# APPLICATIONS OF THE BOUNDARY ELEMENT METHOD WITH THE CONVOLUTION QUADRATURE

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***Abstract.*** *The study of elastodynamic problems is important because, practically, all areas of engineering work with these type of analysis and the criteria involve, in many situations, conditions of security and/or confiability. In this work it is presented a computational model based on the Boundary Element Method (BEM) and the Convolution Quadrature Method (CQM) for the study of elastodynamic problems. The BEM is an important tool in the development of solutions for elastodynamic problems and its research is in continuous evolution. The approach used is based on a direct formulation in the time domain that is solved evaluating the classic BEM formulation in conjunction with conventional step-by-step time integration schemes. In the CQM formulation, the convolution integral of the boundary integral equation is numerically approximated by a quadrature formula whose weights depend only on the Laplace transformed fundamental solution and a linear multistep method, producing the direct solution to the problem in the time domain. Numerical examples are presented, confirming that the formulation is stable and accurate.*

***Keywords****: Boundary Element Method, Convolution Quadrature Method, Elastodynamic*

## 1. INTRODUCTION

The study of the dynamic response of elastic solids is of fundamental importance in several branches of engineering, such as machine foundation design, earthquake engineering, and others. In the past few decades, a great deal of work has been done to develop various techniques for obtaining the response of an elastic medium subjected to dynamic excitation. The Boundary Element Method (BEM) (Brebbia et al., 1984) has been emerged as an effective numerical technique for solving a wide class of engineering problems. In this method, as the name signifies, for linear elastostatic problems only the boundary of the domain needs to be modeled, thereby reducing the problem dimensionality by one. Moreover, in the absence of body forces the boundary integral representation is an exact formulation of the problem, and the only approximations are those due to the numerical implementation of those integral equations.

The Time Domain-Boundary Element Method (Mansur, 1983) is known to be well suited to treat wave propagation problems. The time-dependent fundamental solution is available and the usage of ansatz functions with respect to time yields a time-stepping procedure after an analytical time integration within each time step. Another important procedure is the Convolution Quadrature Method (CQM) developed by Lubich (1988). In the CQM formulation, the convolution integrals are approximated by a quadrature rule whose weights are determined by the Laplace transformed fundamental solutions and a linear multi-step method. It is important to note that the CQM approach was initially introduced as "Operational Quadrature Method" (OQM) as can be found in Schanz and Antes (1997); Abreu et al. (2003) and Vera-Tudela and Telles, (2003a).

In recent years the authors have been working with the Boundary Element Method and the Convolution Quadrature Method to solve 2D elastodynamic problems. Some publications related to this topic (Vera-Tudela and Telles, 2003 and 2005; Abreu et al., 2001 and 2003) have been developed and studies have been devoted to solve critical problems such as the excessive computational time during processing. This excessive computational time processing is fundamentally due to computation and storage of the coefficients matrices of the formulation and also due to the computation of the temporal convolution over N time steps requiring $O(N^2)$ operations and O(N) memory allocation per expansion coefficient. In the present work, a Fast Fourier Transform technique is implemented in the calculus of the quadrature weights of the CQM formulation in order to reduce the operations to O(NlogN).

In the following section, the CQM-BEM formulation is reviewed and numerical example illustrating the efficiency of the proposed formulation is presented.

## 2. THE CONVOLUTION QUADRATURE METHOD

The Convolution Quadrature Method was developed by Lubich (1988, 1988a, 1994) and the basic steps are outlined below. An extension to 2D elastodynamics will be discussed in the next section. A convolution integral is:

$$y(t) = u^*(t) * f(t) = \int_0^{t^+} u^*(t-\tau) f(\tau) d\tau \tag{1}$$

where $u^*(t)$ is the fundamental solution in the Time-Domain Boundary Element Method (TD-BEM); f(t) is a time dependent function and y(t) is called the convolution function. Equation (1) can be discretized using CQM as:

$$y(n\,\Delta t) = \sum_{k=0}^{n} \omega_{n-k}(\Delta t)\, f(k\,\Delta t) \qquad n = 0, 1, \dots, N \tag{2}$$

In Equation (2), N is the total number of time steps and $\omega_n$ represents the quadrature weights (or integration weights) that constitute the coefficients of the power series that approximate the Laplace transform $\hat{u}^*(s)$ of the fundamental solution $u^*(t)$, that is:

$$\hat{u}^*(s) = \hat{u}^*\left(\frac{\gamma(z)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n(\Delta t)\, z^n \tag{3}$$

where $s = \frac{\gamma(z)}{\Delta t}$ and z is a complex variable. It is also required that $|\hat{u}^*(s)| \to 0$ when $|s| \to \infty$ for $\Re\,(s) \geq q$ , (q is a real positive number).

The coefficients of the series in Eq. (3) are furnished by the Cauchy integral formula given below:

$$\omega_n(\Delta t) = \frac{1}{2\pi\, i} \int_{|z|=\rho_r} \hat{u}^*\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1} dz = \frac{\rho_r^{-n}}{L} \sum_{l=0}^{L-1} \hat{u}^*\left(\frac{\gamma\left(\rho_r\, e^{il 2\pi/L}\right)}{\Delta t}\right) e^{-inl\, 2\pi/L} \tag{4}$$

where $\rho_r$ is the radius of a circle in the domain of analyticity of the function $\hat{u}^*(s)$.

If a polar coordinate system is adopted, the integral presented in Eq. (4) can be approximated by means a trapezoidal rule with L equal steps $(2\pi/L)$.

The function $\gamma(z)$, previously used in Eqs. (3) and (4), is the quotient of the polynomials generated by a linear multistep method. If the method is employed for approximating a certain function, say x(t), which, by its turn, is the solution of the first order differential equation

$$\frac{dx(t)}{dt} = s\, x(t) + f(t)\,, \qquad \text{with} \qquad x(0) = 0 \tag{5}$$

one has,

$$x(t) \approx \sum_{j=0}^{k} \alpha_j\, x_{n-j} = \Delta t \sum_{j=0}^{k} \beta_j \left(s\, x_{n-j} + f\big((n-j)\,\Delta t\big)\right) \tag{6}$$

and then

$$\gamma(z) = \frac{\alpha_0 + \cdots + \alpha_k z^k}{\beta_0 + \cdots + \beta_k z^k} \tag{7}$$

The function $\gamma(z)$ clearly characterizes the multistep method and must be $A(\alpha) -$ stable with positive angle $\alpha$, stable in a neighborhood of infinity, strongly zero – stable, and consistent of order p. If an error $\delta$ is assumed in the computation of $\hat{u}^*(s)$ of Eq. (4), the choice L=N and $\rho_r = \sqrt{\delta}$ leads to an error in $\omega_n$ of order $O\left(\sqrt{\delta}\right)$ (Lubich, 1994).

## 3. THE TIME DOMAIN FORMULATION WITH THE CONVOLUTION QUADRATURE METHOD

Consider an elastic solid enclosed by a boundary surface, subjected to specified external dynamic loadings and in the absence of body force. The condition of dynamic equilibrium of a body is expressed by the equation:

$$\mu\, u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho\, \ddot{u}_i \tag{8}$$

where $\mu$ and $\lambda$ are Lamé´s constants, $\rho$ is the mass density and $\ddot{u}_i$ are acceleration components. To uniquely formulate the dynamic problem, boundary and initial conditions must be imposed which specify the state of displacements and velocities at time $t_0$. According to the usual procedure of the boundary element formulation, the integral equation (Mansur, 1983) with null initial conditions can be expressed as follows:

$$4\pi C_{ij}(\xi)\, u_j(\xi,t) = \int_0^{t^+}\int_\Gamma u^*_{ij}(x,t;\xi,\tau)\, p_j(x,\tau)\, d\Gamma(x) d\tau - \int_0^{t^+}\int_\Gamma p^*_{ij}(x,t;\xi,\tau)\, u_j(x,\tau)\, d\Gamma(x) d\tau \tag{9}$$

where $u^*_{ij}$ and $p^*_{ij}$ are boundary displacements and tractions, respectively and $C_{ij}$ is the usual free coefficient dependent on the location of $\xi$ (interior or boundary). The source point where the response is computed was denoted as $\xi$.

According to the CQM procedure, the convolution integrals presented in Eq. (9) can be approximated by

$$\int_0^{t^+}\int_\Gamma u^*_{ij}(x,t;\xi,\tau)\, p_j(x,\tau)\, d\Gamma(x) d\tau = \sum_{k=0}^{n} {}^{n-k}g^e_{ij}(x,\xi,\Delta t)\ {}^{k}p^e_j(x) \qquad n = 0, 1, \ldots, N \tag{10}$$

And

$$\int_0^{t^+}\int_\Gamma p^*_{ij}(x,t;\xi,\tau)\, u_j(x,\tau)\, d\Gamma(x) d\tau = \sum_{k=0}^{n} {}^{n-k}h^e_{ij}(x,\xi,\Delta t)\ {}^{k}u^e_j(x) \qquad n = 0, 1, \ldots, N \tag{11}$$

The CQM weights g and h in Eqs. (10) and (11), respectively, are computed with the expressions,

$${}^{n}g^e_{ij}(x,\xi,\Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1}\int_\Gamma \hat{u}^*_{ij}\left(x,\xi,\frac{\gamma\left(\rho_r\, e^{il2\pi/L}\right)}{\Delta t}\right)\, \Phi^e(x) d\Gamma(x)\, e^{-inl\,2\pi/L} \tag{12}$$

and

$${}^{n}h^e_{ij}(x,\xi,\Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1}\int_\Gamma \hat{p}^*_{ij}\left(x,\xi,\frac{\gamma\left(\rho_r\, e^{il2\pi/L}\right)}{\Delta t}\right)\, \Phi^e(x) d\Gamma(x)\, e^{-inl\,2\pi/L} \tag{13}$$

where $\Phi^e(x)$ represents the interpolation function employed in the boundary discretization (in fatc, quadratic interpolation were employed here).

Equation (9) is rewritten in a discretized form as:

$$4\pi C_{ij}(\xi) u_j(\xi,t_n) = \sum_{e=1}^{E}\sum_{k=0}^{n} {}^{n-k}g^e_{ij}(x,\xi,\Delta t)\ {}^{k}p^e_j(x) - \sum_{e=1}^{E}\sum_{k=0}^{n} {}^{n-k}h^e_{ij}(x,\xi,\Delta t)\ {}^{k}u^e_j(x) \tag{14}$$

Equation (14) can now be written for all boundary nodes in terms of global matrices to give the complete system of equations:

$$\mathrm{C}\,\mathrm{u}^{\mathrm{n}} = \sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{G}^{\mathrm{n-k}}\,\mathrm{p}^{\mathrm{k}} - \sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{H}^{\mathrm{n-k}}\,\mathrm{u}^{\mathrm{k}} \tag{15}$$

Here, C is a quasi diagonal matrix that is formed by the coefficients $C_{ij}(\xi)$; n and k correspond to the variables of the time discretization $t_n = n\,\Delta t$ and $t_k = k\,\Delta t$, respectively.

The system of equations is solved step-by-step. Thus, for the first time step

$$(\mathrm{C} + \mathrm{H}^0)\,\mathrm{u}^1 = \mathrm{G}^0\mathrm{p}^1 + (\mathrm{G}^1\mathrm{p}^0 - \mathrm{H}^1\,\mathrm{u}^0) \tag{16}$$

The columns of the matrices H and G in Eq. (16) must be reordered considering the values of the known and prescribed boundary conditions, obtaining the following expression,

$$\mathrm{A}^0\mathrm{y}^1 = \mathrm{f}^1 + \mathrm{f}^0 \tag{17}$$

where $f^1$ is formed by the boundary contributions at $t = \Delta t$ and

$$\mathrm{f}^0 = \mathrm{G}^1\,\mathrm{p}^0 - \mathrm{H}^1\,\mathrm{u}^0 \tag{18}$$

For an interval $t_n = n\,\Delta t$ the expression can be written as

$$\mathrm{A}^0\mathrm{y}^{\mathrm{n}} = \mathrm{f}^{\mathrm{n}} + \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{f}^{\mathrm{k}} \tag{19}$$

and

$$\mathrm{f}^{\mathrm{k}} = \mathrm{G}^{\mathrm{n-k}}\,\mathrm{p}^{\mathrm{k}} - \mathrm{H}^{\mathrm{n-k}}\,\mathrm{u}^{\mathrm{k}} \tag{20}$$

## 4. FUNDAMENTAL SOLUTION

The fundamental solution for a elastodynamic 2D problem in the Laplace domain (Barra, 1996) is written as:

$$\hat{u}^*_{ij}(x,\xi,s) = \frac{1}{\rho\,c_s^2}\left[\varphi(r)\delta_{ij} - \chi(r) r_{,i}\,r_{,j}\right] \tag{21}$$

and the fundamental tractions are,

$$\begin{aligned}\hat{p}^*_{ij}(x,\xi,s) = \Bigg\{ & \left[\frac{d\varphi(r)}{dr} - \frac{\chi(r)}{r}\right]\left(\delta_{ij}\frac{\partial r}{\partial n} + r_{,j}\;n_i\right) - 2\,\frac{\chi(r)}{r}\left(n_j\,r_{,i} - 2\,r_{,i}\;r_{,j}\frac{\partial r}{\partial n}\right) \\ & -2\,\frac{d\chi(r)}{dr}\,r_{,i}\;r_{,j}\frac{\partial r}{\partial n} + \left(\frac{c_p^2}{c_s^2} - 2\right)\left(\frac{d\varphi(r)}{dr} - \frac{d\chi(r)}{dr} - \frac{\chi(r)}{r}\right) r_{,i}\;n_j\Bigg\}\end{aligned} \tag{22}$$

where the functions $\chi(r)$ and $\varphi(r)$ are defined as follow

$$\chi(r) = k_2\left(\frac{sr}{c_s}\right) - \frac{c_s^2}{c_p^2}\,k_2\left(\frac{sr}{c_p}\right) \tag{23}$$

and

$$\varphi(r) = k_0\left(\frac{sr}{c_s}\right) + \left(\frac{sr}{c_s}\right)^{-1}\left[k_1\left(\frac{sr}{c_s}\right) - \frac{c_s}{c_p}\,k_1\left(\frac{sr}{c_p}\right)\right] \tag{24}$$

where r is the distance between $\xi$ and x; $c_p$ is the P-wave velocity and $c_s$ is the S-wave velocity; $k_j$ is the modified Bessel function of the second kind (Abramowitz and Stegun, 1972) and $\delta_{ij}$ is the Kronecker delta.

## 5. THE FFT APPLIED TO THE CQM-BEM

In this section, the procedures adopted to reduce the time processing for the matrix construction are explained. It was observed that a great computational effort in the CQM-BEM algorithm is dedicated principally in the storage of the matrix of coefficients H and G.

The quadrature weights g and h can be obtained in an efficient form using de FFT algorithm (Cooley and Turkey, 1965; Brigham, 1974), then defining

$$^{l}\widehat{U}{}^{*e}_{ij} = \int_{\Gamma_e} \hat{u}^*_{ij}(r, s_l)\,\Phi^e(x)\,d\Gamma \quad \text{it is obtained} \quad {}^{n}g^e_{ij}(\xi, \Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1} {}^{l}\widehat{U}{}^{*e}_{ij}\, e^{-\alpha\, n\, l} \tag{25}$$

$$^{l}\widehat{P}{}^{*e}_{ij} = \int_{\Gamma_e} \hat{p}^*_{ij}(r, s_l)\,\Phi^e(x)\,d\Gamma \quad \text{it is obtained} \quad {}^{n}h^e_{ij}(\xi, \Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1} {}^{l}\widehat{P}{}^{*e}_{ij}\, e^{-\alpha\, n\, l} \tag{26}$$

In such a way, to obtain the sub matrices g and h is sufficient to determine the discrete Fourier transform of ${}^{l}\widehat{U}^*$ and ${}^{l}\widehat{P}^*$ $(l = 0, 1, 2, \ldots L - 1)$ and multiply by the factor $\rho_r^{-n}/L$. Using the FFT algorithm, every CQM quadrature weights can be obtained with a number of operations of order $N\log(N)$ (note that L=N).

It is important to note that, when the Gauss quadrature formula is applied to the integrals of the Eqs. (25) and (26), the FFT subroutine is executed once for every Gauss point. In order to reduce this looping, the FFT subroutine is executed after the numerical integration of the element.

The storage of the matrices is an important point to consider since as the FFT transform of the vectors ${}^{l}\widehat{U}^*$ and ${}^{l}\widehat{P}^*$ are complex numbers only half of the coefficients must be actually stored (Brigham, 1974). In other words, the elements of the vector $\hat{u}^*(r, S)$ are complex numbers and their real component $\mathrm{Re}(\hat{u}^*(r, S))$ is symmetric and their imaginary component $\mathrm{Im}\big(\hat{u}^*(r, S)\big)$ is antisymmetric. This consideration is, also, valid for $\hat{p}^*(r, S)$. These coefficients can be determined for the first $L/2$ values of S $\big(s_0, s_1, \ldots, s_{L/2}\big)$ and the coefficients that remains can be allocated for the values of S for $\big(s_{L/2+1},\ s_{L/2+2}, \ldots, s_{L-1}\big)$. In consequence, the computational cost of the storage of the sub matrices of Eqs. (25) and (26) is reduced in a half.

The algorithm implemented initially with the discrete Fourier transform (DFT) can be modified as follows:

FFT-1: The computation of the sub matrices g and h in Eqs. (25) and (26) can be implemented with the FFT algorithm. Thus, the number of operations have an order $N\log(N)$.

FFT-2: When this computation is performed, every time-dependent loop that initially executes N operations is reduced to N/2 operations. The remaining is the complex conjugate value.

## 6. NUMERICAL EXAMPLES

This example concerns a rectangular plate (Chien and Wu, 2001; Chien et al., 2003) subjected to a cyclic load at one of its end (Fig. 1). The material properties are: Poisson´s ratio $\nu = 0$, Young´s modulus $E = 1\ \mathrm{N/m^2}$, and mass density $\rho = 1.0\ \mathrm{kg/m^3}$. The dimensions of the rectangular plate are $h = 20\mathrm{m}$ and $b = 1\mathrm{m}$.

Initially the boundary element discretization consists of 44 quadratic elements, one internal point and dual nodes in the four edges of the plate. The total time of loading is 128s and the wave velocity is $c_p = 1\mathrm{m/s^2}$.

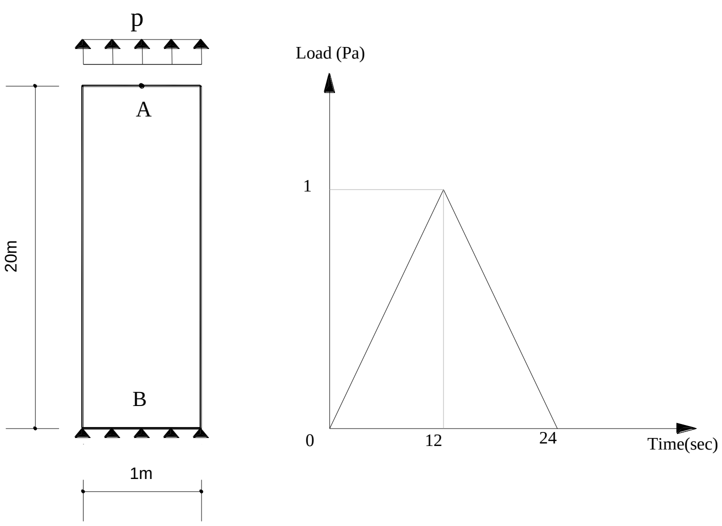


Figure 1. A long and rectangular plate under to a cyclic load.

The CQM-BEM displacements of the point A are compared with the exact solution, and the result are shown in Fig. 2.

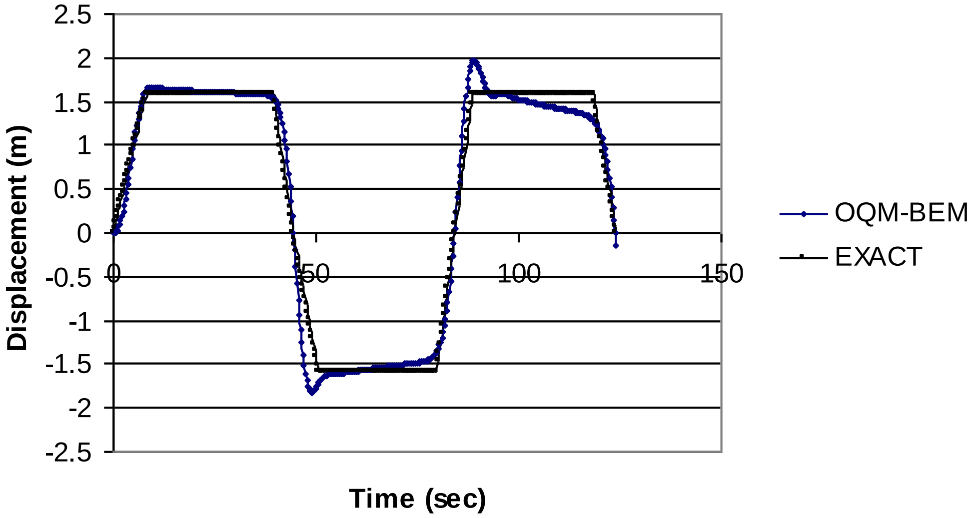


Figure 2. Time variations of vertical displacement of point A of a rectangular plate under a triangular tensile load.

In Figs. (2) and (3) the results obtained with the CQM-BEM formulation are compared with a reference solution named here as "exact" (Chien and Wu, 2001). Note that in these figures the nomenclature is OQM-BEM where OQM is from the initial nomenclature introduced for the CQM.

From Figs. (2) and (3) one can observe that the proposed formulation gives accurate results presenting typical numerical oscillations around the discontinuities. These results were obtained with the modified algorithms FFT-1 and FFT-2 aforementioned. It is observed a numerical damping since the first period of analysis. This numerical error is the combined effect of two kinds of error: the first is caused by the finite precision of computations involving floating-point or integer values and the second is the truncation error.

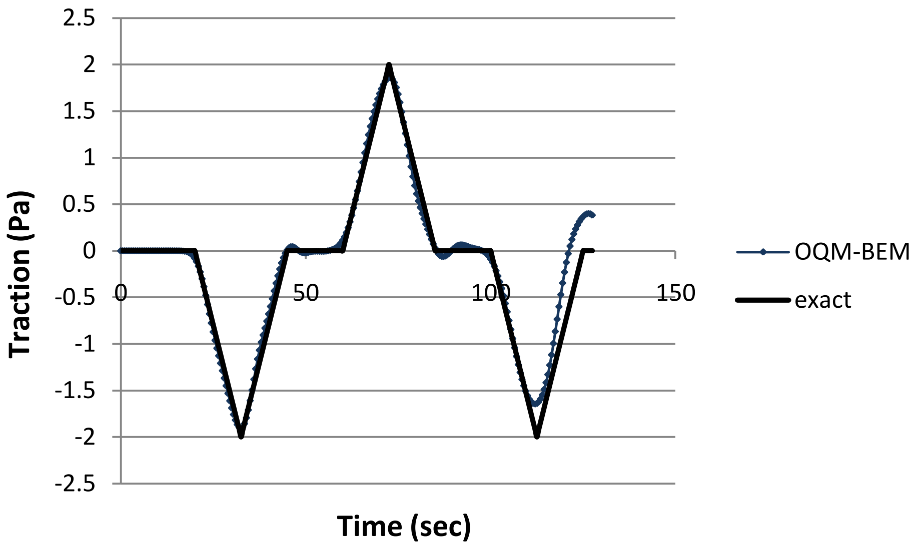


Figure 3. Time variation of normal traction at point B of a rectangular plate under a triangular load.

Figure 4 shows the gain in the CPU-time when the FFT algorithm is implemented. It is important to note that when the number of Fourier coefficients is incremented to N=10, the results lost stability. This is caused due to a limitation in the PC memory capacity; an alternative is considering the use of parallel processing.

When the FFT-2 algorithm is implemented after the FFT-1 algorithm, it can be observed a poor reduction in the computational cost. This is caused because the convolution still represents an important demand in the program. In this way, it can be implemented techniques to reduce the time processing in the convolution.

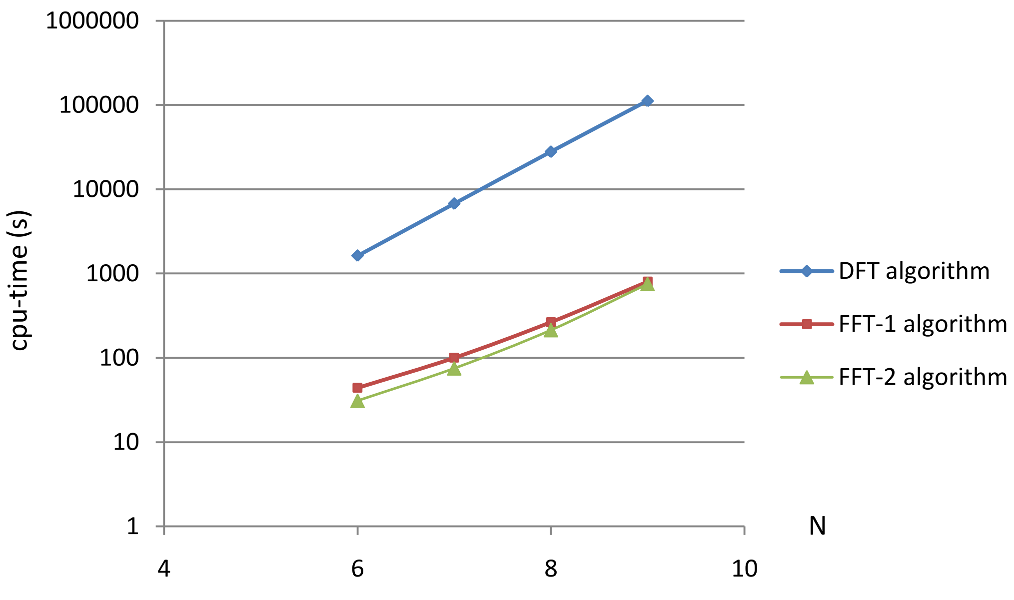


Figure 4. Cpu-time spent for computing the response of the plate of Fig. 1 by three approaches via CQM-BEM versus the number of Fourier coefficients N.

In the Fig. (5) can be observed the interval of the number de elements in test computed. Above this interval, the processing time increases too much. This is a limitation of the PC computer because exist a limit in the availability of memory and is an indicative that other techniques must be adopted, for example a parallel processing. In fact, the algorithm has an important number of loops and the use of several CPU´s is important to make the program run faster. Thus, the element and time discretization can be refined.

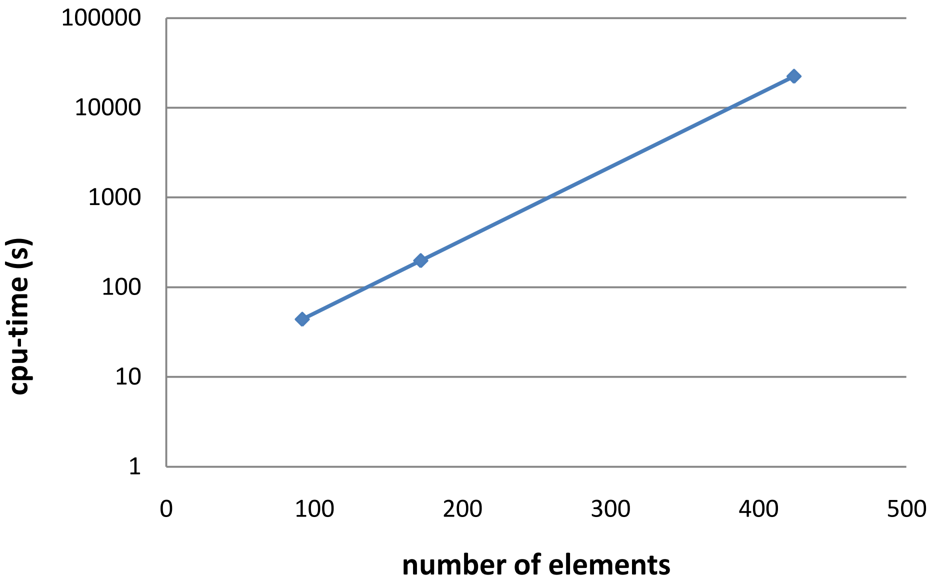


Figure 5. Cpu-time spent for computing the response of the plate of Fig. 1 versus the number of elements in the discretization.

## 7. CONCLUSIONS

In this work, it was presented a Boundary Element formulation and a Convolution Quadrature technique to solve 2D elastodynamics problems with the objective to analyze the computational cost. After the implementation of the two modifications FFT-1 and FFT-2 the computational cost was reduced. The numerical results when compared with the exact solution have shown good accuracy and stability. This results from Fourier coefficient N=10 presents an excessive increment in the time processing. If the number of elements is augmented the same problems is presented. It was also observed that the convolution algorithm represents a great part of the computational cost. Hence, it would be the first target of further developments.

A hardware limitation was observed in the sense that the problem analyzed maintains stability with 44 elements and a maximum of 512 time steps. Above this reference values the results are deteriorated. Parallel techniques can be an important solution and the authors intend to develop soon.

## 8. ACKNOWLEDGEMENTS

This work has been carried out with the support of the FAPERJ and the Fundo Setorial de Infra-Estrutura (CT-INFRA) with the MCT/CNPq – Programa Primeiros Projetos and Apoio às Instituições de Ensino e Pesquisa Sediadas no Estado do Rio de Janeiro; the Universidade Federal Rural do Rio de Janeiro with “Grupos Emergentes” project.

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