A COMPARISON BETWEEN TWO DIFFERENT PLATE THEORIES APPLIED IN THE CASE STUDY OF AN ORTHOTROPIC PLATE

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Abstract. *This article aims at evaluating and comparing both deformation and stress results in a square plate made of an orthotropic laminate material, using two different plate theories: the Classical Plate Theory (whose hypotheses are similar to those assumed in the Bernoulli-Euler beam theory) and the First Order Shear Deformation Theory, also known as the Mindlin-Reissner Plate Theory (whose hypotheses are similar to those assumed in the Timoshenko beam theory). Square plates simply supported in the four edges and subjected to a concentrated force in the center are analyzed using both formulations, for different length to thickness ratios. Some analytical results are also compared to finite element solutions showing a good agreement. The analytical solution shows that plates made of orthotropic materials have a greater influence of shear deformation than plates made of isotropic materials, mainly for those plates with small length to thickness ratio.*

Keywords: *Composite, orthotropic materials, plate theory, shear deformation*

1. INTRODUCTION

Composite laminate plates are generally made of laminas with orthotropic mechanical properties. The use of laminate composites made of polymers (e.g.: epoxy and polyethylene) reinforced with fibers (e.g.: carbon or graphite, glass and aramid) has been increasing in the aerospace, wind turbine blade and automobile structural projects (Romariz, 2008).

According to Abrate (1998) transversal shear stresses need to be considered in numerical simulations of thick plates, otherwise the numerical simulation results can be severely affected. Besides being important in the static analysis, this consideration also needs to be done in the impact analysis of laminate plates when high frequencies vibration modes are excited. In addition to that, transversal shear stresses are very important to predict delamination failure on laminate plates. For thin plates, Mendonça (2005) mentions that errors of 5%, or less, can be expected if a plate has a length to thickness ratio over 100 and the shear deformations are not considered. However, the same is not valid for plates having a small length to thickness ratio, since in this case the shear stresses play an important role in the mechanical behavior.

In this article two plate theories, based on two different kinematics assumptions about the transversal shear stress, are investigated: the Classical Plate Theory, that does not consider the influence of transversal shear stresses, and the First Order Shear Deformation Theory, that does consider the shear influence on deformation results.

The main purpose of this paper is to solve analytically a simple supported plate under a static concentrated load in order to show the dependence of the deformation results on the length to thickness ratio, and notice that plates made of orthotropic lamina materials have a greater influence of shear deformation than plates made of isotropic materials.

2. STATIC LOADING IN BEAMS

According to Abrate (1998), the maximum transverse displacement "*w*" at the center of a simply supported beam, subjected to a uniformly distributed static load, can be predicted using a higher-order theory as:

$$w = \frac{5}{24}\frac{qL^4}{EI}\left[1 + \frac{17}{40}\frac{E}{G}\frac{h^2}{L^2}\right] \tag{1}$$

where L is the beam length, h is the beam height, q is the linear pressure load, E is the Young modulus, G is the shear modulus and I is the second moment of inertia of the beam cross-section.

The second term of Eq. (1) into the brackets is the ratio between the deflection due to shear deformation and that due to bending. As it can be noticed, the effect of shear deformation becomes smaller as the ratio h/L is reduced. Yet, another important effect of shear deformation depends on the ratio between the Young modulus E and the shear modulus G. For isotropic materials $E/G = 2(1+\nu)$, and if we consider a Poisson ratio of $\nu \cong 0.3$, it results $E/G = 2.6$, which is not a high ratio. However, if we consider a typical graphite-epoxy composite (instead of an isotropic material), a unidirectional laminate with fibers oriented in the x-direction can provide $E_1/G_{12} = 25.2$. Therefore, at a first glance, we could say that the effect of shear deformation in graphite-epoxy laminate materials is almost 10 times more significant than that observed in typical isotropic materials. By similarity, this greater influence of shear deformation on displacements should also be expected in laminate plates.

3. CLASSICAL PLATE THEORY (CPT)

In this section some useful relations related to the classical plate theory (CPT) will be briefly presented. Let *u, v, w* be the displacements of the points of the plate in *x, y* and *z* directions respectively, as shown in Fig. 1. Also, let u_0, v_0, w_0 be the displacements at the middle of the plate (that is, the mid-plane displacements).

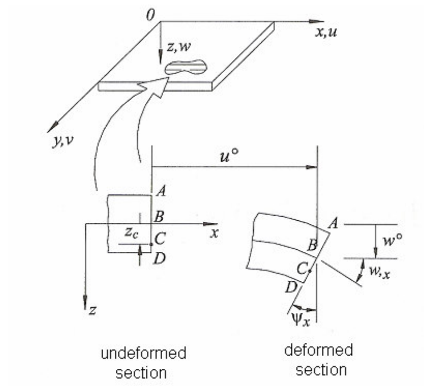


Figure 1. Plate deformation. (Mendonça, 2005)

The Kirchoff's kinematical relations are adopted, as follows:

$$\psi_x = -\frac{\partial w_0}{\partial x}\ ,\ \psi_y = -\frac{\partial w_0}{\partial y} \tag{2}$$

Therefore, the effects of shear deformations are neglected.

$$\gamma_{zx} = \gamma_{yz} = 0 \tag{3}$$

The normal stress σ_z is very small if compared to the stresses σ_x, σ_y, τ_{xy}. It leads to:

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \tag{4}$$

The deformation compatibility equations are given by:

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa\} \tag{5}$$

where $\{\kappa\}$ is the plate curvature vector and $\{\varepsilon^0\}$ is the deformation vector at the mid-plane points.

Assuming that each lamina is in a plane stress state, the constitutive equation in the lamina principal directions is given by:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{6}$$

where:

$$Q_{11} = \frac{E_1^2}{E_1 - \nu_{12}^2 E_2}\ ,\ Q_{12} = Q_{21} = \frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2}\ ,\ Q_{22} = \frac{E_1 E_2}{E_1 - \nu_{12}^2 E_2}\ ,\ Q_{66} = G_{12} \tag{7}$$

If the *X* direction is rotated θ degrees (in the clockwise direction) from the principal direction '1' of the lamina, the constitutive equations for each lamina are given by (see, e.g., Jones, (1999)):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{8}$$

where the matrix terms given in Eq. (8) are defined by:

$$\begin{aligned}
\overline{Q_{11}} &= Q_{11}k^4 + 2k^2l^2\left(Q_{12}+2Q_{66}\right) + Q_{22}l^4 \\
\overline{Q_{12}} &= \left(Q_{11}+Q_{22}-4Q_{66}\right)k^2l^2 + Q_{12}\left(k^4+l^4\right) \\
\overline{Q_{22}} &= Q_{11}l^4 + 2k^2l^2\left(Q_{12}+2Q_{66}\right) + Q_{22}k^4 \\
\overline{Q_{16}} &= \left(Q_{11}-Q_{12}\right)k^3l + \left(Q_{12}-Q_{22}\right)kl^3 - 2kl\left(k^2-l^2\right)Q_{66} \\
\overline{Q_{26}} &= \left(Q_{11}-Q_{12}\right)kl^3 + \left(Q_{12}-Q_{22}\right)k^3l + 2kl\left(k^2-l^2\right)Q_{66} \\
\overline{Q_{66}} &= \left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)k^2l^2 + Q_{66}\left(k^4+l^4\right)
\end{aligned} \tag{9}$$

where $k = \cos\theta$ e $l = \text{sen } \theta$.

The forces and moments in a differential plate element are shown in Fig. 2.

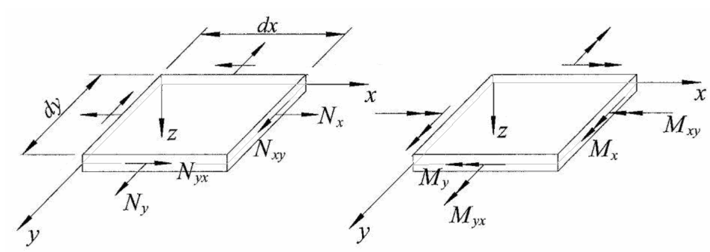


Figure 2. Forces and moments in a differential element.

$$\{N\}^T = \{N_x \quad N_y \quad N_{xy}\} \tag{10}$$

$$\{M\}^T = \{M_x \quad M_y \quad M_{xy}\} \tag{11}$$

Applying Eq. (5) and Eq. (8) in the generalized loads equations of the laminate plate gives:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\kappa\} \end{Bmatrix} \tag{12}$$

where the laminate stiffness matrix are given by:

$$[A] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left[\overline{Q}\right]_k dz = \sum_{k=1}^{n} \left[\overline{Q}\right]_k \left(z_k - z_{k-1}\right) \tag{13}$$

$$[B] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left[\overline{Q}\right]_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left[\overline{Q}\right]_k \left(z_k^2 - z_{k-1}^2\right) \tag{14}$$

$$[D] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left[\overline{Q}\right]_k z^2 dz = \frac{1}{3} \sum_{k=1}^{n} \left[\overline{Q}\right]_k \left(z_k^3 - z_{k-1}^3\right) \tag{15}$$

where n is the number of laminas, k represents each lamina, and $\overline{Q}$ is the lamina stiffness matrix from Eq. (8).

The differential equation of equilibrium gives:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = I_1 \frac{\partial^2 w_0}{\partial t^2} \tag{16}$$

where p is the pressure applied to the plate in z direction, and I_1 is the mass per unit length.

For a symmetric lay-up, $[B] = [0]$. Then, after substituting (12) into (16), the transverse displacements must satisfy a single equation of motion:

$$p = D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12} + D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4 w_0}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w_0}{\partial x \partial y^3} + I_1\frac{\partial^2 w_0}{\partial t^2} \tag{17}$$

4. FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)

In this section some basic relations of the First-Order Shear Deformation Theory (also called Mindlin-Reissner plate theory) will be briefly given. To start with, the kinematical relations adopted in this case are:

$$\psi_x = \gamma_1 - \frac{\partial w_0}{\partial x}, \quad \psi_y = \gamma_2 - \frac{\partial w_0}{\partial y} \tag{18}$$

Therefore, the two transverse shear strains are assumed constant through the thickness. The deformation compatibility equations are given by Eq. (5), besides the following equations related to the shear strains:

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_0}{\partial x} + \psi_x \\ \frac{\partial w_0}{\partial y} + \psi_y \end{Bmatrix} \tag{19}$$

The constitutive equations for each lamina are given by Eq. (8), besides the following relation related to shear stresses and shear strains:

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} G_{23} & 0 \\ 0 & G_{31} \end{bmatrix} \begin{Bmatrix} \gamma_{yx} \\ \gamma_{zx} \end{Bmatrix} \tag{20}$$

The shear forces in a differential plate element are showed in Fig. 3.

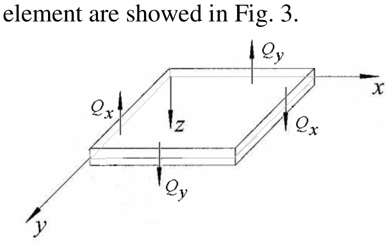


Figure 3. Shear forces in a differential element.

Applying Eq. (19) and Eq.(20) in the generalized load equations of the laminate plate, leads to:

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \frac{\partial w_0}{\partial y} + \psi_y \\ \frac{\partial w_0}{\partial x} + \psi_x \end{Bmatrix} \tag{21}$$

where:

$$A_{ij} = c_1 c_2 \sum \left[\overline{Q_{i,j}}\right]_k (z_k - z_{k-1}) \text{, for i, j = 4, 5} \tag{22}$$

Abrate (1998) suggests that the constants c_1 and c_2 be assumed equal to 5/6. These constants are the equivalent counterparts of the correction factor for the distribution of shear stress in rectangular cross-section beams proposed in the Timoshenko's beam theory.

As described in Abrate (1998), for a symmetric layup, $[B] = [0]$. Then, after substituting Eq. (12) and (21) into the differential equations of equilibrium, the following three differential equations are obtained:

$$\frac{\partial}{\partial x}\left[A_{45}\left(\psi_y + \frac{\partial w_0}{\partial y}\right) + A_{55}\left(\psi_x + \frac{\partial w_0}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[A_{44}\left(\psi_y + \frac{\partial w_0}{\partial y}\right) + A_{45}\left(\psi_x + \frac{\partial w_0}{\partial x}\right)\right] + p = I_1 \frac{\partial w_0}{\partial t} \quad (23)$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[D_{11}\frac{\partial \psi_x}{\partial x} + D_{12}\frac{\partial \psi_y}{\partial y} + D_{16}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[D_{16}\frac{\partial \psi_x}{\partial x} + D_{26}\frac{\partial \psi_y}{\partial y} + D_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\right] \\ &- A_{45}\left(\psi_y + \frac{\partial w_0}{\partial y}\right) - A_{55}\left(\psi_x + \frac{\partial w_0}{\partial x}\right) = I_3 \frac{\partial^2 \psi_x}{\partial t^2}\end{aligned} \quad (24)$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[D_{16}\frac{\partial \psi_x}{\partial x} + D_{26}\frac{\partial \psi_y}{\partial y} + D_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[D_{12}\frac{\partial \psi_x}{\partial x} + D_{22}\frac{\partial \psi_y}{\partial y} + D_{26}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\right] \\ &- A_{44}\left(\psi_y + \frac{\partial w_0}{\partial y}\right) - A_{45}\left(\psi_x + \frac{\partial w_0}{\partial x}\right) = I_3 \frac{\partial^2 \psi_y}{\partial t^2}\end{aligned} \quad (25)$$

where I_3 is the rotary inertia.

5. NUMERICAL SIMULATION

In order to compare the differences between the two approaches, a simply supported composite plate, of thickness h, and subjected to a concentrated load of 1N by patch load, as shown in Fig. 4.a, will be analyzed. The patch load length s is $2.h$. The laminate plate has 10 layers. The laminate lay up is [0, 90, 0, 90, 0, 0, 90, 0, 90, 0], as show in Fig. 4.b.

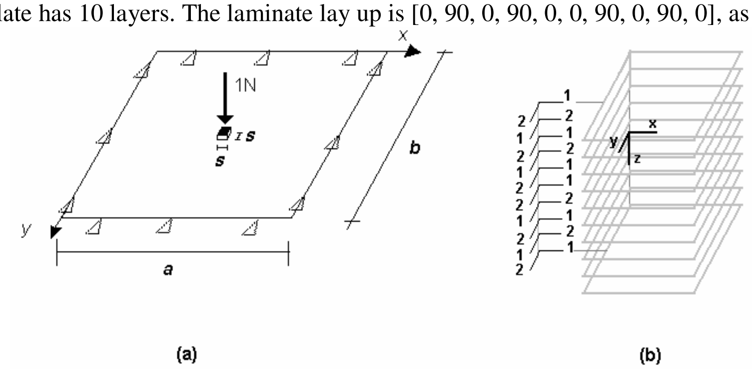


Figure 4. (a) Simply supported plate subjected to concentrated load. (b) Laminate lay-up and lamina directions.

The lamina resin is epoxy and it is reinforced with carbon fibers. The laminate properties are presented in Tab. 1.

Table 1 – Laminate properties.

Symbol	Value	Property
E_1	120.0 GPa	Elastic modulus in the lamina direction 1
E_2	7.9 Gpa	Elastic modulus in the lamina direction 2.
$G_{12}= G_{23}= G_{13}$	5.5 GPa	Shear modulus in the lamina plane 12, 23, 13
ν_{12}	0.3	Poisson coefficient 12.
t_{lamina}	0.269mm	Lamina thickness

In order to simplify the analyses, only square plates were considered. (see plate dimensions in Tab. 2).

Table 2 – Plate dimensions (*a* and *b* in meters).

a = *b*	*a/h*
0,0125	4,6
0,025	9,3
0,050	18,6
0,100	37,2
0,200	74,3

The boundary conditions are:

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \tag{26}$$

$$M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0 \tag{27}$$

5.1 Analytical solutions

Within the classical plate theory, the transverse displacements and the transverse load into double Fourier series are written by:

$$w(x,y) = \sum_{m,n}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{28}$$

$$p(x,y) = \sum_{m,n}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{29}$$

This analytical solution is called Navy solution, and it is presented in several references, e.g., Lekhnitskii (1968) and Timoshenko and Woinowsky-Krieger (1959). Using the Classical Plate Theory (CPT), substituting Eq. (28) and Eq. (29) into Eq.(17), the equations of equilibrium and the boundary conditions are satisfied when transverse displacements are given by:

$$W_{mn} = p_{mn}\frac{a^4}{\pi^4}\left[\frac{1}{D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2r^2 + D_{22}n^4r^4}\right] \tag{30}$$

where $r = a/b$.

For a uniform pressure p distributed over a square patch centered with side s, p_{mn} is given by:

$$p_{mn} = \frac{16p}{mn\pi^2}\sin\left(\frac{m\pi x_0}{a}\right)\sin\left(\frac{n\pi y_0}{b}\right)\sin\left(\frac{m\pi s}{2a}\right)\sin\left(\frac{n\pi s}{2b}\right) \tag{31}$$

Substituting the W_{mn} into Eq. (26), the displacements of the plate are calculated within the CPT theory.
Based on the derivations of the Eq. (26), the curvature vector for the plate is given by:

$$\kappa = \begin{Bmatrix} \sum\limits_{m,n}^{\infty} W_{mn}\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ \sum\limits_{m,n}^{\infty} W_{mn}\left(\frac{n\pi}{b}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ -2\sum\limits_{m,n}^{\infty} W_{mn}\frac{mn\pi^2}{ab}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \end{Bmatrix} \tag{32}$$

Now, using the FSDT, the solution is taken in the form:

$$w_0 = \sum_{m,n=1}^{\infty} W_{mn} \sin\left(m\pi \frac{x}{a}\right)\sin\left(n\pi \frac{y}{b}\right) \quad (33)$$

$$\psi_x = \sum_{m,n=1}^{\infty} X_{mn} \cos\left(m\pi \frac{x}{a}\right)\sin\left(n\pi \frac{y}{b}\right) \quad (34)$$

$$\psi_y = \sum_{m,n=1}^{\infty} Y_{mn} \sin\left(m\pi \frac{x}{a}\right)\cos\left(n\pi \frac{y}{b}\right) \quad (35)$$

Substituting Eq. (33) to (35) into Eq. (23) to (25) gives three coupled algebraic equations for each combination of m and n values:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} q_{mn} \\ 0 \\ 0 \end{Bmatrix} \quad (36)$$

where:

$$K_{11} = A_{55}\left(\frac{m\pi}{a}\right)^2 + A_{44}\left(\frac{n\pi}{b}\right)^2,\ K_{12} = \frac{m\pi}{a},\ K_{13} = A_{44}\frac{n\pi}{b},\ K_{22} = D_{11}\left(\frac{m\pi}{a}\right)^2 + D_{66}\left(\frac{n\pi}{b}\right)^2 + A_{55},$$

$$K_{23} = D_{12}\frac{mn\pi^2}{ab} + D_{66}\frac{mn\pi^2}{ab},\ K_{33} = D_{66}\left(\frac{m\pi}{a}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^2 + A_{44}$$

For each (m, n), the system of three equations and the coefficients W_{mn}, X_{mn} and Y_{mn} can be solved. The displacement can be defined substituting these terms in the Eq. (33). The derivations of the Eq. (34) and (35) can define the plate curvature:

$$\kappa = \begin{Bmatrix} \sum_{m,n=1}^{\infty} X_{mn}\frac{m\pi}{a}\sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{b}\right) \\ \sum_{m,n=1}^{\infty} Y_{mn}\frac{n\pi}{b}\sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{b}\right) \\ -\sum_{m,n=1}^{\infty} X_{mn}\frac{n\pi}{b}\cos\left(m\pi\frac{x}{a}\right)\cos\left(n\pi\frac{y}{b}\right) - \sum_{m,n=1}^{\infty} Y_{mn}\frac{m\pi}{a}\cos\left(m\pi\frac{x}{a}\right)\cos\left(n\pi\frac{y}{b}\right) \end{Bmatrix} \quad (37)$$

5.2 A Finite Element Analysis

A square plate (100mm x 100mm) was meshed using laminate plate elements with 2.0 mm length. There are 2500 elements. The 1N load was applied on the four central elements. The Fig. 5 presents the model.

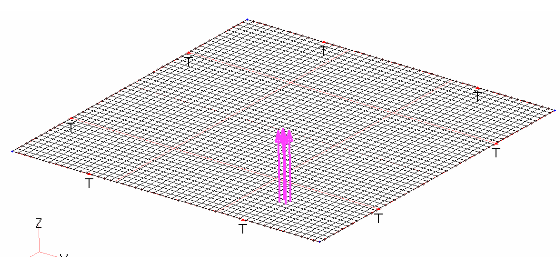


Figure 5. Laminate plate modeled with FEM,

The analysis was performed with the FE code Nastran 2005. The plate theory of this code is the FSDT.

6. RESULTS AND ANALYSIS

The center plate displacements obtained with the analytical solutions are presented in Tab. 3. The number of *m* and *n* were 499.

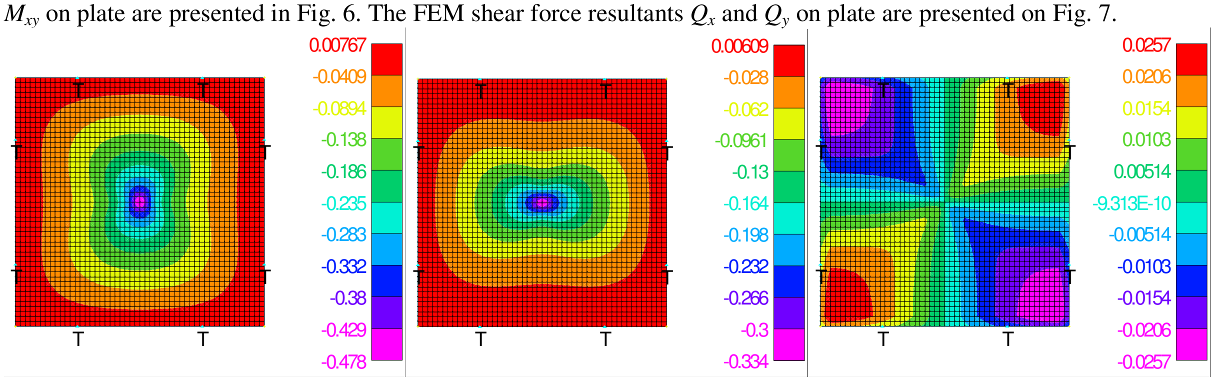
Table 3- Center plate displacement from the analytical solutions.

$a = b$	a/h	w_{FSDT}	w_{CPT}	$(w_{FSDT}/ w_{CPT} -1)$
0.0125	4.6	4.32E-08	2.27E-08	90.5%
0.025	9.3	1.38E-07	1.06E-07	30.0%
0.050	18.6	4.87E-07	4.44E-07	9.7%
0.100	37.2	1.86E-06	1.80E-06	3.0%
0.200	74.3	7.30E-06	7.23E-06	0.9%

w_{FSDT} is the displacement with FSDT.
w_{CPT} is the displacement with CPT.

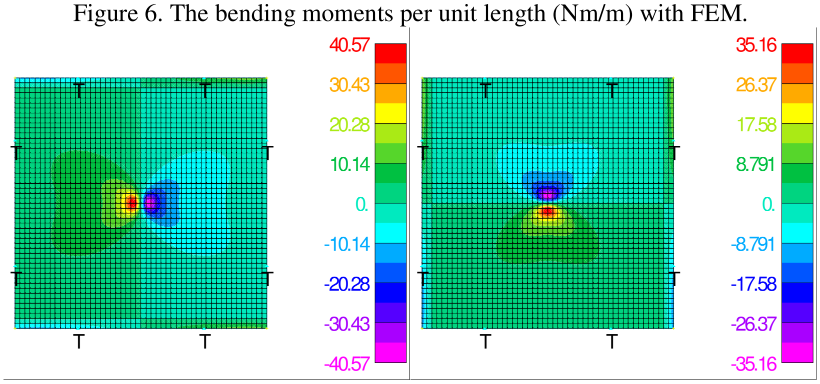
According to Tab. 3, the greater the plate side dimension, the least are the differences between w_{FSDT} and w_{CPT} . So as the ratio *a*/*h* increases, the influence of deformation due the shear stress decreases. This influence is less than 1%, when the ratio *a*/*h* is greater than 74. The influence of shear deformation for metals is less than 1% when the ratio *a*/*h* is greater than 10. It can be concluded that carbon-epoxy laminates are more sensible to shear deformation than metals. Based on Eq. (1), the greater ratio *E*/*G* of carbon epoxy (E_1/G_{12} =21.8) compared to metals (*E*/*G* =2.6) explain this structural behavior. This shear deformation influence is almost 8.4 times the metallic plate one. Then, based on Tab. 3, this laminate can be classified thin, if the ratio *a*/*h* is greater than 74. This ratio is applied to carbon-epoxy plate $[0,90,0,90,0]_s$. Other lay-up or material laminates can have different ratios *a*/*h*. It is dependent of the relation E_1/G_{12} of each lamina.

In order to validate the results obtained with the analytical analysis, a FEM analysis was made for a plate with dimensions a = b = 0.100 m. The center plate displacement was 1,85e-06 m. This result is similar to the FSDT analytical solution, confirming that the analytical solution gives good results. The FEM bending moments M_x, M_y and M_{xy} on plate are presented in Fig. 6. The FEM shear force resultants Q_x and Q_y on plate are presented on Fig. 7.



(a) M_x (b) M_y (c) M_{xy}

Figure 6. The bending moments per unit length (Nm/m) with FEM.



(a) Q_x (b) Q_y

Figure 7. The shear force resultants per length unit (N/m) with FEM.

The FEM bending moments at center plate were compared to the FSDT analytical solution on Tab. 5.

Table 5– Comparing analytic solution and FEM bending moments at center plate.

Load	FEM	Analytic - FSDT
M_x	0.478 Nm/m	0.463 Nm/m
M_y	0.334 Nm/m	0.332 Nm/m
M_{xy}	0 Nm/m	0 Nm/m

The FEM shear force resultants near the patch boundary were compared to the FSDT analytical solution on Tab. 6.

Table 6 – Comparing analytical and FEM shear force resultant at patch boundary.

Load	FEM	Analytic – FSDT
$Q_{x\ (x=a/2-s/2,\ y=b/2)}$	49.48 N/m	52.34 N/m
$Q_{y\ (x=a/2,\ y=b/2-s/2)}$	42.78 N/m	48.28 N/m

Figure 6 and Table 5 show that the M_x is greater than M_y at the center plate. As the plate is orthotropic, the laminate in direction *X* have 60% of the fibers, and in direction *Y* has 40% of the fibers. Consequently, there is a greater stiffness in *X* direction, and this direction is more loaded than *Y* direction.

Figure 7 and Table 6 show the Q_x and Q_y near the boundary patch. On this region, the shear forces are higher. It is also noticed that the load on *X* direction is higher than in *Y* direction, due the greater stiffness of the plate on *X* direction. Similar to a beam loaded at its center, the shear force resultant Q_x and Q_y is asymmetric to the symmetric planes *Y* and *X* respectively. The FEM and FSDT load results are in good agreement as we can see from Tab. 5 and Tab. 6. The small differences may be attributed to the form as the load is applied to the plate: concentrated forces are applied in the nodes that represent the patch load in the finite element model, whereas a distributed load is considered in the analytical solution.

7. CONCLUSIONS

The displacements and stresses obtained with the analytical solution with the two plate theories (CPT and FSDT) give data to compare the influence of the shear deformations on laminate plates. CPT does not consider shear stress deformation, whereas FSDT considers a constant shear stress along the thickness of the plate. The comparisons between the FSDT and FEM results show a good agreement between them, indicating that the analytical solution developed in this paper gives acceptable results.

It was concluded that the shear deformation influence in carbon-epoxy plates is much greater than that observed in metallic plates. For the latter, the shear deformation influence is less than 1% when the ratio *a*/*h* is greater than 10. For the carbon-epoxy laminate plate evaluated in this paper, the shear deformation influence is less than 1% when the ratio *a*/*h* is greater than 74. The greater influence of shear deformation in composite material plates can be explained using the beam high-order theory as a parallel, where it can be observed that the influence of shear deformation on displacements depends not only on the thickness to length ratio but also on the *E*/*G* ratio.

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10. RESPONSIBILITY NOTICE

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