t)$.

$$\begin{aligned}\delta W(t) &= \delta W_c(t) + \delta W_{nc}(t)\\ &= -\delta U(t) + \int_0^L f(x,t)\delta w(x,t)dx\end{aligned} \tag{6}$$

The change in potential energy equal the work done by the conservative actions due to the moment and the shear force

$$U(t) = \frac{1}{2}\int_0^L M(x,t)\frac{\partial \phi}{\partial x}dx + \frac{1}{2}\int_0^L \gamma Q(x,t)dx \tag{7}$$

$$= \frac{1}{2}\int_0^L EI(x)\left(\frac{\partial \phi}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L \gamma^2 sA(x)Gdx \tag{8}$$

The variation $\delta U(t)$ is given by

$$\delta U(t) = \int_0^L EI(x)\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right)dx + \int_0^L sA(x)G\gamma\delta\gamma dx \tag{9}$$

where the number of variables is reduced next by substituting

$$\delta\gamma = \delta\left(\frac{\partial w}{\partial x} - \phi\right) \tag{10}$$

Introduce Eq. 5 and 6 into Hamilton's variational principle

$$\begin{aligned} &\int_{t_1}^{t_2}\left\{\left[\int_0^L \rho(x)A(x)\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right)dx + \int_0^L J(x)\frac{\partial \phi}{\partial t}\delta\left(\frac{\partial \phi}{\partial t}\right)dx\right]\right. \\ - &\left[\int_0^L EI(x)\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right)dx + \int_0^L sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right)\delta\left(\frac{\partial w}{\partial x} - \phi\right)dx\right] \\ + &\left.\int_0^L f(x,t)\delta w(x,t)dx\right\}dt = 0 \end{aligned} \tag{11}$$

where the variation δ operates on the function that immediately follows.

Perform the usual interchanges and integration by parts and combine terms to find

$$\begin{aligned} &\int_{t_1}^{t_2}\left[\int_0^L \left\{\frac{\partial}{\partial x}\left[sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right)\right] - \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} + f(x,t)\right\}\delta w dx\right. \\ + &\int_0^L \left\{\frac{\partial}{\partial x}\left(EI(x)\frac{\partial \phi}{\partial x}\right) + sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right) - J(x)\frac{\partial^2 \phi}{\partial t^2}\right\}\delta\phi dx \\ - &\left.\left(EI(x)\frac{\partial \phi}{\partial x}\right)\delta\phi\Big|_0^L - \left[sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right)\right]\delta w\Big|_0^L\right]dt = 0 \end{aligned} \tag{12}$$

To proceed note that $\delta\phi$ and δw are arbitrary for $0 < x < L$. It then follows that the governing equations of motion for the vibration of a Timoshenko beam are

$$\frac{\partial}{\partial x}\left[sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right)\right] - \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} + f(x,t) = 0 \tag{13}$$

$$\frac{\partial}{\partial x}\left(EI(x)\frac{\partial \phi}{\partial x}\right) + sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right) - J(x)\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{14}$$

with possible boundary conditions defined by

$$\left[EI(x)\frac{\partial \phi}{\partial x}\right]\delta\phi\Big|_0^L = 0 \tag{15}$$

$$\left[sA(x)G\left(\frac{\partial w}{\partial x} - \phi\right)\right]\delta w\Big|_0^L = 0 \tag{16}$$

It is possible to eliminate ϕ in combining Eq. 13 and Eq. 14, leading to an equation governing $w(x,t)$. Now considering the uniforme beam, where $A(x) = A$, $I(x) = I$, and $\rho(x) = \rho$ and after a bit of algebra, find the governing equation to be

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \left(J + \frac{EI\rho A}{sAG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{J^2}{sAG}\frac{\partial^4 w}{\partial t^4} = f(x,t) + \frac{J}{sAG}\frac{\partial^2 f}{\partial t^2} - \frac{EI}{sAG}\frac{\partial^2 f}{\partial x^2} \tag{17}$$

2.2 Simplified eigenvalue problem

In order to be able to analytically tackle the eigenvalue problem, some reasonable simplifying assumptions are needed. When the cross-sectional dimensions are much smaller than the length the shear distortion effect and the rotary inertia effect are reasonably neglected. Also, for eigenvalue problem, external forces $f(x,t)$ are set equal zero.

Therefore, Eq. 13 and Eq. 14 become

$$\frac{\partial}{\partial x}\left[sA(x)G\left(\frac{\partial w}{\partial x}-\phi\right)\right]-\rho(x)A(x)\frac{\partial^2 w}{\partial t^2}=0 \tag{18}$$

$$\frac{\partial}{\partial x}\left(EI(x)\frac{\partial \phi}{\partial x}\right)+sA(x)G\left(\frac{\partial w}{\partial x}-\phi\right)=0 \tag{19}$$

Solve Eq. 19 for $sA(x)G(\partial w/\partial x-\phi)$, and substitute this into Eq. 18. Due to the above assumption of no shear distortion

$$\frac{\partial w}{\partial x}=\phi(x,t)+\gamma(x,t) \quad \Rightarrow \quad \frac{\partial \phi}{\partial x}=\frac{\partial^2 w}{\partial x^2} \quad to \quad \gamma(x,t)=0 \tag{20}$$

The resulting governing equation for $w(x,t)$ is the Euler-Bernoulli beam with variable properties

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]=-\rho(x)A(x)\frac{\partial^2 w}{\partial t^2} \tag{21}$$

or the uniform Euler-Bernoulli beam

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{22}$$

3. MATHEMATICAL FORMULATION

3.1 General solution

For a stepped beam with n different cross-sections and assuming normal modes of vibration with circular frequency ω, the expression for the mode shape $X_i(x_i)$ can be obtained as (Neguleswaran, 2002):

$$c_i^2\frac{\partial^4 w_i(x_i,t)}{\partial x_i^4}+\frac{\partial^2 w_i(x_i,t)}{\partial t^2}=0 \tag{23}$$

where

$$c_i=\sqrt{\frac{EI_i}{\rho A_i}}, \quad i=1,2,\ldots,n \tag{24}$$

Assume the product solution to ith cross-section

$$w_i(x_i,t)=X_i(x_i)T(t) \tag{25}$$

differentiate and substitute this solution into the governing equation, $X_i(x_i)$ is governed by

$$X_i^{iv}(x_i)-\left(\frac{\omega}{c_i}\right)^2 X_i(x_i)=0 \tag{26}$$

by definition

$$\beta_i^4=\frac{\omega^2}{c_i^2}=\frac{\rho A_i\omega^2}{EI_i} \tag{27}$$

The general solution of Eq. 26 is

$$X_i(x_i)=B_{b_1}\sin\beta_i x_i+B_{b_2}\cos\beta_i x_i+B_{b_3}\sinh\beta_i x_i+B_{b_4}\cos\beta_i x_i \quad 0\le x_i\le L_i \tag{28}$$

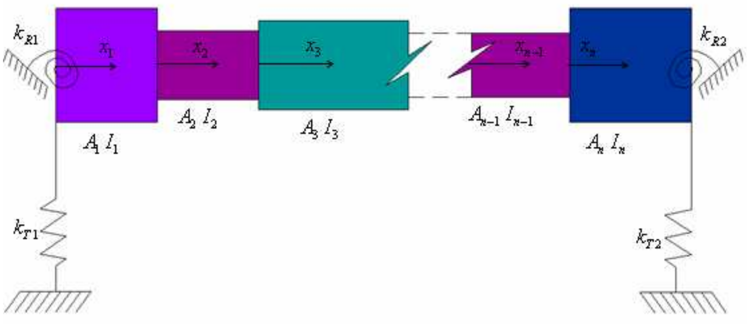

Figure 2. Stepped Beam.

3.2 Boundary conditions

The eigenvalue problem must be solved for a particular set of boundary conditions, resulting in expressions for the eigenfunctions $X_i(x_i)$ and frequencies ω which the structure can accommodate in free vibration. The boundary conditions for the structural system under cosideration, Fig. 2 are as follows:

at $x_1 = 0$,

- bending moment

$$EI_1 \frac{d^2 X_1(x_1)}{dx_1^2}\bigg|_{x_1=0} = k_{R1} \frac{dX_1(x_1)}{dx_1}\bigg|_{x_1=0} \tag{29}$$

- shear force

$$EI_1 \frac{d^3 X_1(x_1)}{dx_1^3}\bigg|_{x_1=0} = -\, k_{T1} X_1(x_1)|_{x_1=0} \tag{30}$$

at $x_n = L_n$,

- bending moment

$$EI_n \frac{d^2 X_n(x_n)}{dx_n^2}\bigg|_{x_n=L_n} = -\, k_{R2} \frac{dX_n(x_n)}{dx_n}\bigg|_{x_n=L_n} \tag{31}$$

- shear force

$$EI_n \frac{d^3 X_n(x_n)}{dx_n^3}\bigg|_{x_n=L_n} = k_{T2} X_n(x_n)|_{x_n=L_n} \tag{32}$$

The continuity conditions at the junction are

- displacement

$$X_{p-1}(x_{p-1})|_{x_{p-1}=L_{p-1}} = X_p(x_p)|_{x_p=0}\,, \quad p = 2, \ldots, n \tag{33}$$

- rotation

$$\frac{dX_{p-1}(x_{p-1})}{dx_{p-1}}\bigg|_{x_{p-1}=L_{p-1}} = \frac{dX_p(x_p)}{dx_p}\bigg|_{x_p=0} \tag{34}$$

- bending moment

$$I_{p-1} \frac{d^2 X_{p-1}(x_{p-1})}{dx_{p-1}^2}\bigg|_{x_{p-1}=L_{p-1}} = I_p \frac{d^2 X_p(x_p)}{dx_p^2}\bigg|_{x_p=0} \tag{35}$$

- shear force

$$I_{p-1}\frac{d^3X_{p-1}(x_{p-1})}{dx_{p-1}^3}\bigg|_{x_{p-1}=L_{p-1}} = I_p\frac{d^3X_p(x_p)}{dx_p^3}\bigg|_{x_p=0} \tag{36}$$

Application of the boundary conditions, Eq. 29 to Eq. 36, to the solution function Eq. 28 leads to a system of homogeneous algebraic equations in the unknowns B_{Bi}. In order to have a non-trivial solution, the determinant of the coefficient matrix must vanish identically.

4. Numerical Results

As exemples, results from two different stepped beams, one with two cross-sections and other with three cross-sections at both elastic end supports are presented. Numerical results for the first and the first three natural frequencies, for different end support was compared to available literature. Numerical results presented in this paper agree with others author and show the validity of the approach.

4.1 Beam with two cross-sections

Table 1 and Tab. 2 show the first non-dimensional natural frequency of stepped beam, $\widehat{\beta}_{1,1}$. They were calculated assuming a stepped beam with lengths $L_1 = L_2 = L/2$, and different moments of inertia ratio, starting with $\bar{I}_1 = 0.1$, and finishing with $\bar{I}_1 = 10$. The moment of inertia ratio is the relation $\bar{I}_1 = I_2/I_1$ where I_1 is the moment of inertia of the first beam cross-section and I_2 is the momento of inertia of the second beam cross-section. It is possible to note that when $\bar{I}_1 = 1$ the both beam cross-section are equal in size and in this case the beam is continuous.

The indexes used in $\widehat{\beta}_{1,k}$ indicate that 1 is related to the first step beam and k is different natural frequencies.

$$\widehat{\beta}_{1,k} = \beta_{1,k}L \tag{37}$$

Here the non-dimensional natural frequencies $\widehat{\beta}_{1,k}$ and the natural frequencies ω_n are related to Eq. 38

$$\omega_n = \left(\frac{\widehat{\beta}_{1,k}}{L}\right)^2\sqrt{\frac{EI_i}{\rho A_i}} \tag{38}$$

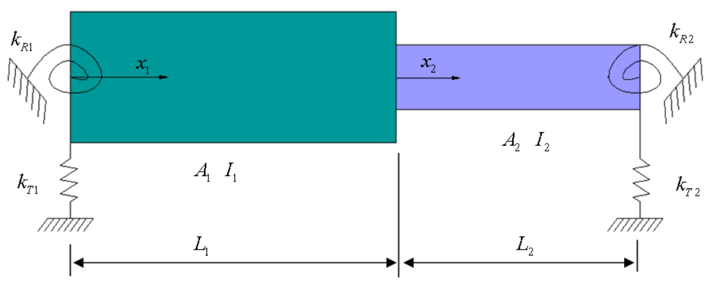

Figure 3. Beam with two cross-sections.

Table 1. First non-dimensional natural frequency of a stepped beam: on elastic end/free supports ($R_2 = T_2 = \infty$)

End supports	$R_1 = T_1$	$R_2 = T_2$	$\hat{\beta}_{1,1}$		
			$\bar{I}_1 = 0.1$	$\bar{I}_1 = 1$	$\bar{I}_1 = 10$
Free-free	∞	∞	0	0	0
—	500	∞	0.34821	0.29263	0.22976
—	5	∞	1.09088	0.91389	0.71583
—	0.05	∞	2.17505	1.81072	1.3883
Clamped-free	0	∞	2.23551	1.87510	1.43628

Table 2. First non-dimensional natural frequency of a stepped beam clamped ($R_1 = T_1 = 0$)/elastic end supports.

support	$R_1 = T_1$	$R_2 = T_2$	$\hat{\beta}_{1,1}$				
			$I_1 = 0.1$	$I_1 = 0.5$	$I_1 = 1$	$I_1 = 5$	$I_1 = 10$
Clamped-free	0	∞	2.23551	2.00987	1.87510	1.56119	1.43628
—	0	500	2.23663	2.01208	1.87866	1.57413	1.45941
—	0	50	2.24656	2.03147	1.90954	1.67694	1.62735
—	0	5	2.33168	2.1873	2.13952	2.20142	2.29838
—	0	0.5	2.70056	2.77289	2.87787	3.24695	3.40801
—	0	0.05	3.42543	3.83194	4.0691	4.56259	4.74954
—	0	0.005	3.88099	4.42004	4.65386	5.03687	5.20451
Clamped-clamped	0	0	3.94537	4.50112	4.73004	5.09501	5.26124

4.2 Beam with three cross-sections

The first three non-dimensional natural beam's frequencies with three cross-section are showed in Tab. 3 to Tab. 5. The beam lengths are $L_1 = 0.200$ (m), $L_2 = 0.300$ (m), and $L_3 = 0.500$ (m). The main dimensions related to cross-section depends on beam type, that is, type 1,and type 2 are retangular cross-section beam, and type 3 is circular cross-section beam, as follows:

- for type 1: $b_1 = 0.005$ (m), $b_2 = 0.006$ (m), and $b_3 = 0.009$ (m)
- for type 2: $h_1 = 0.005$ (m), $h_2 = 0.006$ (m), and $h_3 = 0.009$ (m)
- for type 3: $d_1 = 0.005$ (m), $d_2 = 0.006$ (m), and $d_3 = 0.009$ (m)

Table 3. First three non-dimensional frequencies of a stepped beam with three cross-section - type 1.

Classical end supports	R1	T1	R2	T2	Type 1		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
Clamped-free	0	0	∞	∞	1.66100	4.56222	7.84841
Free-free	∞	∞	∞	∞	0	4.77621	7.91134
Clamped-sliding	0	0	0	∞	2.20800	5.42355	8.62546
Clamped-pinned	0	0	∞	0	3.80416	7.04505	10.1739

Table 4. First three non-dimensional frequencies of a stepped beam with three cross-section - type 2.

Classical end supports	R1	T1	R2	T2	Type 2		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
Clamped-free	0	0	∞	∞	1.71452	5.16922	9.41405
Free-free	∞	∞	∞	∞	0	5.57601	9.51969
Clamped-sliding	0	0	0	∞	2.57846	6.32019	10.2630
Clamped-pinned	0	0	∞	0	4.39742	8.43703	11.8825

Table 6 shows the first non-dimensional frequencies considering different inertia moments, and different boundary condition, e.i. classical and non-classical boundary condition.

5. SUMMARY

In this paper, the non-dimensional natural frequencies for a transversely vibration Euler-Bernoulli beam to one and two step changes in cross-section, different cross-sections, retangular and circular, and on classical and/or elastic end supports are discussed. Although some results are present for the three types of beams, Tab. 3 to Tab. 5, the method developed is applicable to any type of step changes in cross-section with different elastic end support conditions.

Numerical results from stepped beam on elastic end supports model confirm the validity of the approach.

Table 5. First three non-dimensional frequencies of a stepped beam with three cross-section - type 3.

Classical end supports	R1	T1	R2	T2	Type 3		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
Clamped-free	0	0	∞	∞	1.50383	4.93207	9.42708
Free-free	∞	∞	∞	∞	0	5.54988	9.59866
Clamped-sliding	0	0	0	∞	2.42957	6.28097	10.1805
Clamped-pinned	0	0	∞	0	4.18529	8.50978	11.7550

Table 6. First non-dimensional natural frequency of a stepped beam on elastic end supports.

End Support	R1=T1	R2=T2	$\hat{\beta}_{1,1}$		
			$I_1 = I_2 = 0.5$	$I_1 = I_2 = 1$	$I_1 = I_2 = 2$
Free-free	∞	∞	4.36930	4.73004	5.13886
—-	500	500	0.52586	0.53256	0.58453
—-	5	5	1.61745	1.65950	1.81846
—-	0.05	0.05	3.64443	4.02777	4.47737
Clamped-clamped	0	0	4.36930	4.73004	5.13886

6. ACKNOWLEDGEMENTS

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