INTEGRAL TRANSFORM SOLUTION OF ONE-DIMENSIONAL EIGENVALUE PROBLEMS USING A DOMAIN ENCLOSING TECHNIQUE

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Abstract. *The solution of eigenvalue problems in irregular geometries is a key feature towards the development of the Generalized Integral Transform Technique for handling arbitrarily shaped domains. Although such solutions have been attempted with success in previous studies, a coincident domain approach was always adopted. This limits the application to problems defined within a class of irregular domains. In order to augment the number of irregular domains that can be handled by integral transforms, an alternative technique is herein introduced. The method consists of tackling the original eigenvalue problem using an auxiliary problem defined in a regular geometry that encloses the irregular domain. This paper provides a formal solution using an enclosing domain solution for a general one-dimensional Sturm-Liouville problem. Then, with the purpose of validating the methodology, test case results for a simple case whose exact solution is known are computed and compared with exact values. The results show very good agreement, and demonstrate that the convergence rate also depends on the number of significant digits used in the calculation.*

Keywords: *Integral Transform; Eigenvalue Problem; Irregular Domain; Symbolic Computation*

1. NOMENCLATURE

$A_{i,j}$	coefficients matrix
$B_{i,j}$	coefficients matrix
$D_{i,j}$	coefficients matrix
$S_{i,j}$	coefficients matrix
$\mathcal{B}$, $\mathcal{B}^*$	boundary condition operators
a, b	boundary of original problem
d	eigenvalue problem parameter
k	parameter in diffusion term
N	norm of eigenfunctions
w	weight function of original eigenfunctions
w^*	weight function of auxiliary eigenfunctions

Greek Symbols

α^*, β^*	boundary condition parameters
α, β	boundary condition parameters
$\delta_{i,j}$	Kronecker delta
Ψ_n	original eigenfunctions
Ω_i	auxiliary eigenfunctions
μ_n	eigenvalues of original problem
γ_i	eigenvalues of auxiliary problem

2. INTRODUCTION

The analysis and solution of convection-diffusion, as well as other problems represented by partial differential systems, can involve a considerable computational effort, especially if nonlinearity and multidimensional effects are present. Further difficulties arise if there are complexities in the geometries considered. In this context, different approaches have been proposed, ranging from full numerical solutions using traditional discretization techniques to analytical methods, the former being limited to simpler situations. Between these two extremes hybrid methodologies are available, combining the flexibility of numerical solutions with the accuracy of analytical approaches. One such technique is the so-called Generalized Integral Transform (GITT) (Cotta, 1993, 1998; Cotta and Mikhailov, 1997). This technique is based on obtaining solutions using orthogonal eigenfunction expansions. Nevertheless, if complex geometries are considered, a strategy for handling domain irregularities becomes necessary. A common option for dealing with this problem was applied to problems involving heat and fluid flow within irregularly shaped channels, heat conduction in fins of arbitrary geometry as well as problems with temperature dependent thermal conductivity (Aparecido and Cotta, 1990, 1992; Aparecido, Cotta et al., 1989; Barbuto and Cotta, 1997; Cotta and Ramos, 1998; Guerrero, Quaresma et al., 2000). All these applications, either of elliptic or parabolic mathematical nature, the domain irregularities are handled by adopting individual auxiliary problems in each coordinate direction that maps the irregular domain boundaries exactly.

A different strategy involves employing multidimensional eigenvalue problems defined within the considered irregular domain itself. This transfers the task of handling with complex geometries from the original PDE system to the associated eigenvalue problem, as demonstrated in (Sphaier and Cotta, 2002). As expected, this approach involves the solution of a multidimensional eigenvalue problem in an irregular geometry. The solution of eigenvalue problems in irregular domains represents a challenging task, even for well established numerical methods, especially when higher order eigenfunctions are needed, due to their highly oscillatory nature. Nevertheless, the solution to such a difficult eigenvalue problem can also be obtained using the GITT. As a matter of fact, a general methodology for solving multidimensional eigenvalue problems via integral transforms in a class arbitrary geometries was proposed in (Sphaier and Cotta, 2000). This strategy becomes particularly interesting for linear convection-diffusion problems, since direct analytical solutions can be obtained once the solution to the multidimensional eigenproblem is accomplished. Although the first methodology can seem more suitable for non-linear problems, if the physical nature of the problem require the calculation of eigenvalues defined within arbitrarily shaped regions, an approach similar to the one developed in (Sphaier and Cotta, 2000) needs be applied.

The methodology presented in (Sphaier and Cotta, 2000) consists of using auxiliary eigenfunctions, defined within a domain coinciding with the original irregular one to solve the problem. Although the auxiliary eigenfunctions are defined within an irregular domain, they are constructed in a simple fashion, using one-dimensional eigenfunctions. This coincident domain approach is capable of solving a class of geometries; however it cannot be applied to certain situations. In order to circumvent this limitation, extending the solution of eigenvalue problems in arbitrary domains to a broader class of geometries, an alternative approach is herein proposed. The idea behind this approach is to solve the eigenvalue problem defined within the irregular domain by using an auxiliary problem defined within a regular domain that encloses the original irregular boundaries. While this alternative technique is still in a preliminary stage, the purpose of this work is to apply it within a one-dimensional framework, in order to verify its feasibility. Hence, a one-dimensional solution methodology to a Sturm-Liouville problem is formally presented. Then a test-case problem of know exact solution is later selected for illustrating the behavior of the proposed solution. The eigenvalues calculated with the proposed methodology are compared to the exact ones, and very reasonable agreement is seen.

3. METHODOLOGY

3.1 Original and auxiliary eigenvalue problems

In order to demonstrate the proposed methodology a general one-dimensional Sturm-Liouville problem is considered:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right) + (\mu^2\,w(x) - d(x))\,\Psi(x) = 0, \qquad \text{for} \qquad a \le x \le b, \tag{1}$$

$$\mathcal{B}\,\Psi = 0, \qquad \text{for} \qquad x = a, \tag{2}$$

$$\mathcal{B}\,\Psi = 0, \qquad \text{for} \qquad x = b, \tag{3}$$

where the boundary condition operator $\mathcal{B}$ is defined as:

$$\mathcal{B} \equiv \left(\alpha(x) + \beta(x)\,k(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{4}$$

It is know that this type of problem possesses the following orthogonality property:

$$\int_a^b w(x)\,\Psi_n(x)\,\Psi_m(x)\,\mathrm{d}x = \delta_{n,m}\,N(\mu_n), \tag{5}$$

in which Ψ_m and Ψ_n are eigenfunctions respectively corresponding to the eigenvalues μ_m and μ_n, $\delta_{n,m}$ is the Kronecker delta and $N(\mu_n)$ is the norm, defined as:

$$N(\mu_n) \equiv \int_a^b w(x)\,\Psi_n(x)^2\,\mathrm{d}x. \tag{6}$$

Analytical solutions to such problem are straightforward in many cases, and they can be found on several sources. Nevertheless, in this investigation, an alternate solution route, using an auxiliary set of orthogonal eigenfunctions is sought. The auxiliary eigenfunctions, denoted as $\Omega_i(x) = \Omega(x;\gamma_i)$ are normalized according to the following orthonormality relation:

$$\int_0^1 w^*(x)\,\Omega_i(x)\,\Omega_j(x)\,\mathrm{d}x = \delta_{i,j}, \tag{7}$$

where it is assumed that $0 \le a < b \le 1$, such that the domain for the original problem is enclosed by the auxiliary one.

3.2 Transform pair

The goal of the current methodology is to use the basis provided by the auxiliary eigenfunctions to write an expansion for the original eigenfuntions in the form:

$$\Psi(x) = \sum_{i=1}^{\infty} \bar{\Psi}_i\,\Omega_i(x), \qquad \text{for} \qquad a \le x \le b\,. \tag{8}$$

This expression is termed the inversion formula. Based on the previous expansion and the orthogonality of the auxiliary eigenfunctions, the following integral transform formula is obtained:

$$\bar{\Psi}_i = \int_0^1 w^*(x)\,\Psi(x)\,\Omega_i(x)\,\mathrm{d}x\,. \tag{9}$$

This transform is different than traditional integral transforms, in which the transformation is performed within the same domain (termed *coincident domain transforms*). In the form provided by equation (9) the transformation is termed a *enclosing domain transform*.

3.3 Transformation of original problem

The original problem is transformed integrating within its original domain:

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right)\Omega_i(x)\,\mathrm{d}x \,+\, \mu^2\int_a^b w(x)\,\Psi(x)\,\Omega_i(x)\,\mathrm{d}x \,-\, \int_a^b d(x)\,\Psi(x)\Omega_i(x)\,\mathrm{d}x \,=\, 0\,, \tag{10}$$

The diffusive term can be transformed, as usual, using Green's second formula:

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right)\Omega_i(x)\,\mathrm{d}x \,=\, \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)\,\frac{\mathrm{d}\Omega_i}{\mathrm{d}x}\right)\Psi(x)\,\mathrm{d}x \,+\, \left[k(x)\,(\Psi'\,\Omega_i - \Psi\,\Omega_i')\right]\Big|_{x=a}^{x=b}. \tag{11}$$

Simplifications can be performed on the boundary term (last on right) using the boundary conditions from the original problem; nevertheless the information about the boundary conditions from the auxiliary problem cannot be used since these are defined in another boundary ($x = 0$ and $x = 1$). The second term on the right side of the above equation can be simplified with information from the chosen auxiliary problem; however to keep the analysis in a general form, particular simplifications will be avoided at this point.

Substituting the inversion formula in equations (11) and (10) yields:

$$\sum_{i=1}^{\infty}\left(\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k\,\frac{\mathrm{d}\Omega_j}{\mathrm{d}x}\right)\Omega_i\,\mathrm{d}x \,+\, \mu^2\int_a^b w\,\Omega_i\,\Omega_j\,\mathrm{d}x \,-\, \int_a^b d\,\Omega_i\,\Omega_j\,\mathrm{d}x\right)\bar{\Psi}_j \,=\, 0, \tag{12}$$

$$\sum_{i=1}^{\infty}\left(\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k\,\frac{\mathrm{d}\Omega_j}{\mathrm{d}x}\right)\Omega_i\,\mathrm{d}x \,-\, \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k\,\frac{\mathrm{d}\Omega_i}{\mathrm{d}x}\right)\Omega_j\,\mathrm{d}x\right)\bar{\Psi}_j \,=\, \sum_{i=1}^{\infty}\bar{\Psi}_j\,\left[k\,(\Omega_j'\,\Omega_i - \Omega_j\,\Omega_i')\right]\Big|_{x=a}^{x=b} \tag{13}$$

Then, introducing the following coefficients:

$$A_{i,j} \,=\, \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(k\,\frac{\mathrm{d}\Omega_j}{\mathrm{d}x}\right)\Omega_i\,\mathrm{d}x, \qquad B_{i,j} \,=\, \int_a^b w\,\Omega_i\,\Omega_j\,\mathrm{d}x, \tag{14}$$

$$D_{i,j} \,=\, \int_a^b d\,\Omega_i\,\Omega_j\,\mathrm{d}x, \qquad S_{i,j} \,=\, \left[k\,(\Omega_j'\,\Omega_i - \Omega_j\,\Omega_i')\right]\Big|_{x=a}^{x=b}, \tag{15}$$

the previous equations can be written as:

$$\sum_{i=1}^{\infty}\left(A_{i,j} \,+\, \mu^2\,B_{i,j} \,-\, D_{i,j}\right)\bar{\Psi}_j \,=\, 0, \tag{16}$$

$$\sum_{i=1}^{\infty}(A_{i,j} - A_{j,i})\,\bar{\Psi}_j \,=\, \sum_{i=1}^{\infty} S_{i,j}\,\bar{\Psi}_j. \tag{17}$$

In matrix/vector form these are equivalent to:

$$\left(\boldsymbol{A} \,+\, \mu^2\,\boldsymbol{B} \,-\, \boldsymbol{D}\right)\bar{\boldsymbol{\Psi}} \,=\, \boldsymbol{0}, \tag{18}$$

$$\left(\boldsymbol{A} - \boldsymbol{A}^T - \boldsymbol{S}\right)\bar{\boldsymbol{\Psi}} \,=\, \boldsymbol{0}. \tag{19}$$

However, the second equations implies in:

$$\boldsymbol{A} \,=\, \boldsymbol{A}^T + \boldsymbol{S}, \tag{20}$$

which allows equation (21) to be written as:

$$\left(\boldsymbol{A}^T + \boldsymbol{S} + \mu^2\,\boldsymbol{B} - \boldsymbol{D}\right)\bar{\boldsymbol{\Psi}} \,=\, \boldsymbol{0}. \tag{21}$$

This systems represents an algebraic eigenvalue problem, which can be used to determine the original eigenvalues μ and transformed eigenfunctions (given by the eigenvectors of the algebraic problem). Defining the matrix below

$$\boldsymbol{M} \,=\, \boldsymbol{B}^{-1}\left(\boldsymbol{A}^{T} + \boldsymbol{S} - \boldsymbol{D}\right), \tag{22}$$

system (21) can be rewritten in the traditional form:

$$\left(\boldsymbol{M} \,-\, \mu^{2}\,\boldsymbol{I}\right)\bar{\boldsymbol{\Psi}} \,=\, \boldsymbol{0}. \tag{23}$$

Equation (23) allows a direct calculation of the eigenvalues μ_n. They can be evaluated as the square root of eigenvalues of the tensor $\boldsymbol{M}$. The eigenfunctions $\Psi_n(x)$ are determined using the inversion formula (8), where for each eigenvalue μ, the corresponding eigenfunction is reconstructed using the components of the associated eigenvector $\bar{\boldsymbol{\Psi}}$.

3.3.1 Simplifications in the boundary matrix S

The form of the boundary matrix ($\boldsymbol{S}$) in the form previously presented did not take into account information regarding the boundary conditions of the original problem. If such information is considered the coefficients of this matrix can be simplified. Employing boundary conditions (2,3) the following relations can be obtained:

$$[k\,(\Psi'\,\Omega_i \,-\, \Psi\,\Omega_i')]|_{x=a}^{x=b} \,=\, -\left[k\,\Psi\left(\frac{\alpha}{\beta\,k}\,\Omega_i \,+\, \Omega_i'\right)\right]\Bigg|_{x=a}^{x=b}, \tag{24}$$

$$[k(\Psi'\,\Omega_i \,-\, \Psi\,\Omega_i')]|_{x=a}^{x=b} \,=\, \left[k\,\Psi'\left(\Omega_i \,+\, \frac{\beta\,k}{\alpha}\,\Omega_i'\right)\right]\Bigg|_{x=a}^{x=b}, \tag{25}$$

where the first formula should be applied for $\alpha = 0$ whereas the second one should be used for $\beta = 0$. However, a general expression can be obtained combining both expressions:

$$[k(\Psi'\,\Omega_i \,-\, \Psi\,\Omega_i')]|_{x=a}^{x=b} \,=\, \left[\frac{k\,(\Psi' - \Psi)\,(\alpha\,\Omega_i \,+\, \beta\,k\,\Omega_i')}{\alpha \,+\, \beta\,k}\right]\Bigg|_{x=a}^{x=b}, \tag{26}$$

where it should be noted that, different than encountered in transforms within coincident domains, the boundary conditions for the auxiliary eigenvalue problem are not substituted into the boundary term resulting from Green's formula.

Using the inversion formula yields:

$$\sum_{i=1}^{\infty} \bar{\Psi}_j\,\left[k\,(\Omega_j'\,\Omega_i - \Omega_j\,\Omega_i')\right]\big|_{x=a}^{x=b} \,=\, \sum_{i=1}^{\infty} \bar{\Psi}_j\,\left[\frac{k\,(\Omega_j' - \Omega_j)\,(\alpha\,\Omega_i \,+\, \beta\,k\,\Omega_i')}{\alpha \,+\, \beta\,k}\right]\Bigg|_{x=a}^{x=b}, \tag{27}$$

showing that the coefficients $S_{i,j}$ can be given by:

$$S_{i,j} \,=\, \left[\frac{k\,(\Omega_j' - \Omega_j)\,(\alpha\,\Omega_i \,+\, \beta\,k\,\Omega_i')}{\alpha \,+\, \beta\,k}\right]\Bigg|_{x=a}^{x=b}, \tag{28}$$

noting that for cases with $\alpha = 0$ or $\beta = 0$ (Dirichlet or Neumann boundary conditions), alternate expressions can be employed:

$$S_{i,j} \,=\, \left(k\,\Omega_j'\left(\Omega_i \,+\, \frac{\beta\,k}{\alpha}\,\Omega_i'\right)\right)\Bigg|_{x=a}^{x=b}, \qquad \text{for} \quad \alpha \neq 0\,, \tag{29}$$

$$S_{i,j} \,=\, -\left(k\,\Omega_j\left(\frac{\alpha}{\beta\,k}\,\Omega_i \,+\, \Omega_i'\right)\right)\Bigg|_{x=a}^{x=b}, \qquad \text{for} \quad \beta \neq 0\,. \tag{30}$$

4. TEST PROBLEM

In order to test the current methodology, a simplified version of the original problem (Helmholtz Equation) with Dirichlet boundary conditions is selected:

$$\Psi''(x) \,+\, \mu^{2}\,\Psi(x) \,=\, 0, \qquad \text{for} \qquad a \leq x \leq b, \tag{31}$$

$$\Psi \,=\, 0, \qquad \text{for} \qquad x \,=\, a, \tag{32}$$

$$\Psi \,=\, 0, \qquad \text{for} \qquad x \,=\, b, \tag{33}$$

This problem has well known analytical solutions in the form

$$\Psi_n(x) \,=\, \sin(\mu_n\,(x-a)), \qquad \text{with} \qquad \mu_n \,=\, \frac{n\,\pi}{b-a}. \tag{34}$$

An auxiliary eigenvalue problem in a form similar to the original one is chosen:

$$\Omega''(x) \,+\, \gamma^2\,\Omega(x) \,=\, 0, \qquad \text{for} \qquad 0 \le x \le 1, \tag{35}$$

$$\mathcal{B}^*\,\Omega \,=\, 0, \qquad \text{for} \qquad x \,=\, 0, \tag{36}$$

$$\mathcal{B}^*\,\Omega \,=\, 0, \qquad \text{for} \qquad x \,=\, 1, \tag{37}$$

where the operator $\mathcal{B}^*$ is defined as:

$$\mathcal{B}^* \,\equiv\, \left(\alpha^*(x) \,+\, \beta^*(x)\,k^*(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{38}$$

However, different combination of the boundary conditions parameters are analyzed for comparison purposes. Regardless of the boundary conditions, for the selected test case, some coefficients are simplified, yielding:

$$B_{i,j} \,=\, \int_a^b \Omega_j\,\Omega_i\,\mathrm{d}x, \qquad A_{i,j} \,=\, -\gamma_j^2\,B_{i,j}, \qquad D_{i,j} \,=\, 0, \qquad S_{i,j} \,=\, \left[(\Omega_j' \,-\, \Omega_j)\Omega_i\right]_{x=a}^{x=b}. \tag{39}$$

The different boundary conditions and the resulting auxiliary eigenfunction, for the analyzed cases are described below:

- Case 1: $\Omega(0) = \Omega(1) = 0$.

$$\Omega_i(x) \,=\, \sqrt{2}\,\sin(\gamma_i\,x), \qquad \gamma_i \,=\, n\,\pi, \qquad n = 1, 2, \ldots \tag{40}$$

- Case 2: $\Omega'(0) = \Omega(1) = 0$.

$$\Omega_i(x) \,=\, \sqrt{2}\,\cos(\gamma_i\,x), \qquad \gamma_i \,=\, (n-1/2)\,\pi, \qquad n = 1, 2, \ldots \tag{41}$$

- Case 3: $\Omega(0) = \Omega'(1) = 0$.

$$\Omega_i(x) \,=\, \sqrt{2}\,\sin(\gamma_i\,x), \qquad \gamma_i \,=\, (n-1/2)\,\pi, \qquad n = 1, 2, \ldots \tag{42}$$

- Case 4: $\Omega'(0) = \Omega'(1) = 0$.

$$\Omega_i(x) \,=\, \sqrt{2}\,\cos(\gamma_i\,x), \qquad \gamma_i \,=\, n\,\pi, \qquad n = 1, 2, \ldots \tag{43}$$

$$\Omega_i(x) \,=\, 1, \qquad \gamma_i \,=\, 0, \qquad (\text{for} \quad n = 0) \tag{44}$$

5. RESULTS AND DISCUSSION

The solutions given in the previous sections were implemented in the Mathematica system (Wolfram, 2003) and are now presented. The first ten eigenvalues are calculated and compared to the exact solution, obtained from equations (34), for different truncation orders (i_{max}) and different values of working precision (WP). The working precision is the number of decimal digits used in the computations. Table 1 and 2 present the results calculated for cases 1, 2, 3 and 4, using $a = 0.25$ and $b = 0.75$. As can be seen, the first eigenvalues converge faster than the last ones. It is also seen that as the truncation order is increased, the required working precision (WP) is also increased. Hence, one can observe a convergence with both truncation order and working precision. Next, tables 3 and 4 present the calculated results for the same four cases, but setting $a = 0.1$ and $b = 0.9$. Analyzing these results one again sees that higher truncation orders are required for the convergence of larger eigenvalues, and a higher working precision is needed as the truncation order is increased.

Comparing the convergence rate resulting from the four different boundary condition cases, one notices that all cases present the same behavior with respect to the truncation order. However, different values of working precision are required for each case. For $a = 0.25$ and $b = 0.75$, case 4 presents the worse convergence rate with WP, followed by case 1. Cases 2 and 3 present similar convergence rates with WP, with case 2 having the best overall performance. Repeating this analysis for $a = 0.1$ and $b = 0.9$, one sees less disparity between all cases; however, it can be seen that case 4 is again the worse option. In addition, comparing the results from the two different domains (given by the values of a and b) it is seen that the domain with $a = 0.25$ and $b = 0.75$ needs higher values of working precision for convergence. This suggests that using an enclosing boundary closer to the original boundary might lead to a better convergence. Nevertheless, the convergence rate with truncation order is the same for the two different analyzed domains.

6. CONCLUSIONS

This paper presented a different approach for solving problems in irregular geometries, consisting of employing a basis of eigenfunctions defined within a domain that encloses the original region. At the current stage of development, a formal solution of one dimensional eigenvalue problems using an auxiliary problem defined within an enclosing region was presented. The methodology was tested on a simple problem with known exact solution for two different domains, and four different types of auxiliary eigenfunctions were employed. The calculated eigenvalues were compared with the known exact values and a convergence analysis was performed. It was seen that the convergence rate depends not only the truncation order, but also on the number of decimal places used in the computations (denoted working precision). The results showed the natural tendency in which the lager eigenvalues require larger truncation orders. All tested cases, for both domains, presented the same convergence behavior with truncation order. Nonetheless, a very different behavior was noticed when the working precision (WP) required for each case was analyzed. The domain in which the boundaries were closer to the the enclosing domain presented a better convergence rate with the WP. Also, the auxiliary eigenfunctions based on mixed boundary condition types (Dirichlet at one end and Neumann at the other) presented better results.

The present methodology is a first step towards the development of an integral transform solution strategy using eigenfunction expansions based on enclosing domains. The results obtained in this work indicates that the method could be applied to multidimensional problems as well. Hence, future developments will be aimed at expanding the methodology to handle 2D and 3D problems. In addition, the solution of general convection-diffusion problems using eigenfunction bases in enclosing domains shall also be performed.

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9. RESPONSIBILITY NOTICE

Table 1. Eigenvalues convergence for case 1 & 2 with $a = 0.25$ and $b = 0.75$.

i_{max}	WP	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
case 1 with $a = 0.25$ and $b = 0.75$.											
10	10	45.1740	157.914	430.263	631.655	791.499	1428.35	1870.17	4131.51	-10214.8	-17395.1
10	15	39.4785	157.914	355.307	631.655	987.775	1427.89	2838.00	4131.72	-11282.8	-17410.3
10	20	39.4785	157.914	355.307	631.655	987.775	1427.89	2838.00	4131.72	-11282.8	-17410.3
15	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
15	15	146.278	157.914	399.829	631.655	752.438	1177.78	1421.22	1558.95	2217.67	2527.55
15	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2527.36	3201.50	4257.65
20	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
20	15	157.914	631.655	1421.22	2526.62	3947.84	-3.7×10^6	-3.2×10^7	-7.8×10^7	1.4×10^8	4.4×10^8
20	20	39.4784	157.914	355.306	526.473	631.655	986.961	1421.22	1905.14	1934.44	2526.62
25	15	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
25	20	21.0530	157.914	171.012	631.655	794.453	1278.33	1421.22	1752.42	complex	complex
25	25	15.5066	141.347	157.914	592.704	631.655	1103.34	1421.22	1622.11	2200.33	2526.62
25	30	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
30	20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
30	25	38.3729	157.914	357.132	631.655	996.811	1218.13	1421.22	complex	complex	1934.86
30	30	38.7866	157.914	354.758	631.655	989.977	1216.39	1421.22	complex	complex	1935.08
30	35	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
35	25	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
35	30	15.6147	140.236	157.914	406.538	631.655	1276.61	1421.22	complex	complex	1847.84
35	35	14.2651	128.314	157.914	357.453	631.655	991.396	1421.22	1568.08	2164.52	2526.62
35	40	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
40	30	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
40	35	16.3457	138.933	157.914	398.654	631.655	781.245	1421.22	1846.90	2522.28	2526.62
40	40	13.8189	124.098	157.914	343.365	631.655	670.846	1421.22	1523.94	2147.03	2526.62
40	45	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
case 2 with $a = 0.25$ and $b = 0.75$.											
10	10	39.4785	157.914	355.306	631.656	988.289	1445.59	2847.55	4591.13	-11188.2	-20924.4
10	15	39.4785	157.914	355.306	631.656	988.289	1445.60	2847.55	4591.12	-11188.2	-20924.4
10	20	39.4785	157.914	355.306	631.656	988.289	1445.60	2847.55	4591.12	-11188.2	-20924.4
15	10	41.0026	158.513	362.807	531.261	693.264	962.556	1268.27	1521.44	1875.95	2582.95
15	15	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2527.64	3211.02	4265.45
15	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2527.64	3211.02	4265.45
20	10	complex	complex	384.495	complex	complex	complex	complex	1284.07	2244.81	2464.15
20	15	37.0887	157.308	355.472	633.063	986.791	1422.47	1936.07	2509.97	3195.94	complex
20	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
25	10	-7.6×10^7	-7.6×10^7	3.0×10^8	3.1×10^8	6.3×10^9	6.3×10^9	-6.8×10^9	-6.9×10^9	4.7×10^{10}	4.8×10^{10}
25	15	complex	complex	364.694	complex	complex	complex	complex	1159.25	complex	complex
25	20	33.5367	161.357	369.527	485.432	673.386	985.955	1292.48	1531.65	1908.99	2327.62
25	25	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
30	10	1514.11	1522.65	2194.61	2209.97	-2308.98	2369.86	2518.30	2562.65	2589.53	2793.35
30	15	91.3830	181.682	344.387	564.878	724.503	1040.72	1329.26	1553.06	2123.19	2602.45
30	20	85.2149	170.573	310.121	524.265	690.203	966.537	1293.09	1533.70	1840.69	2306.27
30	25	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
35	15	348.804	386.131	421.961	1012.28	1087.32	1097.32	1194.50	1893.50	2074.18	2098.06
35	20	complex	complex	340.455	complex	complex	complex	complex	1094.50	complex	complex
35	25	33.4499	160.347	378.815	449.241	663.243	993.985	1269.98	1514.56	1901.3	2365.77
35	30	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
40	15	118.843	224.371	-248.138	277.972	668.842	complex	complex	complex	complex	complex
40	20	89.5126	173.528	345.949	567.996	736.085	1070.28	-1161.77	1375.49	1618.77	2080.00
40	25	83.3932	168.375	303.568	516.647	683.333	955.221	1285.02	1527.76	1933.17	2365.58
40	30	39.3985	157.840	355.435	631.716	986.934	1421.23	1934.49	2526.63	3197.77	3947.93
40	35	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
exact		39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84

Table 2. Eigenvalues convergence for cases 3 & 4 with $a = 0.25$ and $b = 0.75$.

i_{max}	WP	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
case 3 with $a = 0.25$ and $b = 0.75$.											
10	10	39.4785	157.914	355.306	631.656	988.286	1445.67	2843.31	4598.02	-11379.6	-20672.9
10	15	39.4785	157.914	355.306	631.656	988.286	1445.67	2843.31	4598.01	-11379.6	-20672.9
10	20	39.4785	157.914	355.306	631.656	988.286	1445.67	2843.31	4598.01	-11379.6	-20672.9
15	10	38.6644	160.086	362.555	529.058	694.100	962.579	1267.03	1526.430	1858.45	2617.51
15	15	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2527.640	3211.03	4265.22
15	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2527.640	3211.03	4265.22
20	10	89.5080	174.641	323.034	536.539	701.244	988.377	1302.31	1483.40	1962.77	2368.9
20	15	36.6261	158.314	354.733	631.013	986.346	1421.08	1931.47	2513.25	3218.03	complex
20	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
25	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
25	15	complex	complex	368.849	complex	complex	complex	complex	1195.53	complex	complex
25	20	33.5397	161.354	369.529	485.441	673.379	985.965	1292.49	1531.65	1909.00	2327.64
25	25	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
30	10	596.455	1681.39	1714.96	2043.99	2140.39	2284.66	2481.09	2577.54	2585.42	2717.59
30	15	94.0018	178.790	344.758	563.233	723.504	1040.21	1327.29	1551.82	2077.80	2125.26
30	20	85.2178	170.571	310.121	524.270	690.199	966.537	1293.09	1533.69	1840.65	2306.30
30	25	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
35	15	1599.09	1796.17	1833.65	2185.00	3054.22	3082.59	3287.68	3331.72	3374.68	3504.97
35	20	complex	complex	320.479	complex	complex	complex	complex	1156.36	complex	complex
35	25	33.4512	160.346	378.814	449.246	663.240	993.988	1269.99	1514.55	1901.31	2365.77
35	30	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
40	15	-113.927	224.534	245.019	complex	complex	complex	complex	complex	complex	1794.55
40	20	91.6020	178.651	341.830	576.829	736.631	1075.99	1375.99	-1503.22	1608.44	2079.31
40	25	83.3945	168.375	303.568	516.650	683.330	955.223	1285.02	1527.76	1933.18	2365.58
40	30	39.4768	157.919	355.297	631.665	986.953	1421.22	1934.45	2526.61	3197.76	3947.83
40	35	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
40	40	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
case 4 with $a = 0.25$ and $b = 0.75$.											
10	10	39.4784	157.914	355.306	462.612	631.656	1454.72	4693.00	7079.47	-17260.3	-20874.3
10	15	39.4784	157.914	355.306	631.655	991.157	1438.79	3151.59	4587.22	-13811.2	-20802.4
10	20	39.4784	157.914	355.306	631.655	991.157	1438.79	3151.58	4587.22	-13811.2	-20802.4
15	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
15	15	39.4784	157.929	355.306	579.800	631.673	986.960	1364.18	1421.18	1934.44	2546.74
15	20	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2529.18	3206.93	4382.92
20	15	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
20	20	39.4784	54.4207	355.306	440.617	780.577	986.960	1187.76	1597.77	1934.44	2213.75
20	25	39.4784	161.282	355.306	631.343	986.960	1419.57	1934.44	2531.11	3197.75	3981.01
20	30	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
25	20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
25	25	39.4784	228.040	355.306	523.084	913.851	986.960	1314.41	1702.49	1934.44	2771.26
25	30	39.4784	181.361	355.306	431.917	767.766	986.960	1179.77	1663.21	1934.44	2164.30
25	35	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
30	25	39.4784	complex	complex	355.306	-846.205	857.793	986.960	1239.04	1691.64	complex
30	30	39.4784	58.6220	228.907	355.306	843.555	986.960	1235.91	complex	complex	1682.74
30	35	39.4784	50.2357	200.245	355.306	755.221	986.960	1170.42	1655.58	1934.44	2041.51
30	40	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
35	30	39.4784	162.407	-294.321	355.306	complex	complex	complex	complex	986.960	1508.39
35	35	39.4784	57.6807	230.039	355.306	518.566	986.960	1287.03	1742.94	complex	complex
35	40	39.4784	49.1781	195.608	355.306	436.430	986.960	1164.96	1653.63	1934.44	2214.51
35	45	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
40	35	39.4784	154.071	355.306	629.829	717.867	986.960	1420.60	complex	complex	1761.45
40	40	39.4784	158.292	355.306	631.055	717.213	986.960	1430.17	1784.95	1934.44	2527.32
40	45	39.4784	157.914	355.306	488.246	631.655	986.960	1353.18	1421.22	1934.44	2510.83
40	50	39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84
exact		39.4784	157.914	355.306	631.655	986.960	1421.22	1934.44	2526.62	3197.75	3947.84

Table 3. Eigenvalues convergence for cases 1 & 2 with $a = 0.1$ and $b = 0.9$.

i_{max}	WP	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
case 1 with $a = 0.1$ and $b = 0.9$.											
10	10	18.5866	64.3669	-89.5586	151.764	252.976	399.137	-400.153	560.301	760.649	986.96
10	15	18.5866	64.3669	-89.5586	151.764	252.976	399.137	-400.153	560.301	760.649	986.96
10	20	18.5866	64.3669	-89.5586	151.764	252.976	399.137	-400.153	560.301	760.649	986.96
15	10	15.4371	61.7999	138.904	247.054	385.702	555.440	755.699	986.960	1249.28	1544.84
15	15	15.4371	61.7999	138.904	247.054	385.702	555.440	755.699	986.960	1249.28	1544.84
15	20	15.4371	61.7999	138.904	247.054	385.702	555.440	755.699	986.960	1249.28	1544.84
20	10	15.4224	61.6873	138.799	246.746	385.543	555.170	755.645	986.960	1249.13	1542.15
20	15	15.4224	61.6873	138.799	246.746	385.543	555.170	755.645	986.960	1249.13	1542.15
20	20	15.4224	61.6873	138.799	246.746	385.543	555.170	755.645	986.960	1249.13	1542.15
25	10	15.4811	61.6795	138.234	246.739	385.414	555.175	755.632	986.960	1249.1	1542.12
25	15	15.4213	61.6852	138.791	246.741	385.532	555.166	755.642	986.960	1249.12	1542.13
25	20	15.4213	61.6852	138.791	246.741	385.532	555.166	755.642	986.960	1249.12	1542.13
30	10	40.4698	113.673	204.035	407.534	728.753	732.376	986.960	1049.71	1168.44	1861.93
30	15	15.4213	61.6851	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
30	20	15.4213	61.6851	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	10	986.960	1172.81	1639.92	1902.37	2091.16	2789.95	3078.92	3264.16	3304.89	3454.66
35	15	15.4213	61.6851	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	20	15.4213	61.6851	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
40	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
40	15	12.6519	61.6856	119.455	246.739	555.167	577.725	964.269	986.960	1442.37	1542.13
40	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
case 2 with $a = 0.1$ and $b = 0.9$.											
10	10	17.4998	-63.6883	66.5607	146.397	254.781	392.374	563.037	759.556	1094.27	-2448.56
10	15	17.4998	-63.6883	66.5607	146.397	254.781	392.374	563.037	759.556	1094.27	-2448.56
10	20	17.4998	-63.6883	66.5607	146.397	254.781	392.374	563.037	759.556	1094.27	-2448.56
15	10	15.4364	61.7466	138.902	246.900	385.717	555.290	755.761	987.037	1249.33	1544.59
15	15	15.4364	61.7466	138.902	246.900	385.717	555.290	755.761	987.037	1249.33	1544.59
15	20	15.4364	61.7466	138.902	246.900	385.717	555.290	755.761	987.037	1249.33	1544.59
20	10	15.4219	61.6871	138.795	246.746	385.538	555.170	755.642	986.962	1249.13	1542.14
20	15	15.4219	61.6871	138.796	246.746	385.537	555.170	755.643	986.961	1249.13	1542.14
20	20	15.4219	61.6871	138.796	246.746	385.537	555.170	755.643	986.961	1249.13	1542.14
25	10	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
25	15	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
25	20	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
30	10	15.4211	61.6850	138.791	246.740	385.534	555.169	755.653	986.954	1249.11	1542.13
30	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
30	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	10	50.4849	108.668	212.020	314.497	477.972	625.542	844.621	1043.62	1308.46	1568.97
35	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
40	10	51.0333	122.390	complex	complex	443.473	614.086	825.124	1033.09	1294.16	1555.43
40	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
40	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
exact		15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13

Table 4. Eigenvalues convergence for cases 3 & 4 with $a = 0.1$ and $b = 0.9$.

i_{max}	WP	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
		case 3 with $a = 0.1$ and $b = 0.9$.									
10	10	16.6794	65.0525	-126.964	144.706	253.481	391.582	562.552	759.530	1095.61	-2318.00
10	15	16.6794	65.0525	-126.964	144.706	253.481	391.582	562.552	759.529	1095.61	-2318.00
10	20	16.6794	65.0525	-126.964	144.706	253.481	391.582	562.552	759.529	1095.61	-2318.00
15	10	15.4359	61.7448	138.899	246.896	385.712	555.287	755.760	987.037	1249.33	1544.57
15	15	15.4359	61.7448	138.899	246.896	385.712	555.287	755.760	987.037	1249.33	1544.57
15	20	15.4359	61.7448	138.899	246.896	385.712	555.287	755.760	987.037	1249.33	1544.57
20	10	15.4219	61.6871	138.795	246.746	385.538	555.170	755.642	986.962	1249.13	1542.14
20	15	15.4219	61.6871	138.795	246.746	385.537	555.170	755.643	986.961	1249.13	1542.14
20	20	15.4219	61.6871	138.795	246.746	385.537	555.170	755.643	986.961	1249.13	1542.14
25	10	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
25	15	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
25	20	15.4213	61.6851	138.791	246.740	385.532	555.165	755.642	986.960	1249.12	1542.13
30	10	15.4211	61.6850	138.791	246.740	385.534	555.169	755.653	986.954	1249.11	1542.13
30	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
30	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	10	50.4850	108.667	212.020	314.495	477.974	625.538	844.625	1043.61	1308.47	1568.95
35	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
35	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
40	10	51.0338	122.389	complex	complex	443.479	614.080	825.128	1033.08	1294.16	1555.43
40	15	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
40	20	15.4213	61.6850	138.791	246.740	385.531	555.165	755.642	986.960	1249.12	1542.13
		case 4 with $a = 0.1$ and $b = 0.9$.									
10	10	15.4422	61.7092	138.838	246.74	385.847	556.311	794.983	1072.08	-1559.16	-2388.64
10	15	15.4422	61.7092	138.838	246.74	385.847	556.311	794.983	1072.08	-1559.16	-2388.64
10	20	15.4422	61.7092	138.838	246.74	385.847	556.311	794.983	1072.08	-1559.16	-2388.64
15	10	15.4214	61.6944	138.792	246.74	385.533	555.201	755.687	987.295	1249.41	1543.94
15	15	15.4214	61.686	138.792	246.74	385.533	555.194	755.688	987.293	1249.41	1543.94
15	20	15.4214	61.686	138.792	246.74	385.533	555.194	755.688	987.293	1249.41	1543.94
20	10	13.1034	61.6838	145.534	246.74	390.386	555.17	758.952	986.963	1250.6	1541.93
20	15	15.4213	61.685	138.791	246.74	385.532	555.165	755.643	986.962	1249.13	1542.13
20	20	15.4213	61.685	138.791	246.74	385.532	555.165	755.643	986.962	1249.13	1542.13
25	10	62.926	66.5679	246.74	302.313	554.054	611.358	986.576	1012.93	1513.1	1535.59
25	15	15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.961	1249.12	1542.13
25	20	15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.961	1249.12	1542.13
30	10	—	—	—	—	—	—	—	—	—	—
30	15	15.4213	61.6847	138.791	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13
30	20	15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13
35	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
35	15	15.4211	138.789	246.74	246.872	385.532	675.772	755.641	1089.42	1249.12	1585.22
35	20	15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13
40	10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
40	15	13.6667	61.864	246.74	281.68	555.22	605.148	986.936	1004.05	1504.53	1542.2
40	20	15.4213	61.685	138.792	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13
40	25	15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13
exact		15.4213	61.685	138.791	246.74	385.531	555.165	755.642	986.96	1249.12	1542.13