NONLINEAR VIBRATIONS OF HYPERELASTIC ANNULAR MEMBRANES WITH CONTINUOUSLY VARYING DENSITY OR THICKNESS

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Abstract. *This work presents an investigation of the nonlinear free vibration response of a pre-stretched annular hyperelastic membrane with varying density or thickness. The membrane is subjected to finite deformations and it is composed of an isotropic, homogeneous, incompressible and hyperelastic material, which is modeled as a neo-Hookean material. The variation of the membrane density or of the membrane thickness in the radial direction is investigated. First the solution of the membrane under a uniform radial stretch is obtained. Then the equations of motion of the pre-stretched membrane are derived. From the linearized equations of motion, the natural frequencies and mode shapes of the membrane are obtained analytically. The transversal displacements are described by hypergeometric functions. Then the natural modes are used to approximate the nonlinear deformation field using the Galerkin method. The results are compared with the results evaluated for the same membrane using a nonlinear finite element formulation. Excellent agreement is observed up to very large deflections. The results show the strong influence of the stretching ratio on the linear and nonlinear oscillations of the membrane.*

Keywords*: annular membranes, hyperelastic material, non-linear vibrations, varying density.*

1. INTRODUCTION

Membranes have received considerable attention recently due to their applications in several engineering areas, including space applications (Jenkins, 2001; Ruggiero e Inman 2006), actuators and sensors (Kofod, 2001); robotics (Pei et. al, 2004), bio-engineering devices (Gonçalves et al., 2003) and civil engineering structures (Hsieh and Plaut, 1990). Also membranes play a significant role in nature due its high load-carrying capacity per unit weight. The analysis of membrane mechanics is an important topic in nonlinear continuum mechanics. In particular the study of hyperelastic membranes under finite deformations, such as elastomeric membranes and most biological tissues, is a rather challenging subject and, in such cases, elasticity in the fully non-linear range must be employed. The pioneering works of Rivlin (1948) and Green and Adkins (1960) on non-linear elasticity set up the basis for the analysis of structures under large deformations. Strain-invariant constitutive models are usually used to describe the behavior of hyperelastic materials, the simplest constitutive model is the neo-Hookean model, which can be viewed as a simplification of the Mooney-Rivlin law.

Several analytical and numerical analyses are found in literature on the linear vibrations of membranes with continuously varying density and different geometries, for example Laura et al (1998), Gutierrez et al (1998), Wang (1998), Buchanan and Peddieson (1999) and Buchanan (2005). On the influence of varying thickness on the free vibration characteristics of the membrane, little is know in literature. However, both variations are found in practical applications and particularly in nature.

Gottlieb (2000) presents an analytical solution for the free linear vibrations of a circular membrane with varying density in the radial direction. Polynomial and logarithmic variations of the density are considered. He also studies the influence of discontinuous variations of the density along the radial direction of the membrane on the vibration modes and frequencies.

Circular and annular membranes with the density in the radial direction varying in a polynomial form are studied by Jabareen and Eisenberger (2001). They obtain exact solutions for the natural and frequencies and mode shapes, including both axi-symmetric and non-symmetric modes. The linear equation of motion of the elastic membrane is solved by the method of Frobenius, which can be used to solve certain classes of differential equations with variable coefficients about a regular singular point (Humi and Miller, 1998).

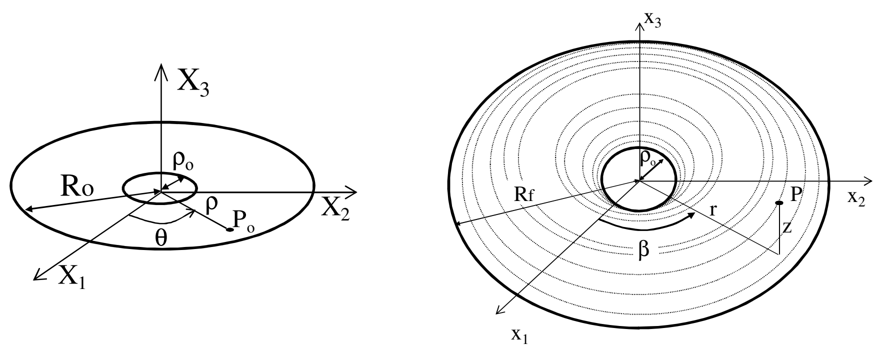
Using also Frobenius method, Willatzen (2002) presents a semi-analytical approximation for the axi-symmetric modes shapes and related natural frequencies for circular and annular membranes with continuously varying density in the radial direction. Several density distributions are considered in their analysis.

Bala Subrahmanyam and Sujith (2001) obtain an analytical solution for the axi-symmetric vibrations of circular and annular membranes with varying density in the radial direction. Through a change of variables, the linear equation of motion of the membrane is transformed into either a Kummer's confluent hypergeometric differential equation whose solution can be expressed in terms of Kummer's confluent hypergeometric function or a Bessel functions (Humi and Miller, 1998; Seaborn, 1991; Magnus, Oberheittinger and Soni, 1966). They observe that the natural frequencies increase as the internal radius of the annular membrane increases and decreases as the variation of the density in the radial direction increases. A review of the literature on the static and dynamic behavior of membranes, both theoretical and experimental, can be found in Jenkins and Leonard (1991), Jenkins (1996) and Jenkins and Korde (2006).

So, the aim of the present work is to study the linear and non-linear free vibrations of a pre-stretched annular hyperelastic membrane with varying density or thickness. The membrane material is assumed to be isotropic and incompressible and its behavior is described by the neo-Hookean constitutive law. A variational formulation, considering finite deformations, is used to derive the equilibrium equations of the membrane under a uniform radial stretch and the equations of motion of the pre-stretched membrane. The linear and non-linear vibrations are analyzed and the influence of the pre-stretch on these results is evaluated. The problem is also analyzed using the finite element software Abaqus 6.5.

2. PROBLEM FORMULATION

First consider a circular hyperelastic membrane of undeformed external radius R_o, undeformed internal radius ρ_o, thickness h and mass density Γ. The membrane is first fixed along the inner edge and uniformly stretched in radial direction, reaching a final radius R_f, and then fixed along this edge. After the application of the static radial traction, the membrane is perturbed in the transverse direction. The deformed and undeformed geometries, co-ordinate system and associated static (r_o, β_o, z_o) and dynamic (u, v, w) displacement components are shown in Fig. 1.



(a) Undeformed (b) Deformed

Figure 1 – Deformed and undeformed configurations of the stretched membrane in the radial direction

The co-ordinates of a material particle at a point P_o on the mid-surface undeformed reference configuration, in a co-ordinate system with the origin at the center of the circular membrane, are given by:

$$X_1 = \rho\cos\theta \qquad X_2 = \rho\sin\theta \qquad X_3 = 0 \tag{1}$$

where, ρ, θ and X_3 are, respectively, the radial, circumferential and transversal co-ordinates.

The co-ordinates of the same typical point, P, at a given instant t, in an arbitrary deformed configuration are given by:

$$x_1 = r(\rho,\theta,t)\cos\beta(\rho,\theta,t) \qquad x_2 = r(\rho,\theta,t)\sin\beta(\rho,\theta,t) \qquad x_3 = z(\rho,\theta,t) \tag{2}$$

where r, β and z are, respectively, the radial, circumferential and transversal co-ordinates of the deformed membrane.

Since the aim of the present analysis is to study the dynamic nonlinear behavior of a pre-stretched membrane, the co-ordinates of a deformed point are considered as the sum of two parts, that is:

$$r(\rho,\theta,t) = r_o(\rho,\theta) + u(\rho,\theta,t) \qquad \beta(\rho,\theta,t) = \theta + \beta_o(\rho,\theta) + v(\rho,\theta,t) \qquad z(\rho,\theta,t) = z_o(\rho,\theta) + w(\rho,\theta,t) \tag{3}$$

where $u(\rho,\theta,t)$, $v(\rho,\theta,t)$ and $w(\rho,\theta,t)$ are the perturbation components in the radial, circumferential and transversal directions, respectively and $r_o(\rho,\theta)$, $\beta_o(\rho,\theta)$ and $z_o(\rho,\theta)$ describes the initial deformed static state.

The strain energy density can be written as a function of the principal stretches λ_1, λ_2 and λ_3 or, alternatively, of the strain invariants I_1, I_2 and I_3 (Green and Adkins, 1960). The three strain invariants of the deformation field can be written in terms of the principal stretches λ_i (i=1, 2, 3) as:

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2 \\ I_3 &= (\lambda_1\lambda_2\lambda_3)^2 \end{aligned} \tag{4}$$

The membrane material is homogeneous, isotropic and incompressible ($I_3 = 1$). There are several constitutive laws in literature particularly adapted to the representation of elastomers. Considering a neo-Hookean material, the energy density function can be described as:

$$W = C_1(I_1 - 3) \tag{5}$$

where C_1 is a material parameter and I_1 is the first strain invariant.

The elastic strain energy, U, is the volume integral of W in the undeformed configuration, which, in the present case, becomes:

$$U = \int_{\rho_o}^{R_o}\int_0^{2\pi}\int_0^{h(\rho)} W(\rho, r, r_{,\rho}, r_{,\theta}, z_{,\rho}, z_{,\theta}, \beta_{,\rho}, \beta_{,\theta}, \rho, \theta)\, \rho \; dz\, d\theta\; d\rho \tag{6}$$

The work of the external applied forces, W_e, considering a radial stretch due to a uniform distributed force f along the circular boundary, is:

$$W_e = 2\pi\,\rho\, f\,(r_o - \rho)\Big|_{\rho=R_o} \tag{7}$$

The kinetic energy, T, is given by:

$$T = \int_{\rho_o}^{R_o}\int_0^{2\pi}\int_0^{h(\rho)} \Gamma(\rho)\frac{(\dot{r}^2 + (r\dot{\beta})^2 + \dot{z}^2)}{2}\rho\; dz\, d\theta\; d\rho \tag{8}$$

where $\Gamma(\rho)$ is the mass density of the material in the natural configuration and $(\bullet) = \partial(\,)/\partial t$.

Then, using Eqs. (7-9), one obtains the Lagrangian function:

$$L = T - U + W_e \tag{9}$$

Based on Eq. (10) and using calculus of variations, the differential equations for each case can be easily obtained.

2.1. Static Analisys

For a uniformly stretched membrane in the radial direction, the radial displacement function - $r_o(\rho)$ – must satisfy the following non-linear differential equilibrium equation:

$$\frac{r_o}{\rho} - \frac{3\rho^3}{r_o^{\,3} r_o'^{\,2}} - r_o' - \rho\, r_o'' + \frac{3\rho^2}{r_o^{\,2} r_o'^{\,3}} - \frac{3\rho^3 r_o''}{r_o^{\,2} r_o'^{\,4}} = 0 \tag{10}$$

and the following boundary conditions:

$$r_o(R_o) = R_f \tag{11}$$

$$r_o(\rho_o) = \rho_o \tag{12}$$

where $(\,)' = \partial(\,)/\partial\rho$.

The nonlinear equilibrium equation of the membrane, Eq. (10), precludes an exact solution. So, the solution must be obtained by approximation techniques. Here the solution is obtained by the shooting method (Press et al., 2007). The differential equation of the boundary value problem is integrated using the Runge-Kutta integration scheme and the free boundary conditions at the initial point are adjusted by the Newton-Raphson method (Soares, 2009). Based on the results obtained by the shooting method, the radius of the extended annular membrane can be described by the following function of ρ:

$$r_o(\rho) = a_1 \ln\rho + a_2\,\rho^2 \ln\rho + a_3\,\rho^2 + a_4\rho + a_5 \tag{13}$$

where a_i are constants that depend on the properties of the deformed membrane. The static transversal and circumferential displacement components, z_o and β_o, are zero. The principal stresses are given by:

$$\sigma_1 = \frac{2C_1(r_o'^4\, r_o^{\,2} - \rho^2)}{r_o'^2\, r_o^{\,2}} \qquad \sigma_2 = \frac{2C_1(r_o'^2 r_o^{\,4} - \rho^4)}{\rho^2 r_o'^2 r_o^{\,2}} \tag{14}$$

2.2. Variable Density

First, the influence of the continuously varying density is analyzed. In this case, the membrane thickness, h_o, is constant and the material density varies in the radial direction according to the law (Jabareen e Eisenberger, 2001; Subrahmanyam e Sujith, 2001; Willatzen, 2002):

$$\Gamma(\rho) = \Gamma_o\,(1 + \kappa\rho^2) \tag{15}$$

where $\Gamma(\rho)$ is the material density at a point with radial coordinate ρ, Γ_o is a constant and the parameter κ describes de variation of the material mass density along the undeformed radius of the membrane. According to this expression, the mass density decreases as ρ increases when $\kappa > 0$ and decreases when $\kappa < 0$, as illustrated in Fig. 2.

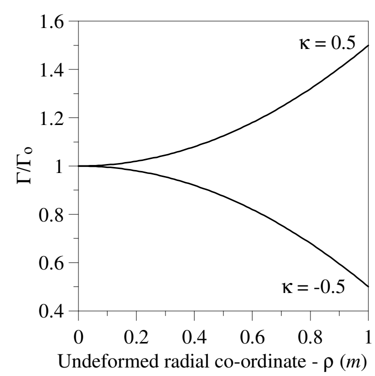


Figure 2 – Variation of the normalized density along the radial direction of the membrane.

2.2.1. Linear Vibration Analisys

The linear equation of motion of the membrane with varying density is obtained by applying Hamilton's principle and is given by (Soares, 2009):

$$2C_1\left[\left(\frac{\rho^2}{r_o'^4 r_o^2} - 1\right)\underline{\underline{\frac{\partial^2 w(\rho,\theta,t)}{\partial\rho^2}}} + \left(\frac{\rho^2}{r_o'^2 r_o^4} - \frac{1}{\rho^2}\right)\underline{\underline{\frac{\partial^2 w(\rho,\theta,t)}{\partial\theta^2}}} + \left(\frac{3\rho}{r_o'^4 r_o^2} - \frac{2\rho^2}{r_o'^3 r_o^3} - \frac{1}{\rho} - \frac{4\rho^2 r_o''}{r_o'^5 r_o^2}\right)\underline{\underline{\frac{\partial w(\rho,\theta,t)}{\partial\rho}}}\right] + \\ + \Gamma_o(1+\kappa\rho^2)\left(\underline{\underline{\frac{\partial^2 w(\rho,\theta,t)}{\partial t^2}}}\right) = 0 \tag{16}$$

The underline is used to distinguish the terms of the differential equations from the coefficients, which are function of the initial stress state.

The transversal displacement w is obtained by transforming the linear equation of motion (17) into an equation that has a known analytical solution. Solving the partial differential equation by separation of variables (ρ, θ, t), the transversal displacement can be written as:

$$w(\rho,\theta,t) = A_{mn}\, G(\rho)\cos(n\theta)\,\cos(\omega_{mn}\, t) \tag{17}$$

where A_{mn} is the modal amplitude; $G(\rho)$, is an unknown function of ρ; m, is the number of half-waves in the radial direction; n, the number of circumferential waves and ω_{mn}, the free vibration frequency associated to mode (m, n).

Substituting (17) into (16), the following linear differential equation in ρ with variables coefficients, known as Whittaker differential equation (Abramowitz e Stegun, 1972), is obtained:

$$\left(-1+\frac{\rho^2}{r_o'^4 r_o^{\,2}}\right)\frac{d^2G(\rho)}{d\rho^2}+\left(-\frac{1}{\rho}-\frac{4\rho^2 r_o''}{r_o'^5 r_o^{\,2}}+\frac{3\rho}{r_o'^4 r_o^{\,2}}-\frac{2\rho^2}{r_o'^3 r_o^{\,3}}\right)\frac{dG(\rho)}{d\rho}+\left(\frac{\Gamma_o(1+\kappa\rho^2)\omega_{mn}^2}{2C_1}-\frac{n^2\rho^2}{r_o'^2 r_o^{\,4}}+\frac{n^2}{\rho^2}\right)G(\rho)=0 \tag{18}$$

Solving equation (19) together with the boundary conditions in the radial direction (Abramowitz e Stegun, 1972), the following expression for the membrane vibration modes is obtained (Soares, 2009):

$$w(\rho,\theta,t) = A_{mn}\left[M_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho^2 I\right)-C\,W_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho^2 I\right)\right]\cos(n\theta)\,\cos(\omega_{mn}\, t) \tag{19}$$

where:

$$K=\frac{\Gamma_o\, k_{mn}\left(r_o'(R_o)\right)^3 R_f^{\,3}}{2\,C_1 R_o\left(R_f^{\,2}\left(r_o'(R_o)\right)^4-R_o^{\,2}\right)}\qquad C=\frac{M_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho_o^{\,2} I\right)}{W_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho_o^{\,2} I\right)} \tag{20}$$

$$\begin{aligned}Z(k_{mn}) = {} & W_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho_o^{\,2} I\right)M_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}R_o^{\,2} I\right)- \\ & M_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}\rho_o^{\,2} I\right)W_n\left(\frac{-I}{4}\sqrt{K/\kappa};\frac{n}{2};\sqrt{K\kappa}R_o^{\,2} I\right)\end{aligned} \tag{21}$$

and M_n and W_n are Whittaker`s function of order n (Abramowitz e Stegun, 1972; Wolfram mathworld, 2008); k_{mn}, is the m-th root of $Z(k_{mn})$ and $r_o'(R_o) = dr_o(R_o)/d\rho$.

By substituting the transversal displacement (19) into Eq. (18), the natural frequencies of the membrane are obtained.

2.2.2. Non-Linear Vibration Analisys

Through the results of the FE analysis of this problem, one can observed that, during the transversal non-linear vibrations of the pre-stretched membrane, the in-plane displacements u and v are rather small compared with the transversal displacement w (Soares, 2009). So, in the following semi-analytical procedure only the transversal displacement is considered. So, the nonlinear equation of motion of the membrane in the transversal direction reduces to:

$$-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial z_{,\rho}}\right)-\frac{\partial}{\partial\theta}\left(\rho\frac{\partial W}{\partial z_{,\theta}}\right)+\rho\,\Gamma_o\,(1+\kappa\rho^2)\frac{\partial^2 w}{\partial t^2}=0 \tag{22}$$

The displacement field is then approximated by a sum of MxN linear vibration modes, Eq. (20), and equation (23) is solved by the Galerkin-Urabe method.

2.3. Thickness Variation

For the membrane with variable thickness, the following variation law in the radial direction is considered:

$$h(\rho) = h_o e^{\eta\rho^2} \tag{23}$$

where h_o is a reference value and η is a constant that describes the variation of the thickness along the radial direction. The membrane material density is constant and equal to Γ_o. The membrane thickness increases from the center to the boundary when $\eta > 0$ and decreases when $\eta < 0$, as illustrated in Fig. 3.

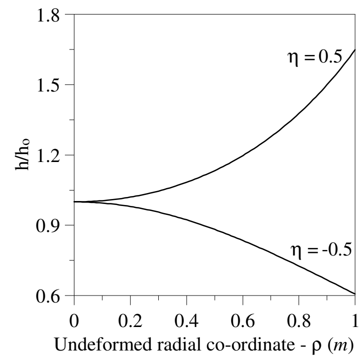


Figure 3 – Variation of the normalized thickness along the radial direction of the membrane.

2.3.1. Linear Vibration Analisys

The linearized equation of motion of the hyperelastic membrane with continuously variable thickness in the radial direction is given as:

$$
\begin{aligned}
&2C_1\left[\left(-1+\frac{\rho^2}{r_o'^4 r_o^2}\right)\frac{\partial^2 w(\rho,\theta,t)}{\partial\rho^2}+\left(-\frac{1}{\rho^2}+\frac{\rho^2}{r_o'^2 r_o^4}\right)\frac{\partial^2 w(\rho,\theta,t)}{\partial\theta^2}+\right.\\
&\left.+\left(-\frac{1}{\rho}-\frac{4\rho^2 r_o''}{r_o'^5 r_o^2}+\frac{3\rho}{r_o'^4 r_o^2}-\frac{2\rho^2}{r_o'^3 r_o^3}+\frac{2\rho^3\eta}{r_o'^4 r_o^2}-2\rho\eta\right)\frac{\partial w(\rho,\theta,t)}{\partial\rho}\right]+\Gamma\left(\frac{\partial^2 w(\rho,\theta,t)}{\partial t^2}\right)=0
\end{aligned}
\tag{24}
$$

By separation of variables the following ordinary differential equation with variable coefficients in the radial direction is obtained:

$$
\left(\frac{\rho^2}{r_o'^4 r_o^2}-1\right)\frac{d^2G(\rho)}{d\rho^2}+\left(\left(\frac{\rho^3}{r_o'^4 r_o^2}-\rho\right)2\eta-\frac{1}{\rho}-\frac{4\rho^2 r_o''}{r_o'^5 r_o^2}+\frac{3\rho}{r_o'^4 r_o^2}-\frac{2\rho^2}{r_o'^3 r_o^3}\right)\frac{dG(\rho)}{d\rho}+\left(\frac{\Gamma\omega_{mn}^2}{2C_1}-\frac{n^2\rho^2}{r_o'^2 r_o^4}+\frac{n^2}{\rho^2}\right)G(\rho)=0 \tag{25}
$$

An approximate solution for the transversal displacement that satisfies the boundary conditions can be written as:

$$
w(\rho,\theta,t)=A_{mn}\frac{e^{-0.5\eta\rho^2}}{\rho}\left[M_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho^2\right)-CW_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho^2\right)\right]\cos(n\theta)\,\cos(\omega_{mn}t) \tag{26}
$$

where:

$$
B=\frac{\left(r_o'(R_o)\right)^3 R_f^{\,3}}{2\,C_1 R_o\left(R_f^{\,2}\left(r_o'(R_o)\right)^4-R_o^{\,2}\right)}\qquad C=\frac{M_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho_o^{\,2}\right)}{W_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho_o^{\,2}\right)} \tag{27}
$$

$$
\begin{aligned}
Z(b_{mn})=&\,Wn\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho_o^{\,2}\right)M_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta R_o^{\,2}\right)-\\
&M_n\left(\frac{-1}{2}-\frac{\Gamma b_{mn}}{4B\eta};\frac{n}{2};-\eta\rho_o^{\,2}\right)W_n\left(\frac{-1}{2}-\frac{\Gamma o_{mn}}{4B\eta};\frac{n}{2};-\eta R_o^{\,2}\right)
\end{aligned}
\tag{28}
$$

and b_{mn} is the m-th root of $Z(b_{mn})$.

Substituting (26) into Eq. (25) and applying the Galerkin method, the natural frequencies of the membrane are obtained.

2.3.2. Non-Linear Vibration Analisys

The nonlinear equation of motion in the transversal direction is given by:

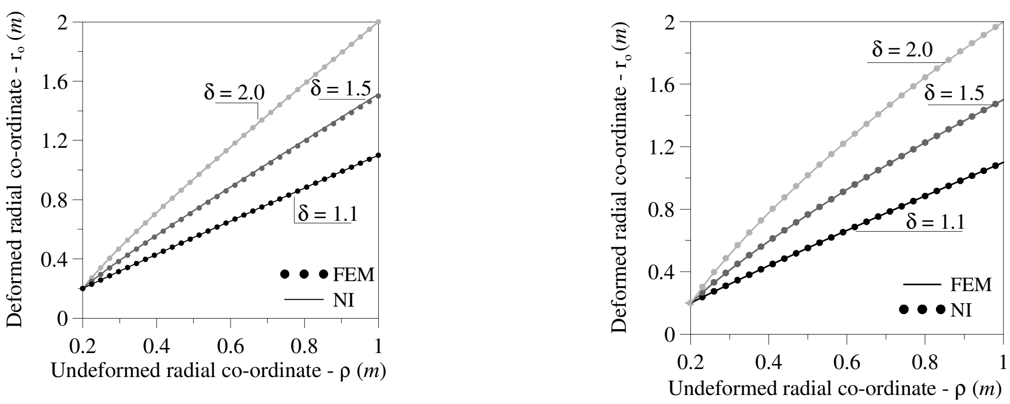
$$-2\eta\,\rho\frac{\partial W}{\partial z_{,\rho}}-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial z_{,\rho}}\right)-\frac{\partial}{\partial\theta}\left(\rho\frac{\partial W}{\partial z_{,\theta}}\right)+\rho\,\Gamma_o\frac{\partial^2 w}{\partial t^2}=0 \tag{29}$$

The transversal displacement can be approximated by a sum of *MxN* linear modes and by applying the Galerkin-Urabe method. Through this procedure the nonlinear frequency-amplitude relation can be obtained.

4. NUMERICAL RESULTS

For the numerical analysis, an annular membrane with undeformed radius $R_o = 1\ m$, internal radius $\rho_o = 0.20\ m$, initial thickness $h_o = 0.001m$ and initial mass density $\Gamma_o = 2200$ Kg/m^3 is considered. The constant of the neo-Hookean, material is taken as $C_1 = 0.17$ MPa (Selvadurai, 2006). This membrane is also analyzed using the finite element software Abaqus 6.5, using the 2304 membrane elements M3D4R for the annular membrane with variable density and 1152 solid elements C3D8RH for the annular membrane with variable thickness.

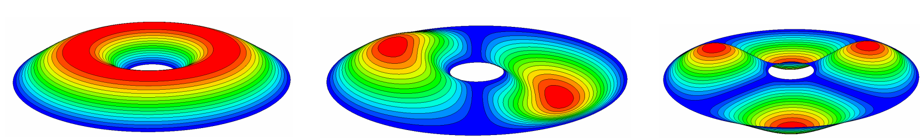
Fig. 4 shows the variation of the deformed radial coordinate, $r_o(\rho)$, as a function of the radial coordinate ρ for three different values of the radial stretching ratio $\delta = R_f/R_o$ obtained through the shooting method and through the FE method, using the FE program ABAQUS. Fig. 4.a shows the results for a membrane with variable density and in Fig. 4.b, for a membrane with variable thickness ($\eta = 0.5$). In the first case, the solution is independent of the value of the parameter κ in Eq. (16).



Variable Density – any value of κ Variable Thickness - $\eta = 0.5$

Figure 4 – Variation of stretching radial co-ordinate of the annular membrane.

The vibration modes and frequencies were computed for increasing values of the κ and η. The analytical (AN) and finite element (FEM) results for the natural frequencies are compared in Tab. 1. The shape of three selected modes is illustrated in Fig. 5.



$n = 0; m = 1$ $n = 1; m = 1$ $n = 2; m = 1$

Figure 5 – Selected vibration modes of the annular membrane.

Table 1 – Vibration Frequencies (rad/s).

Variable Density ($\delta = 1.1$)									
m	n	$\kappa = -0.5$		$\kappa = 0.0$		$\kappa = 0.5$		$\kappa = 0.595$	
		AN	FEM	AN	FEM	AN	FEM	AN	FEM
1	0	35.472	36.108	32.713	32.568	29.486	29.759	29.038	29.305
1	1	39.519	39.776	35.197	35.765	32.462	32.599	31.925	32.089
1	2	49.197	48.765	43.380	43.523	39.720	39.366	38.703	38.709

Variable Thickness ($\delta = 1.1$)							
M	n	$\eta = -0.5$		$\eta = 0.0$		$\eta = 0.5$	
		AN	FEM	AN	FEM	AN	FEM
1	0	31.701	31.639	32.713	32.568	33.190	33.034
1	1	34.702	34.726	35.197	35.765	35.920	36.204
1	2	42.636	42.270	43.380	43.523	43.723	43.778

Fig. 6 illustrates the variation of the lowest natural frequency of the membrane as a function of the stretching ratio δ for different density (Fig. 6.a) and thickness (Fig. 6.b) distributions. In all cases, the frequency increases as δ increases and tends to a constant value for large values of δ. Also, for a given value of δ, the frequency increases as κ increases and as η decreases.

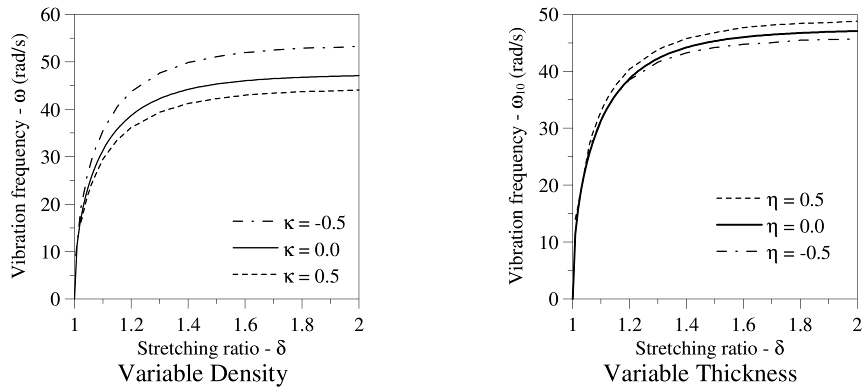


Figure 6 – Frequency of vibration (ω_{10}) – Stretching ratio.

The normalized frequency-amplitude relation for different values of κ and η, associated with the lowest natural frequency that corresponds to the first axi-symmetric mode ($m = 1$ and $n = 0$), is shown in Fig. 7, considering for reference a point with coordinates (0; 0.5) in the membrane configuration for $\delta = 1.1$.

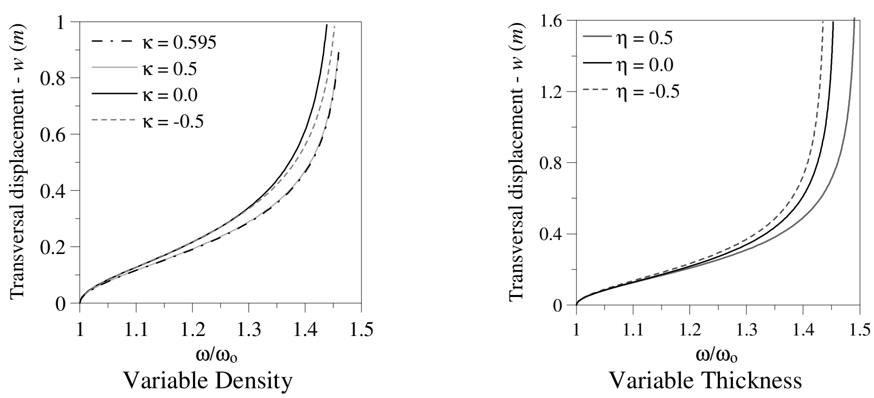


Figure 7 – Normalized frequency-amplitude relation ($\delta = 1.1$).

The frequency-amplitude response is also evaluated from the free lightly damped response of the membrane, discretized by the FE method, using the methodology proposed by Nandakumar e Chatterjee (2005). The analytical results are favorably compared with the FE results in Fig. 8 for a membrane with $\delta = 1.10$, considering two different density distributions and in Fig. 9 for two membranes with variable thickness.

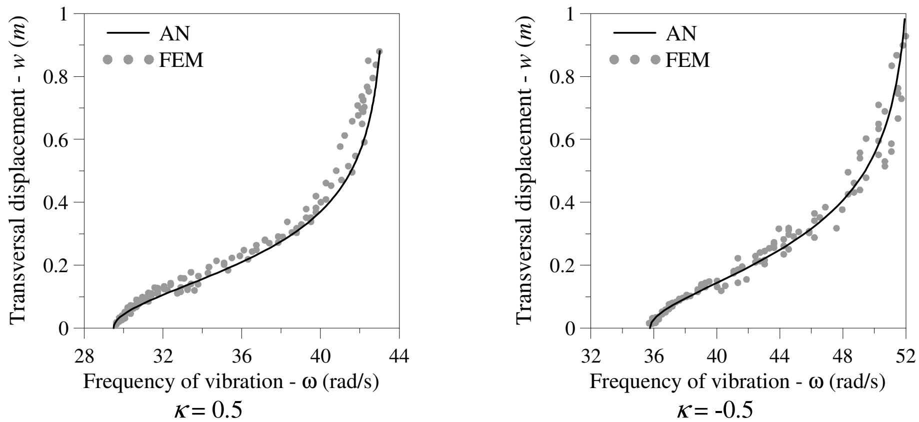


Figure 8 – Frequency of vibration – transversal displacement for membrane with variable density $((\rho,\theta) = (0,0.5))$.

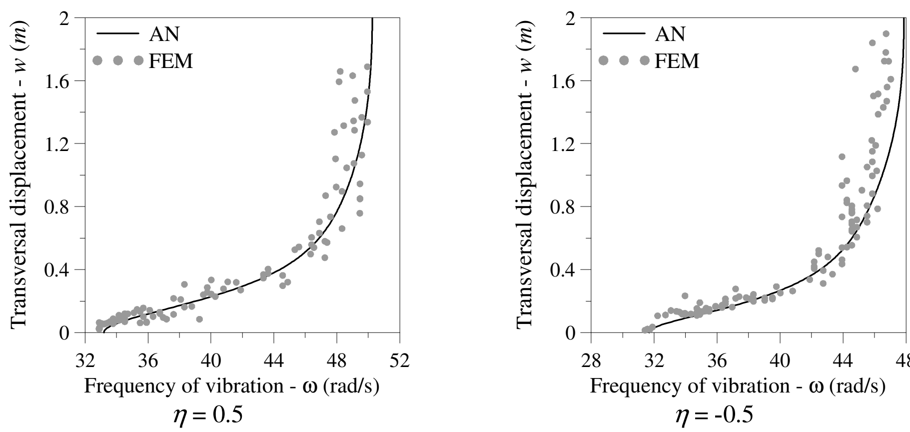


Figure 9 – Frequency of vibration – transversal displacement for membrane with variable thickness $((\rho,\theta) = (0,0.5))$.

4. CONCLUSIONS

The mathematical modeling for the nonlinear vibration analysis of a pre-stretched annular hyperelastic membrane with varying inertial is presented in this paper. The membrane material is considered as homogeneous, isotropic and neo-Hookean. First the solution of the stretched membrane is obtained showing that all relevant quantities are a function of the material constant and the stretching ratio only. Then, the equations of motion of the stretched annular membrane are obtained. By solving the linearized equations of motion, the vibration modes and frequencies of the hyperelastic membrane are obtained and these normal modes are used, together with the Galerkin method, to obtain an approximation of the nonlinear dynamic response. The same problem is also analyzed using the finite element software Abaqus.

The results show that a low order model can give accurate results up to very large deflection. The accuracy of this model is shown by comparisons with numerical values computed by the finite element method. The results highlight the influence of the stretching ratio (deformed radius/ undeformed radius) on the vibration frequencies and nonlinear frequency-amplitude relation. It is shown that a lightly stretched membrane displays a highly nonlinear response and that the nonlinearity decreases as the stretching ratio increases.

5. ACKNOWLEDGEMENTS

The authors acknowledge the financial support of the Brazilian research agencies CAPES, CNPq and FAPERJ.

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