# RECONSTRUCTION OF THE CONVECTION COEFFICIENT FROM NON-INTRUSIVE MEASUREMENTS: REGULARIZATION OF THE INVERSE PROBLEM BY THE TRUNCATED SINGULAR VALUE DECOMPOSITION METHOD

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***Abstract.*** *This paper presents a regularization technique suited to the inverse problem associated with the reconstruction of the internal convection coefficient from external temperature measurements. More precisely, the error function between measured and model temperatures, obtained from a prospective convection coefficient, is minimized by applying the Newton. The Truncated Singular Value Decomposition (TSVD) is used to regularize the corresponding hessian matrix. Numerical simulations were conducted for three different test cases and three different randomly perturbed initial guesses in order to validate our proposal. Results confirm that the reconstructed convection coefficient can be effectively reconstructed.*

***Keywords:*** *Inverse problem, convection coefficient, singular value decomposition, thermal tomography, regularization*

## 1. INTRODUCTION

Industrial process tomography is a technique used to determine the internal properties of an object through non-intrusive external measurements. The main application of thermal tomography is to the mapping of imperfections (fouling, cracks, voids, etc.) in solid materials. Another important application is in the measurement of thermophysical properties in processes involving fluids. It is the case of flowmeters of thermal principle, in which the velocity of the flow is determined by the measurement of the convection coefficient. The principle of thermal tomography operation is usually based on the application of a heat flux at the external boundary of the process or object and on the measurement of the response temperatures at that same boundary.

Among the main advantages of thermal tomography are its low cost and robustness, which make it extremely interesting for industrial applications. However, in the current theoretical framework, the obtained results are associated with large errors. More specifically, thermal tomography is currently capable of producing qualitative internal images of a medium or material.

These difficulties arise from non-linearities and the intrinsic ill-conditioned nature of the inverse problem, which makes it extremely sensitive to the presence of errors.

The construction of an image from signals with the aid of tomographic techniques generally involves solving an inverse problem. A thermal problem is described by a differential equation that governs the temperature inside the domain, and by adequate boundary conditions, which simulate the excitation and measurement process. In the inverse problem studied in this paper, we assume that no access is granted to the surface where the convection coefficient distribution is to be determined. This lack of information will be bridged by introducing supplementary data measured at the accessible boundary of the problem. An error functional, whose minimization corresponds to the sought solution (Borcea, 2003), (Özisik and Orlande, 2000) is defined.

There are several practical situations in which the convection coefficient can not be determined by direct measurements. Thus, the interest in the development of a technique capable of producing good estimates of the convection coefficient from non-intrusive measurements becomes obvious. The objective of this work is to present a new regularization method suited for the solution of the inverse problem that arises when the convection coefficient is to be reconstructed from external temperature measurements obtained in response to an excitation heat flux. According to a functional formulation, the intrinsic ill-conditioned nature of the inverse problem appears in the hessian matrix associated with the Newton or a Newton-like optimization strategy used to refine approximations of the convection coefficient. Our proposition is the obtaining of a pseudo-inverse of the hessian matrix by truncating its singular value decomposition in order to eliminate the problematic directions. Results have confirmed that this approach is feasible and allows a satisfactory reconstruction of the convection coefficient.

## 2. STATEMENT OF THE PROBLEM

Let $\theta$ and $T$ be thermal potentials and $k$ the thermal conductivity of the medium. The ambient temperature is $T_\infty$ and, thus, $\theta = T - T_\infty$. Denoting the domain of the problem by $\Omega$, the heat conduction equation is written as

$$\vec{\nabla} \cdot (-k\vec{\nabla}\theta) = 0 \quad \text{in} \quad \Omega. \tag{1}$$

The interaction between $\Omega$ and the exterior occurs through the boundary $\partial\Omega$ and is defined by relationships between thermal excitation and response. Consider the following mixed boundary condition, i.e., the coupling between conduction and heat convection

$$-k\frac{\partial\theta}{\partial n} = h\theta + q \quad \text{in} \quad \partial\Omega, \tag{2}$$

where $n$ is the normal vector, $h$ is the convection coefficient and $q$ is the heat flux imposed at the boundary. If all the parameters of equations (1)–(2) were known, a mathematically well-posed problem would be constituted, which could solved by conventional numerical techniques. However, this is not the case in thermal tomography; one or more parameters are unknown in parts of the boundary. The strategy for the solution consists in palliating this information deficiency through the measurement of redundant boundary conditions at the accessible parts of the boundary.

The finite element method was chosen to discretize the problem equations from the following residual equation based on a convenient weight function ($v$) (Becker, Carey and Oden, 1981)

$$\int_\Omega k\vec{\nabla}\theta \cdot \vec{\nabla}v\,dx\,dy + \int_{\partial\Omega_2}[h\theta + q]v\,ds = 0. \tag{3}$$

Our approach to solve the inverse problem associated with the reconstruction of the convection coefficient from boundary data is based on an error functional assessing the difference between measured and model temperatures.

Assuming a prospective distribution of h ($h_{num}$), the set of equations (1)–(2) is used to simulate the problem, returning $T_{num}$. In this work, instead of taking the experimental measurements, they are simulated through the same set of equations (1)–(2), however solved for a reference convection coefficient $h_{actual}$, and having $T_{actual}$ as the result.

Thus, two different models were considered: one corresponding to the experimental assembly itself (actual or analogical model) and the other corresponding to the numerical implementation of equations (1)–(2) on a digital computer (mathematical-numerical model) (Rolnik and Seleghim Jr., 2006). The error functional can be written as follows

$$e(h_{num}) = \|T_{actual} - T_{num}\|, \tag{4}$$

where temperatures $T_{actual}$ and $T_{num}$ refer only to the accessible region of the boundary where measurements can be accomplished. In Eq. (4), the euclidian norm is usually adopted, that is,

$$e(h_{num}) = \sqrt{\sum(T_{actual} - T_{num})^2}. \tag{5}$$

## 3. OPTIMIZATION ALGORITHM AND REGULARIZATION STRATEGY

The correct convection coefficient distribution can be found by an optimization method that produces successive corrections $d_k$ to prospective solutions $h_k$ minimizing the error function given by Eq. (5). In mathematical terms, this can be expressed as (Colaço, Orlande and Dulikravich, 2006; Nocedal and Wright, 1999)

$$h_{k+1} = h_k + \lambda_k\,d_k, \tag{6}$$

where $\lambda_k$ is the search step size, $d_k$ is the direction of descent and $k$ is the iteration number. The graphs of $h_k$ or its statistical moments over the iterations correspond to the here called convergence trajectories and reveal extremely important aspects about the ill-conditioned nature of the problem.

The majority of the deterministic optimization methods can be described by Eq. (6), and they differ from each other by the form of calculating the descent direction. For instance, in the Steepest Descent method (Nocedal and Wright, 1999), which is a gradient-based method, $d_k$ are corrections along the gradient of the error function which corresponds to downward movements along the steepest descent. Therefore, the direction of the descent is given by

$$d_k = -\vec{\nabla}e(h_k), \tag{7}$$

where $\vec{\nabla}e$ corresponds to the vector gradient associated with the error surface and calculated at the $k$-th iteration.

The main characteristic of this method is the capacity to converge for the solution even if the initial guess is distant from the global minimum. An important disadvantage is the easiness with which convergence trajectory is trapped by local minima.

A very powerful optimization technique is the Newton method (Nocedal and Wright, 1999), which in addition to using first-order derivative of the error functional, such as the Steepest Descent and the Conjugate Gradient methods, also uses information of the second derivative in order to achieve a faster convergence rate. Therefore, in this method the corrections are given by

$$d_k = -H(h_k)^{-1}\overrightarrow{\nabla} e(h_k), \tag{8}$$

where $H$ is the Hessian matrix containing the second-order derivatives of the error functional in relation to the local values of the convection coefficient

$$H_{i,j} = \frac{\partial^2 e}{\partial h_i \partial h_j}. \tag{9}$$

Although the convergence rate of the Newton method is quadratic, the calculation of the Hessian matrix is computationally expensive. As a result, other methods that approximate the Hessian matrix with simpler and faster computing forms have been developed. In this work, the finite difference method was adopted to calculate the Hessian matrix, with derivatives approximated by standard second-order central differences.

In brief, the sequence that hypothetically leads to the solution (global minimum) is the following:

$$h_{k+1} = h_k - \lambda \cdot \triangle h_k = h_k - \lambda \cdot H(h_k)^{-1}\overrightarrow{\nabla} e(h_k), \tag{10}$$

where $\triangle h$ are corrections in vector $h$.

This method is very powerful and straightforward if the Hessian matrix associated with the optimization problem is well behaved. Since the problem we are dealing with is inverse and, therefore, intrinsically ill-conditioned, $H$ is expected to be problematic in some sense. More precisely, corrections $\triangle h_k$ in Eq. (10), which are obtained by solving the Hessian-gradient problem, may lead to an erratic convergence trajectory.

The Newton method obtains a quadratic convergence rate, however it may fail to converge depending on whether the initial guess is outside of the convergence attraction region (region next the maximum/minimum) or due to the ill-conditioning of the Hessian matrix.

Considering that the error functional was not regularized by the application of some specific techniques, as the Tikhonov method for instance, some equivalent procedure must be applied to the Hessian matrix. The Singular Value Decomposition was adopted in this work to obtain a pseudo-inverse of the Hessian matrix with a better conditioning number: the eigenvectors corresponding to singular or near singular eigenvalues of H are truncated (TSVD). In rough, this corresponds to throwing away sets of equations that are nearly linear-dependent or corrupted by round off errors and, thus, solving Eq. (10) in an average sense. These problematic equations attract the solution to the null space associated with H and, consequently, result in increasingly larger corrections $d_k$ towards infinity.

## 4. NUMERICAL SIMULATIONS AND RESULTS

Numerical simulations were conducted for three different test cases and three different randomly perturbed initial guesses in order to validate our proposal. The exact simulation conditions and the corresponding results will be shown in the sequel.

### 4.1 Test Cases

Consider a two-dimensional domain, that is, a square of unitary sides with $k = 1$ in the whole domain and the other boundary condition parameters according to Fig. 1. The governing equation (1), in Cartesian coordinates, was discretized by the finite element method. The domain was discretized in a computational mesh of $31 \times 31$ points, generating $961$ nodes equally spaced ($\delta x = \delta y = 0.033$) and $1800$ linear triangular elements. Both the temperature variation and the thermal conductivity were assumed linear inside the elements.

The reference problem, which mimics the actual experimental test, was defined by setting a reference convection coefficient distribution ($h_{actual}$) at the inferior side of the domain ($y = 0$). More specifically, by fixing $h_{actual}$, reference temperatures are obtained on the accessible boundary of the domain ($y = 1$), which would be measured in response to the application of the reference heat flux. As the error (4) decreased during the optimization process, the obtained temperatures of the numerical model converged to the reference problem temperatures and, supposedly, $h_{num}$ converged to $h_{actual}$.

Three different distributions of h defined three different test cases in which the first one the actual convection coefficient was adopted as a constant function, that is, $h_{actual} = 5$. In the second test, a cosinusoidal convection coefficient was adopted as $h_{actual} = 1 + cos(2\pi x)$ and in the last test, a triangular convection coefficient was adopted:

$$h_{actual} = \begin{cases} 2 - 2(x/15), & x \leq 15; \\ -2 + 2(x/15), & x > 15. \end{cases}$$

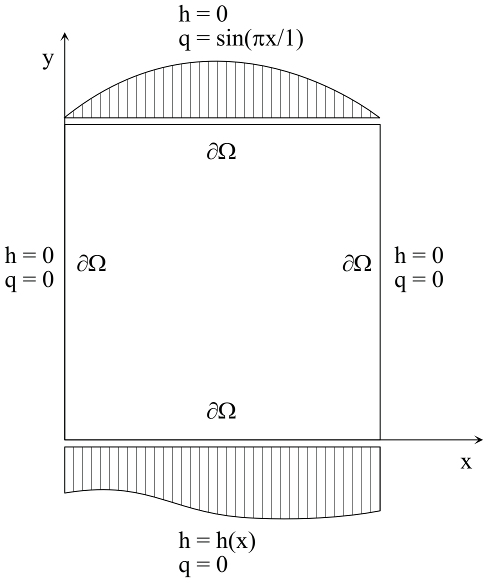

Figure 1. Configuration of the problem simulated numerically.

### 4.2 Hessian Matrix Conditioning and Thresholding Criterion

The singular value decomposition (SVD) is based on the matrix factorization. One of the advantages of SVD is the order reduction of the problem through the elimination of eigenvalues very small (Press et al., 1992).

The Hessian matrix can be written as the product of an orthogonal matrix $U_{m,n}$, a diagonal matrix $W_{n,n}$ with positive or zero elements (the singular values), and the transpose of an orthogonal matrix $V_{n,n}$, i.e.,

$$H = U[diag(w_j)]V^T. \tag{11}$$

The inverses of these matrices are trivial to compute: $U$ and $V$ are orthogonal, so their inverses are equal to their transposes, i.e., $U^{-1} = U^T$ and $V^{-1} = V^T$; $W$ is diagonal, so its inverse is the diagonal matrix whose elements are the reciprocals of the elements $w_j$. From Eq. (11) it follows immediately that the inverse of $H$ is

$$H^{-1} = V[diag(1/w_j)]U^T. \tag{12}$$

This construction may contain singularities if one of the values of $w_j$'s to be zero or (numerically) for it to be so small that its value is dominated by roundoff error and therefore unknowable. If more than one of the $w_j$'s have this problem, then the matrix is even more singular.

This is shown in the following figure where the eigenvalues of the Hessian matrix were calculated at a pathological and a regular region of the optimization surface. The ideal thresholding limits for the cases simulated in this work were determined by trial and error.

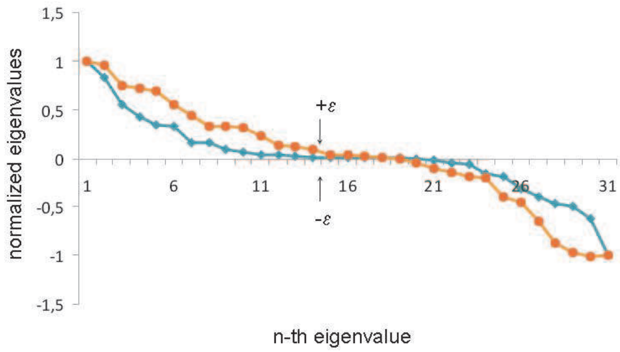

Figure 2. Eigenvalues spectrum of the Hessian matrix for two different regions on the optimization surface (blue dots correspond to a pathological region, orange dots correspond to a regular region).

### 4.3 Convergence Trajectories

As previously mentioned, since the inverse problem is intrinsically ill-conditioned, the corresponding optimization problem is pathological. This can be shown by simulating the convergence trajectory on the error surfaces, whose two-dimensional versions were analyzed in (Brandi and Seleghim Jr, 2008) and (Rolnik and Seleghim Jr., 2006). More specifically, starting from perturbations of $h_{actual}$ in Eq. (4), the sequence given by Newton's recurrence formula (10) should converge monotonically if the error surface were concave. As shown in (Brandi and Seleghim Jr, 2008) our optimization surfaces are pathological, that is, they contain structures such as narrow valleys, multiple local minima, plateaus, etc., and, consequently, the corresponding convergence trajectories are expected to be erratic.

For each case studied we will construct three convergence trajectories by starting from progressively more perturbed versions of $h_{actual}$ given by the formula $h_{inicial} = h_{actual} + \delta$, where $\delta$ corresponds to a uniform random perturbation vector centered around zero, and a standard deviation previously set. This ensures that the inicial guess in Eq. (10) is taken progressively more distant from the correct solution $h_{actual}$.

In the first case, $h_{actual}$ is constant and equal to 5 and the perturbations are set according to the following formulas

$$h = 5 + \begin{cases} \delta \in [-5.0, +5.0], & \text{case 1.1;} \\ \delta \in [-2.0, +2.0], & \text{case 1.2;} \\ \delta \in [-0.15, +0.15], & \text{case 1.3.} \end{cases} \tag{13}$$

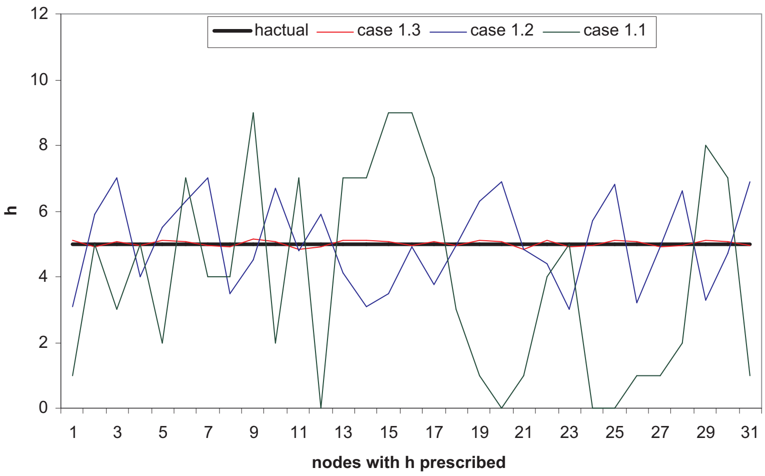


Figure 3. Initial guesses and real solution of the convection coefficient.

In the second case, $h_{actual}$ is a cosinusoidal function and the perturbations are set according to the following formulas

$$h = 1 + cos(2\pi x) + \begin{cases} \delta \in [-2.0, +2.0], & \text{case 2.1;} \\ \delta \in [-1.0, +1.0], & \text{case 2.2;} \\ \delta \in [-0.5, +0.5], & \text{case 2.3.} \end{cases} \tag{14}$$

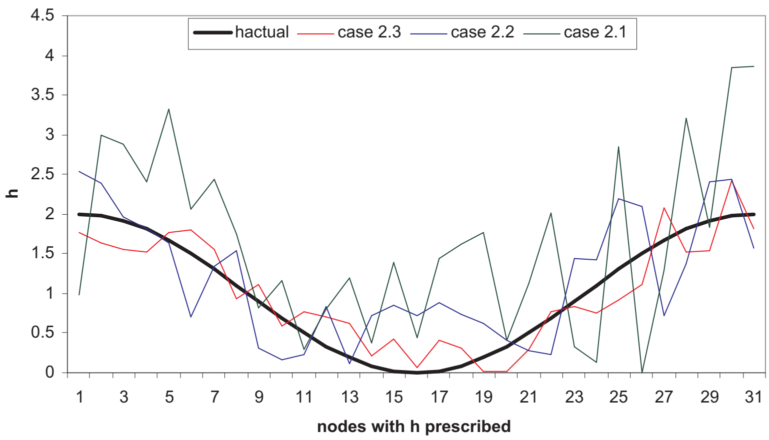


Figure 4. Initial guesses and real solution of the convection coefficient.

In the third case $h_{actual}$ is a triangular function and the perturbations are set according to the following formulas

$$h = \left\{ \begin{array}{ll} 2 - 2(x/15), & x \leq 15 \\ -2 + 2(x/15), & x > 15 \end{array} \right\} + \left\{ \begin{array}{ll} \delta \in [-2.0, +2.0], & \text{case 3.1;} \\ \delta \in [-1.0, +1.0], & \text{case 3.2;} \\ \delta \in [-0.5, +0.5], & \text{case 3.3.} \end{array} \right. \tag{15}$$

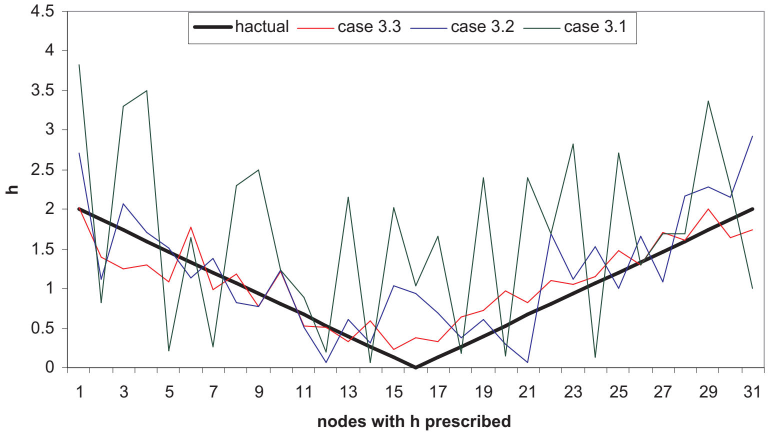


Figure 5. Initial guesses and real solution of the convection coefficient.

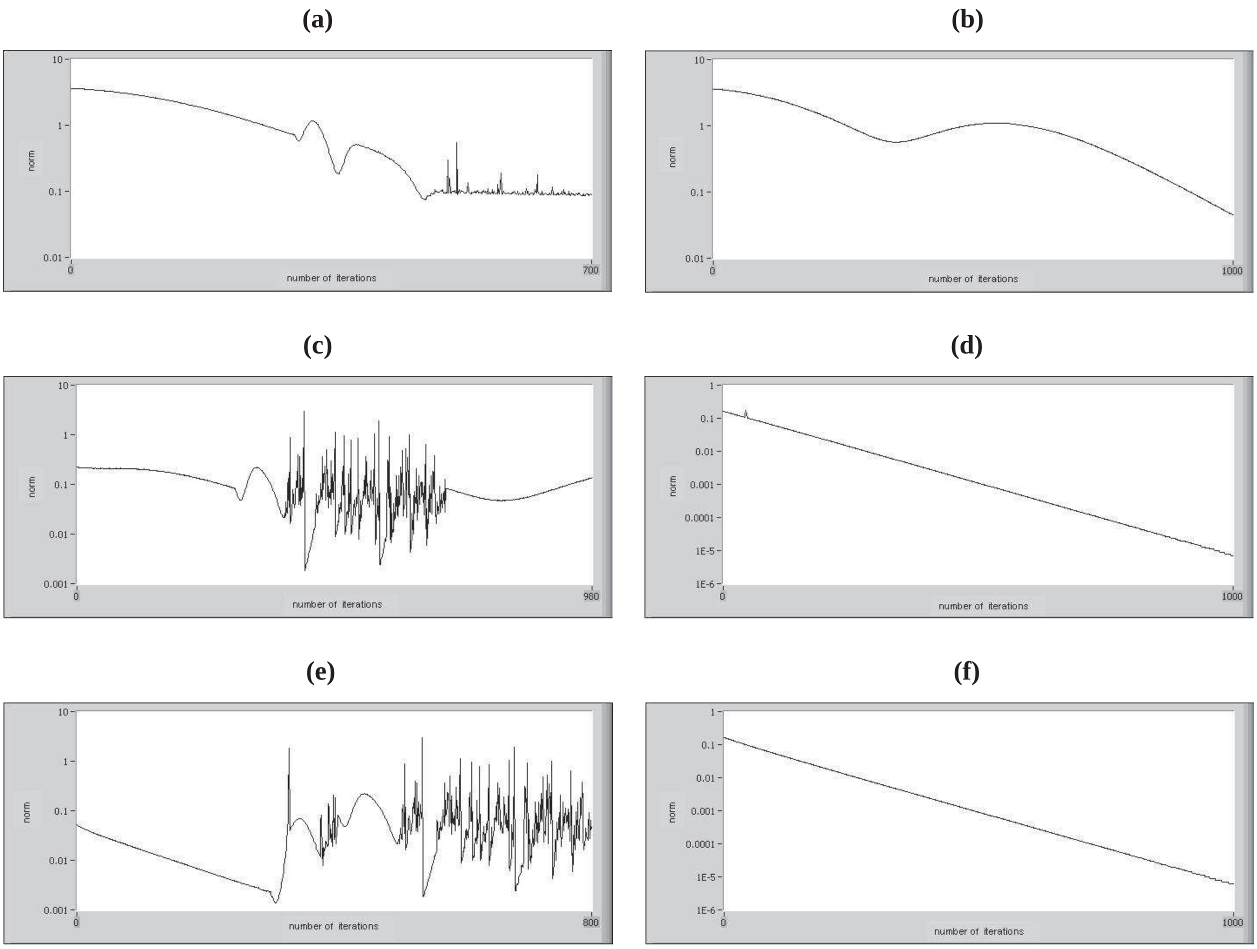


Figure 6. Convergence trajectories corresponding to Eq. (13) using the Newton method (case 1.1=(a), case 1.2=(c), case 1.3=(e)) and using the Newton method with TSVD (case 1.1=(b), case 1.2=(d), case 1.3=(f)).

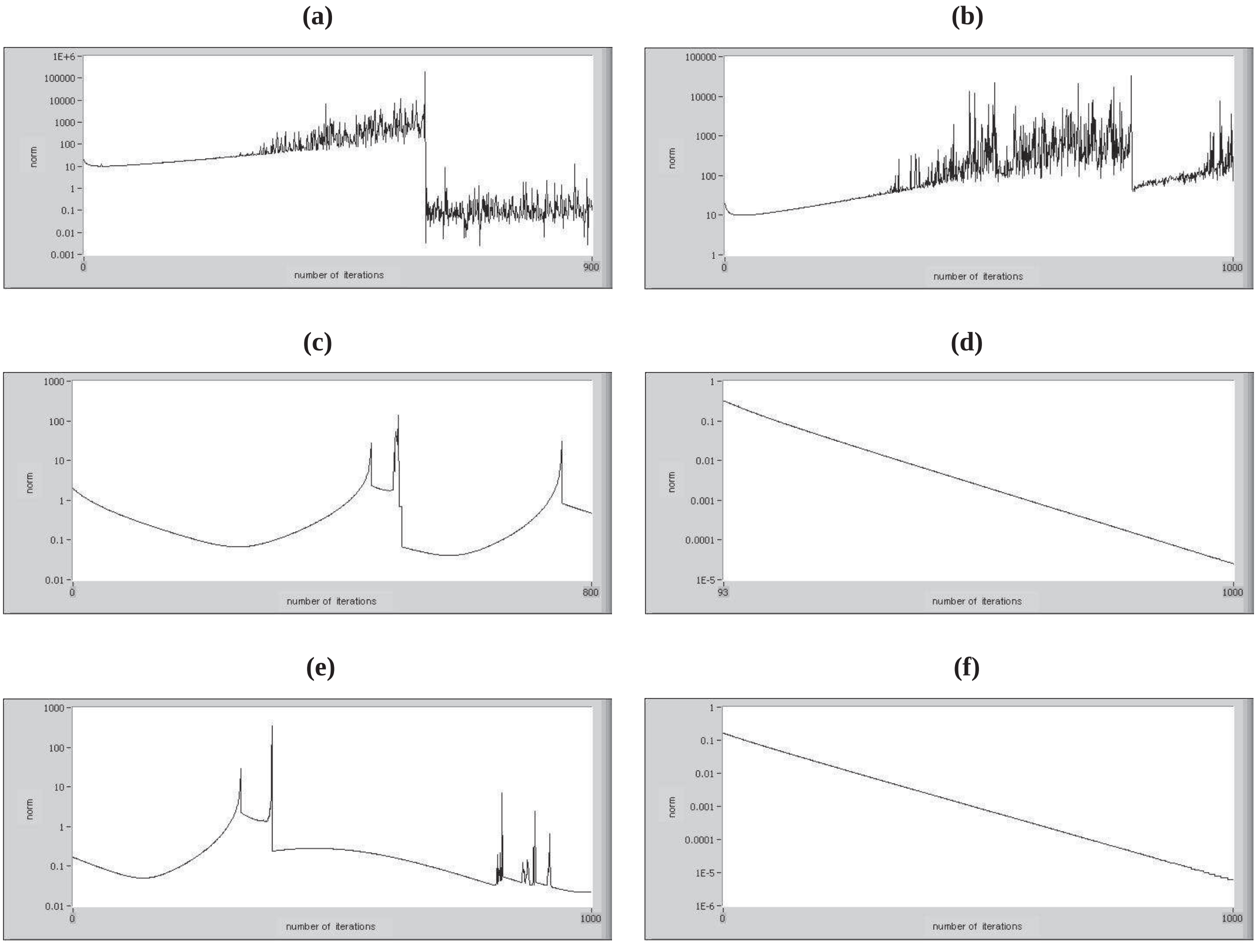


Figure 7. Convergence trajectories corresponding to Eq. (14) using the Newton method (case 2.1=(a), case 2.2=(c), case 2.3=(e)) and using the Newton method with TSVD (case 2.1=(b), case 2.2=(d), case 2.3=(f)).

The resulting convergence trajectories are shown in Figs. 6, 7 and 8. As expected, the proposed technique based on applying the TSVD method to the Hessian matrix of the Newton optimization algorithm was capable of overcoming the convergence problems associated with the intrinsic ill-conditioned nature of the inverse problem considered in this work. In other words, several pathologies of the optimization surface were virtually eliminated which stabilized the convergence trajectory in a way that the global minimum could be attained in a reasonable computational time.

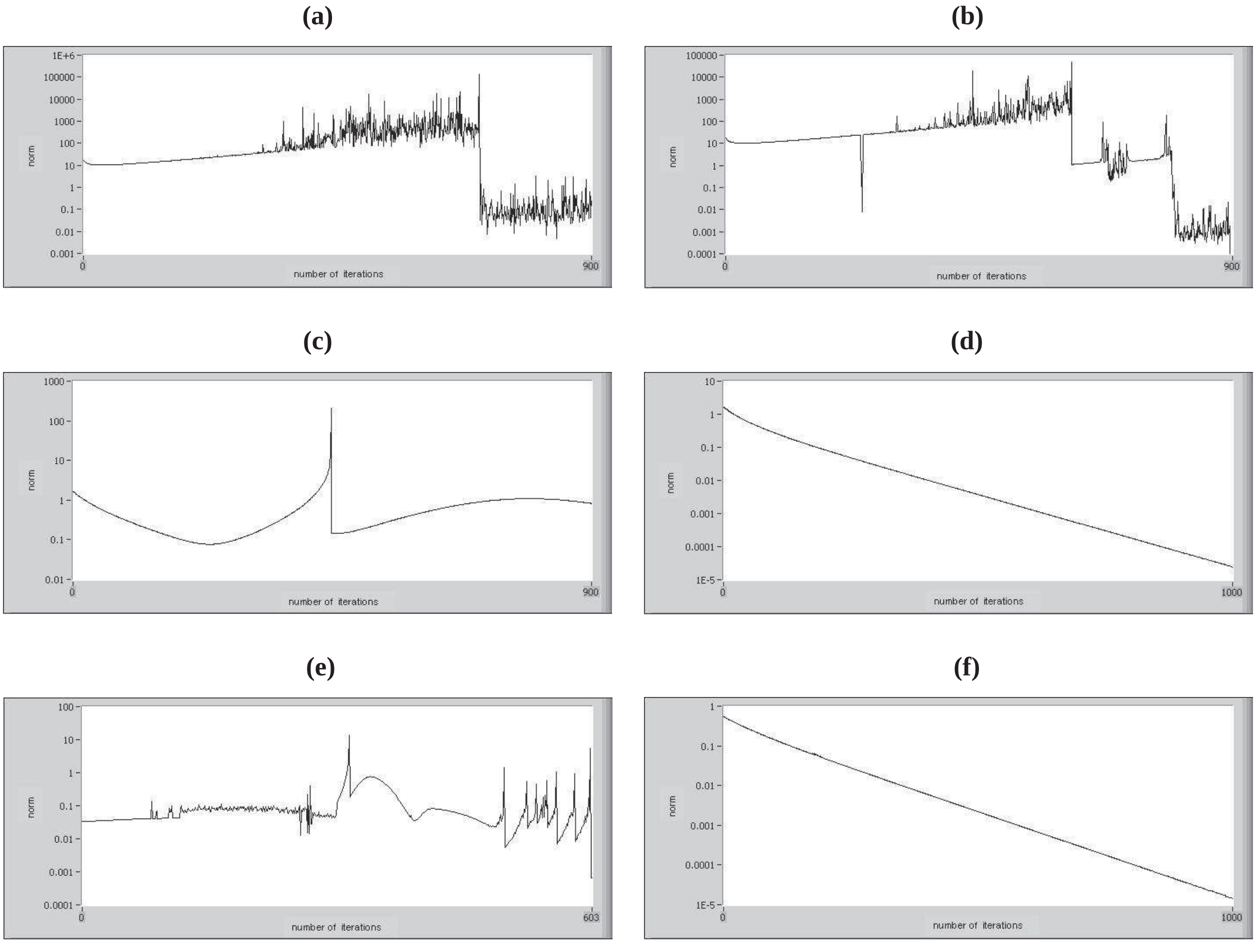


Figure 8. Convergence trajectories corresponding to Eq. (15) using the Newton method (case 3.1=(a), case 3.2=(c), case 3.3=(e)) and using the Newton method with TSVD (case 3.1=(b), case 3.2=(d), case 3.3=(f)) .

## 5. CONCLUSIONS

In this paper the minimization of the error function was based on Newton's method, where the inverse of the Hessian matrix can be replaced by a pseudo-inverse built to come from the technique of TSVD associated with the reconstruction of the internal convection coefficient from external temperature measurements. Three different test cases and three different randomly perturbed initial guesses were analyzed. Results have confirmed that this approach is feasible and allows a satisfactory reconstruction of the convection coefficient.

## 6. ACKNOWLEDGEMENTS

The authors would like to acknowledge FAPESP Grant No. 04/15185-7, for the financial support given to this research.

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