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IMPROVED LUMPED-DIFFERENTIAL FORMULATIONS FOR LAMINAR FLOW IN POLYGONAL CROSS-SECTION DUCTS

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Abstract. *This paper shows how improved lumped approximation techniques can be employed for calculating pressure gradients in laminar flows through polygonal cross-section ducts. The adopted methodology consists in reducing the number of spatial dimensions in the governing equations with approximation rules provided by the Coupled Integral Equations Approach (C.I.E.A.), significantly diminishing the computational effort. The transformed system studied in this work can be directly integrated yielding analytical solutions for the averaged velocity and friction factor. The results of the simplified formulations are compared with solutions to the complete problem (without approximations) and a reasonable agreement is observed. An error analysis of the results indicates regions for applicability of the methodology where accuracy requirements can be maintained. Based on a comparison of the results with solutions previously available in the literature, an analysis of errors leads us to regions of applicability of the solution, where the accuracy requirements can be maintained.*

Keywords: *Lumped Diferential Formulation; Coupled Integral Equations Approach; Friction Factor; Laminar Duct Flow*

1. NOMENCLATURE

$H_{\alpha,\beta}$	Hermite approximation;
a, b, c	geometric parameters in cross-section domain;
z, y	cross-section variables;
x	axial variable;
z_1	function for left boundary;
Y, Z	dimensionless problem variables;
Γ	dimensionless function for left boundary;
P	hydrostatic pressure;
u	axial velocity;
$\bar{u}$	cross-sectional averaged velocity;
U	dimensionless axial velocity;
K	aspect ratio;
G_P	dimensionless pressure gradient;
f	Fanning's friction-factor;
D_H	dimensionless hydraulic diameter;
k_j^*, h_j^*	boundary condition parameters.
Greek Symbols	
α, β, ν	Hermite approximation parameters;
ξ, η	dimensionless boundary parameters;
μ	dynamic viscosity;
ζ_1, ζ_2, ζ_3	solution constants.
Subscripts	
i	Hermite approximation parameter;

2. INTRODUCTION

For centuries, analytical methods were the only solution available for heat and fluid flow problems. After the introduction of computers, numerical methods greatly evolved and its widespread usage became inevitable. Nowadays, with the availability of closed-packages for solving engineering problems (which are mostly based on numerical techniques), analytical methods became old-fashioned, and on several occasions numerical techniques are used for problems that posses analytical solutions. Nevertheless, analytical solutions still play a crucial role in the development of science and engineering, and its relevance should not be overlooked.

The approximation of an integral by a linear combination of the integrand values and its derivatives at the integration limits, was originally developed by Hermite (1878) and first presented by Menning, Auerbach et al. (1983). They were the first to use this two-point approach, deriving it in a fully differential form called $H_{\alpha,\beta}$. It was shown that the already known Obreschkoff formulae didn't present any new features in relation to the $H_{\alpha,\beta}$ method, which generated more accurate solutions to linear ODE systems (initial-value and boundary-value problems) compared to other methods.

Such approximations may simplify the problem at hand to such an extent that even analytical methods can be used. Nevertheless, the error involved in approximations should be controlled for maintaining precision requirements. One technique that take this into account is the Coupled Integral Equations Approach, which is based on the above cited literature. This method was used in a variety of problems such as drying (Dantas, Orlande et al., 2007), ablation (Ruperti, Cotta et al., 2004), fins (Aparecido and Cotta, 1989), hyperbolic heat conduction (Reis, Macêdo et al., 2000), radiative cooling (Su, 2004), conduction with temperature-dependent conductivity (Su, Tan et al., 2009) and heat exchangers (Neto and Cotta, 1993), besides the solutions to diffusion problems obtained by Corrêa and Cotta (1998).

This paper presents an approximate analytical methodology based on the Coupled Integral Equations Approach for solving steady laminar flow in irregular geometries and ultimately obtaining the friction factor. Simulation results for different polygonal cross-section ducts, including rectangular, trapezoidal and hexagonal geometries are carried out. Four different approximation cases are employed and the results are compared with previously published data. An error analysis of the results is carried out and used to guide the selection of which approximation should be used in order to minimize the error for each problem.

3. HERMITE APPROXIMATION

The basis for the Coupled Integral Equations Approach (CIEA) is the Hermite approximation of an integral, denoted, $H_{\alpha,\beta}$, which is given by the general expression:

$$\int_{x_{i-1}}^{x_i} f(x)dx = \sum_{\nu=0}^{\alpha} c_\nu(\alpha,\beta) h_i^{\nu+1} f^{(\nu)}(x_{i-1}) + \sum_{\nu=0}^{\beta} c_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} f^{(\nu)}(x_i) + Error_{\alpha,\beta} \tag{1}$$

where,

$$h_i = x_i - x_{i-1}, \qquad c_\nu(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta-\nu+1)!}{(\nu+1)!(\alpha-\nu)!(\alpha+\beta+2)!} \tag{2}$$

and $f(x)$ and its derivatives $f^{(\nu)}(x)$ are defined for all $x \in [x_{i-1}, x_i]$. It is assumed that $f^{(\nu)}(x_{i-1}) = f_{i-1}^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \alpha$ and $f^{(\nu)}(x_i) = f_i^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \beta$.

This integration formula can provide different approximation levels, from the classical lumped system analysis to improved lumped-differential formulations. A detailed error analysis of the application of the CIEA to diffusion problems using $H_{0,0}$, $H_{0,1}$, $H_{1,0}$, and $H_{1,1}$ Hermite approximations was carried out in (Alves, Sphaier et al., 2000). Since approximations of order higher than $H_{1,1}$ involve derivatives of order higher than one, these are avoided for the sake of simplicity of the methodology. Hence, only the two different approximations below are considered:

$$H_{0,0} \Rightarrow \int_0^h f(x)dx \approx \frac{1}{2}h(f(0)+f(h)), \tag{3}$$

$$H_{1,1} \Rightarrow \int_0^h f(x)dx \approx \frac{1}{2}h(f(0)+f(h)) + \frac{1}{12}h^2(f'(0)-f'(h)), \tag{4}$$

which correspond to the well known trapezoidal and corrected trapezoidal integration rules, respectively.

4. LAMINAR FLOW IN POLYGONAL CROSS-SECTION DUCTS

In this section, the integral approximation rules (3,4) are applied to fully-developed laminar flow in straight ducts with polygonal cross-sections, allowing expressions for the friction-factor in different geometries to be obtained. The tested cross-section geometries were rectangle, trapezoid and hexagon. Figure 1 displays the general problem domain with the adopted geometric parameters. The function $z_1(y)$ describes the right boundary between $y = a$ and $y = c$. The rectangular and trapezoidal geometries are part of a group where a Dirichlet boundary condition is prescribed in $y = 0$ due to the no slip restriction. In the remaining case there is a Neumann boundary condition at $y = 0$ because of the symmetry about the z axis.

The dimensionless fluid-flow problem is the geometry considered is given by the following equations:

$$\frac{\partial^2 U}{\partial Y^2} + K^2\frac{\partial^2 U}{\partial Z^2} = G_P, \qquad \text{for} \qquad 0 \le Y \le 1 \quad \text{and} \quad 0 \le Z \le 1, \tag{5}$$

$$h_j^*\frac{\partial U}{\partial Y} + k_j^* U = 0, \qquad \text{for} \qquad Y = 0 \quad \text{and} \quad 0 \le Z \le 1, \tag{6}$$

$$U = 0, \qquad \text{for} \qquad Y = 1 \quad \text{and} \quad 0 \le Z \le 1, \tag{7}$$

$$\frac{\partial U}{\partial Z} = 0, \qquad \text{for} \qquad Z = 0 \quad \text{and} \quad 0 \le Y \le 1, \tag{8}$$

$$U = 0, \qquad \text{for} \qquad Z = Z_1(Y) \quad \text{and} \quad 0 \le Y \le 1, \tag{9}$$

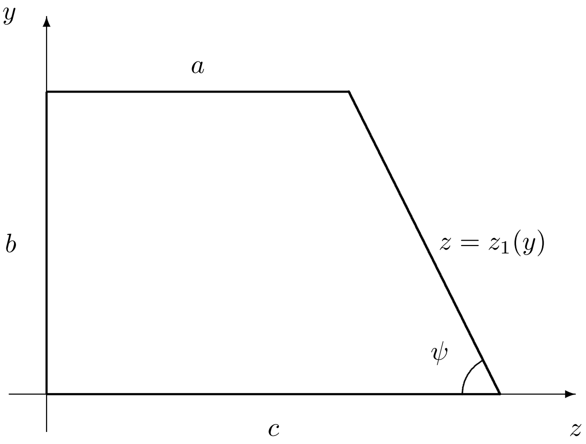

Figure 1. Problem domain

where the boundary condition at $y = 0$ is written in a generalized form, according to the parameters h_j^* and k_j^*. The current geometric group will depend on the value of j: $j = 1$ for the first geometric group (trapezoid and rectangle) and $j = 2$, for the second (hexagon). Hence, the values for the boundary condition parameters are $h_1^* = k_2^* = 0$ and $h_2^* = k_1^* = 1$.

The involved dimensionless groups are given by:

$$Y = \frac{y}{b}, \qquad Z = \frac{z}{a}, \qquad U = \frac{u}{\bar{u}}, \qquad K = \frac{b}{a}, \qquad G_P = \frac{b^2}{\mu\,\bar{u}}\frac{\mathrm{d}P}{\mathrm{d}x}, \tag{10}$$

Where $U = U(Y, Z)$ and $\Gamma(Y) = \xi\,Y + \eta$ is the dimensionless form of $z_1(y)$. Expressions for ξ and η are easily obtained, giving:

$$\xi = -K\,\cot(\psi) \qquad \text{and} \qquad \eta = 1 + K\,\cot(\psi), \tag{11}$$

The expression of Fanning's friction-factor can be readily expressed in terms of the dimensionless pressure gradient as:

$$f\,\mathrm{Re} = -\frac{G_P}{2}\frac{D_H^2}{b^2}, \tag{12}$$

in which G_P is the dimensionless pressure gradient, D_H is the hydraulic diameter and b is a geometric parameter. The hydraulic diameter depends on the type of the duct cross-section geometry, so we express it in a generalized form:

$$\frac{D_H^2}{b^2} = \left(\frac{4 + 2\,K\,\cot(\psi)}{1 + K\,\csc(\psi) + k_j^*\,(1 + K\,\cot(\psi))}\right)^2, \tag{13}$$

where the Reynolds number is based on the hydraulic diameter. The average velocity in the horizontal direction is defined as:

$$U_{av}(Y) = \frac{\int_0^{Z_1} U(Y,Z)\,\mathrm{d}Z}{\xi\,Y + \eta}. \tag{14}$$

4.1 Integration and approximation

With little algebraic manipulation, the integration of equations (5,6,7) within $0 \le Z \le \Gamma$ followed by the substitution of equations (8,9) and (14), yields:

$$\frac{\mathrm{d}^2 U_{av}}{\mathrm{d}Y^2} + \frac{2\,\xi}{\xi\,Y + \eta}\frac{\mathrm{d}U_{av}}{\mathrm{d}Y} + \frac{(K^2 + \xi^2)}{\xi\,Y + \eta}\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma} = G_P, \quad \text{for} \quad 0 \le Y \le 1, \tag{15}$$

$$\eta\,h_j^*\,\frac{\mathrm{d}U_{av}}{\mathrm{d}Y} + (\xi\,h_j^* + \eta\,k_j^*)\,U_{av} = 0, \quad \text{at} \quad Y = 0, \tag{16}$$

$$U_{av} = 0, \quad \text{at} \quad Y = 1, \tag{17}$$

where the following relation between the Y- and Z-derivatives of U at $Z = \Gamma$, obtained from the boundary condition at this location (9), was used for simplification:

$$\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma} = -\,\xi\,\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma}. \tag{18}$$

Equations (15-17) are an exact form of system (5-9) transformed to eliminate the independent variable Z, *i.e.* no approximations have been used. In order to solve equations (15-17) the derivatives of the two-dimensional velocity field U at $Z = \Gamma(Y)$ must be expressed in terms of the Z-averaged velocity, $U_{av}(Y)$. At this point no further exact transformations can be of assistance, and the need for an approximation rule becomes clear. Observing equations (15) and (18), the CIEA methodology should be applied to generate relations between the unknown derivatives ($\partial U/\partial Z|_{Z=\Gamma}$ and $\partial U/\partial Y|_{Z=\Gamma}$) and the averaged velocity $U_{av}(Y)$; however, two different approximation alternatives arise:

1. Apply the CIEA using Hermite formulas for the integrals of $U(Y,Z)$ and $\partial U/\partial Z$.
2. Apply the CIEA using Hermite formulas for the integrals of $U(Y,Z)$ and $\partial U/\partial Y$.

Although there are no differences between the above alternatives regarding the number of approximations utilized, different results can be obtained for each case. As it is shown further, using $H_{0,0}$ or $H_{1,1}$ formulas for $U(Y,Z)$, together with $H_{0,0}$ formulas for the derivatives of U, leads to approximation rules in the following form:

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma} = -\frac{1}{\xi}\,\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma} \approx -\gamma_1\,\frac{U_{av}}{\xi\,Y+\eta} + \frac{\gamma_2}{\xi}\,\frac{\mathrm{d}U_{av}}{\mathrm{d}Y}, \tag{19}$$

where γ_1 and γ_2 are parameters whose values depend on the type of approximation used. These will be determined in the next section. For rectangular ducts, relation (18) does not hold, and only the first approximation alternative is possible. For these type of ducts (with $\psi = \pi/2$), a general approximation relation without γ_2 is obtained:

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma} \approx -\gamma_1\,\frac{U_{av}}{\xi\,Y+\eta} = -\gamma_1\,U_{av}, \tag{20}$$

where the second equality reflects the fact that $\eta = 1$ and $\xi = 0$ for rectangular ducts. Using the general approximations (19,20), the system given by equations (15-17) is solved, yielding:

$$\frac{U_{av}(Y)}{G_P} = \zeta_1\,(\xi\,Y+\eta)^{\gamma_3+\chi} + \zeta_2\,(\xi\,Y+\eta)^{\gamma_3-\chi} + \zeta_3\,(\xi\,Y+\eta)^2, \tag{21}$$

for ducts with $\psi \neq \pi/2$, and the simplified form for rectangular profiles:

$$\frac{U_{av}(Y)}{G_P} = \zeta_1\,\sinh(K\,Y\,\sqrt{\gamma_1}) + \zeta_2\,\cosh(K\,Y\,\sqrt{\gamma_1}) + \zeta_3, \tag{22}$$

where ζ_1, ζ_2 are integration constants (depending on the boundary conditions). The other parameters are given by:

$$\gamma_3 = -\frac{1}{2}\left(1 + \frac{\gamma_2}{\xi^2}\,(K^2+\xi^2)\right) = -\frac{1}{4}\,(1+2\,\gamma_2+\cos(2\,\psi))\,\sec(\psi)^2, \tag{23}$$

$$\zeta_3 = \frac{-1}{K^2\,\gamma_1 + (4\,\gamma_3-4+\gamma_1)\,\xi^2} = \frac{\sin^2(\psi)}{K^2\,(3-\gamma_1+2\,\gamma_2+3\cos(2\,\psi))}, \tag{24}$$

$$\chi = \sqrt{\gamma_1\,\left(1+\frac{K^2}{\xi}\right) + \gamma_3^2} = \sqrt{\gamma_1\,\sec(\psi) + \gamma_3^2}\,. \tag{25}$$

For the rectangular profile ($\psi = \pi/2$):

$$\frac{U_{av}(Y)}{G_P} = \zeta_3\left(1 - \frac{\cosh(K\,(Y-1/2)\,\sqrt{\gamma_1})}{\cosh(K\,\gamma_1/2)}\right), \qquad \text{with} \qquad \zeta_3 = -\frac{1}{K^2\gamma_1}. \tag{26}$$

Using the definition of the cross-sectional averaged velocity:

$$\frac{1}{A}\int_0^b\int_0^{z_1(y)} u\,\mathrm{d}z\,\mathrm{d}y = \bar{u}, \qquad \text{with} \qquad A = \frac{(a+c)}{2}\,b, \tag{27}$$

one can write

$$\frac{1}{A^*}\int_0^1 U_{av}(Y)\,\mathrm{d}Y \;=\; 1 \qquad \text{where} \qquad A^* \;=\; \frac{1\,+\,\eta}{2}, \tag{28}$$

and hence the dimensionless pressure gradient can be calculated from:

$$G_P \;=\; \left(\frac{1}{A^*}\int_0^1 \frac{U_{av}}{G_P}\,\mathrm{d}Y\right)^{-1} \;=\; A^*\,\xi\,\left(\int_\eta^{\xi+\eta}\frac{U_{av}}{G_P}\,\mathrm{d}\Gamma\right)^{-1}, \tag{29}$$

where a change of variable was performed to facilitate integration, which yields:

$$\frac{A^*}{G_p} \;=\; \frac{\zeta_2\xi}{\gamma_3-\chi+1}\left(1-\eta^{\gamma_3-\chi+1}\right)+\frac{\zeta_1\xi}{\gamma_3+\chi+1}\left(1-\eta^{\gamma_3+\chi+1}\right)+\frac{1}{3}\zeta_3\left(1-\eta^3\right)\xi. \tag{30}$$

Once G_P has been calculated, the friction factor is readily obtained from equation (12). This results in an expression for the friction-factor parameterized by the geometric relations (K and ψ), the geometry-type parameters (k_j^* and h_j^*) and by parameters that depend on the type of approximation (γ_1 and γ_2).

For rectangular ducts, a general expression, valid for two levels of approximations (indicated by the value of γ_1), is given by:

$$f\,\mathrm{Re} \;=\; \frac{8\,K^3\,\gamma_1^{3/2}}{(K+2)^2\left(K\sqrt{\gamma_1}-2\tanh\left(\frac{1}{2}\,K\,\sqrt{\gamma_1}\right)\right)}. \tag{31}$$

5. COUPLED INTEGRAL EQUATIONS APPROACH

Now that a general expression for the friction factor is available, four different approximation schemes are tested, each leading to different values for the parameters γ_1 and γ_2. For the sake of simplicity, no Hermite approximations of order higher than $H_{1,1}$ are used and the $H_{1,1}$ approximation is used solely for the integral of $U(Y,Z)$.

5.1 First case - $H_{0,0}/H_{0,0}$, alternative 1

In this case the $H_{0,0}$ approximation is used to yield expressions for the integrals (within the horizontal coordinate) of the velocity profile and its derivative with respect to Z:

$$\int_0^{\Gamma(Y)} U(Y,Z)\,\mathrm{d}Z \;\approx\; \frac{1}{2}\,(\xi\,Y+\eta)\,(U(Y,0)+U(Y,\Gamma(Y))), \tag{32}$$

$$\int_0^{\Gamma(Y)} \frac{\partial U}{\partial Z}\,\mathrm{d}Z \;\approx\; \frac{1}{2}\,(\xi\,Y+\eta)\left(\left.\frac{\partial U}{\partial Z}\right|_{Z=0} \;+\; \left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma}\right). \tag{33}$$

The above equations are solved for the unknown potential and derivative, substituting the boundary information and the definition of average potential, to give:

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma} \;=\; -\frac{4\,U_{av}(Y)}{(\xi\,Y+\eta)} \qquad \text{and} \qquad U(Y,0) \;=\; 2\,U_{av}(Y), \tag{34}$$

which leads to the following values for the γ-parameters:

$$\gamma_1 \;=\; 4, \qquad \gamma_2 \;=\; 0. \tag{35}$$

5.2 Second case - $H_{0,0}/H_{0,0}$, alternative 2

In this case the $H_{0,0}$ approximation is again used to yield expressions for the integral of the velocity profile (32). However, instead of approximating the integral of the velocity's Z-derivative, the $H_{0,0}$ rule is used with the Y-derivative:

$$\int_0^{Z_1(Y)} \frac{\partial U}{\partial Y}\,\mathrm{d}Z \;\approx\; \frac{1}{2}\,(\xi\,Y+\eta)\left(\left.\frac{\partial U}{\partial Y}\right|_{Z=0} \;+\; \left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma(Y)}\right) \tag{36}$$

Solving the above equation, together with (32), for the unknown potential and its derivative, and substituting the boundary information yields:

$$\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma(Y)} \;=\; 2\,\xi\,\frac{U_{av}(Y)}{\xi\,Y+\eta} \;+\; 2\,\frac{\mathrm{d}U_{av}}{\mathrm{d}Y} \;-\; \left.\frac{\partial U}{\partial Y}\right|_{Z=0}, \tag{37}$$

$$U(Y,0) \;=\; 2\,U_{av}(Y), \tag{38}$$

Simplifying gives:

$$\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma(Y)} = 2\,\xi\,\frac{U_{av}(Y)}{\xi\,Y+\eta} \tag{39}$$

which results in the following values for γ_1 and γ_2:

$$\gamma_1 = 2, \qquad \gamma_2 = 0. \tag{40}$$

5.3 Third case - $H_{1,1}/H_{0,0}$, alternative 1

In this case the $H_{0,0}$ approximation is used to yield an expression for the integral of the velocity's Z-derivative (33), and, for the integral of the velocity profile, the $H_{1,1}$ approximation is used as follows:

$$\int_0^{Z_1(Y)} U(Y,Z)\,\mathrm{d}Z \approx \frac{1}{2}\,(\xi\,Y+\eta)\,(U(Y,0)+U(Y,\Gamma)) + \frac{1}{12}\,(\xi\,Y+\eta)^2\left(\left.\frac{\partial U}{\partial Z}\right|_{Z=0} - \left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma}\right) \tag{41}$$

The above equation, together with (33), is solved for the unknown potential and its derivative to give:

$$\left.\frac{\partial U}{\partial Z}\right|_{Z=\Gamma(Y)} = -\frac{3\,U_{av}(Y)}{(\xi\,Y+\eta)} \qquad \text{and} \qquad U(Y,0) = \frac{3}{2}\,U_{av}(Y) \tag{42}$$

leading to the following values for the γ-parameters:

$$\gamma_1 = 3, \qquad \gamma_2 = 0, \tag{43}$$

5.4 Fourth case - $H_{1,1}/H_{0,0}$, alternative 2

In this case the $H_{1,1}$ approximation is again used to yield expressions for the integral of the velocity profile, equation (41). However, now the $H_{0,0}$ rule is applied to the integral of the Y-derivative (as in the second case, equation (36)). Solving equations (33 , 41) and (36) for the unknown potential and its derivative, the following expressions are found:

$$\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma} = 2\,\xi\,\frac{U_{av}(Y)}{\xi\,Y+\eta} + 2\,\frac{\mathrm{d}U_{av}}{\mathrm{d}Y} - \left.\frac{\partial U}{\partial Y}\right|_{Z=0} \tag{44}$$

$$U(Y,0) = \frac{3}{2}\,U_{av}(Y) \tag{45}$$

Simplifying gives:

$$\left.\frac{\partial U}{\partial Y}\right|_{Z=\Gamma} = 2\,\xi\,\frac{U_{av}(Y)}{\xi\,Y+\eta} + \frac{1}{2}\,\frac{\mathrm{d}U_{av}}{\mathrm{d}Y} \tag{46}$$

and the following values for the approximation-type parameters are obtained:

$$\gamma_1 = 2, \qquad \gamma_2 = -\frac{1}{2} \tag{47}$$

6. RESULTS AND DISCUSSION

Now that the solution methodology has been presented, friction factor results for different cross-section geometries are presented.

6.1 Rectangle

For rectangular ducts, the following explicit expressions for the friction-factor are obtained:

$$f\,\mathrm{Re} = \frac{32\,K^3}{(K+2)^2\,(K-\tanh(K))}, \qquad \text{for case 1,} \tag{48}$$

$$f\,\mathrm{Re} = \frac{24\,K^3\,\sqrt{3}}{(K+2)^2\left(K\,\sqrt{3}-2\tanh\left(\frac{K}{2}\,\sqrt{3}\right)\right)}, \qquad \text{for case 3.} \tag{49}$$

A correlation obtained from an analytical solution to these types of ducts is given by (Shah and London, 1978):

$$f\,\mathrm{Re} = 24(1-0.2537\,K^5+0.9564\,K^4-1.7012\,K^3+1.9467\,K^2-1.3553\,K) \tag{50}$$

The results calculated for the two types of approximations are presented in table 1. As can be seen, case 3 ($H_{0,0}/H_{1,1}$) presents better results than case 1 ($H_{0,0}/H_{0,0}$), having a maximum relative error of 7% for $K = 0.405$. It is clear that the error is not uniform with K, and its average value is 6.1% for case 1 and 3.6% for case 3. Note also, that the solution converges to the problem of laminar flow between parallel plates when the aspect ratio approaches zero.

Table 1. Friction-factor in rectangular duct for different aspect ratios.

	case 1		case 3		
K	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	eq. (50)
0.005	23.8807	0.1	23.8806	0.1	23.8385
0.105	21.7610	3	21.7371	3	21.0552
0.205	20.0767	6	19.9938	5	18.9820
0.305	18.7408	7	18.5729	6	17.4495
0.405	17.6856	8	17.4137	7	16.3288
0.505	16.8575	9	16.4682	6	15.5234
0.605	16.2144	8	15.6981	5	14.9624
0.705	15.7225	8	15.0730	3	14.5931
0.805	15.3545	7	14.5679	1	14.3738
0.905	15.0885	6	14.1629	1	14.2663
1.005	14.9065	5	13.8413	3	14.2286

6.2 Trapezoid

The results for a trapezoidal cross-section duct are presented in table 2, for various aspect ratios and trapezoid angles, using all four approximation alternatives. In order to analyze the trapezoidal profiled duct, exact solutions (Aparecido, 1988) using the Generalized Integral Transform Technique (GITT) were used for comparisons. It is clear that for the first approximation alternative (cases 1 and 3), the $H_{0,0}/H_{1,1}$ approximations yield better results than its $H_{0,0}/H_{0,0}$ counterpart. However, the opposite trend is generally seen for the second approximation alternative (cases 2 and 4), with $H_{0,0}/H_{0,0}$ outperforming $H_{0,0}/H_{1,1}$. Another important observation that must be made concerns the approximation alternatives. It can be seen that case 1 gives better results for smaller angles than case 2, and that this behavior is inverted for larger angles. The same can be seen while comparing cases 3 and 4. These findings indicate that the approximation alternative 1 is better for smaller angles, while alternative 2 is better suited for larger angles.

6.3 Hexagon

The results for hexagonal profiles are calculated for different angles and aspect ratios and are presented in table 3, which includes the GITT results presented by Aparecido (1988). As can be seen, in general, for all cases, the error increases with increasing aspect ratio K. For the first approximation alternative (cases 1 and 3) the error increases with decreasing angle, whereas for the second alternative a different behavior is seen. For case 2 the error is minimum for moderate values of ψ (30° and 45°) and is larger for smaller and larger values. A similar behavior is observed for case 4; however the error is smaller for a larger range of ψ (from 30° to 60°). Comparing the different levels of approximation ($H_{0,0}/H_{0,0}$ and $H_{1,1}/H_{0,0}$) for a same approximation alternative, one notes, for the first alternative, that $H_{1,1}/H_{0,0}$ (case 3) gives better results than $H_{0,0}/H_{0,0}$ (case 1). Nevertheless a different trend is seen for the second alternative. In these cases the $H_{1,1}/H_{0,0}$ (case 4) gives better results for moderate angles, but it is the $H_{0,0}/H_{0,0}$ (case 2) which yields better results for larger and smaller values of ψ. Analyzing the presented results one can also note that alternative 2 (cases 2 and 4) is generally better for smaller angles.

7. CONCLUSIONS

This paper presented an alternative approach for calculating friction-factor in steady laminar fluid flow in ducts of different polygonal cross-sections (rectangular, trapezoidal and hexagonal). An approximate analytical methodology, based on the coupled integral equations approach is used. Closed form analytical expressions are obtained for Fanning's friction factor, for various combination of polygon aspect ratio and angle. Results for four different approximation cases are presented, consisting of a combination of two levels of approximations and two approximation alternatives. The data is compared with previously published results, and an error analysis shows that some cases have better performance than others; however this is not uniformly observed. Because of the heterogeneity of the error among the different approximations, an optimum approximation for all cases cannot be determined from this study. Nevertheless, if future research is aimed at obtaining error estimates for different approximations, a rule for selecting the proper approximation type could be devised.

Table 2. Friction-factor in trapezoidal duct.

		case 1		case 2		case 3		case 4		
	K	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	GITT
$\psi = 85°$	0.01	23.7619	0.2	23.7615	0.2	23.7617	0.2	23.7627	0.2	23.8023
	0.10	21.8568	3	21.8133	3	21.8351	3	21.9287	4	21.1842
	0.25	19.4607	6	19.2267	5	19.3437	6	19.8378	8	18.2987
	0.50	16.9604	8	16.2152	3	16.5879	6	18.0678	15	15.6759
	0.75	15.6248	7	14.2595	2	14.9425	3	17.4199	20	14.5763
	1.00	14.9778	5	12.9652	9	13.9726	2	17.2490	21	14.2351
$\psi = 75°$	0.01	23.7548	0.2	23.7543	0.2	23.7545	0.2	23.7544	0.2	23.7908
	0.10	21.8147	3	21.7694	3	21.7920	3	21.7797	3	21.1389
	0.25	19.4312	6	19.1935	5	19.3124	5	19.2473	5	18.3152
	0.50	17.0078	8	16.2765	3	16.6423	5	16.4396	4	15.8047
	0.75	15.7265	7	14.4223	2	15.0748	3	14.7069	0.1	14.6971
	1.00	15.0856	6	13.2039	7	14.1458	1	13.8887	3	14.2519
$\psi = 60°$	0.01	23.7273	0.1	23.7267	0.1	23.7270	0.1	23.7267	0.1	23.7428
	0.10	21.6258	4	21.5716	3	21.5987	3	21.5716	3	20.8917
	0.25	19.1839	6	18.9133	5	19.0486	5	18.9133	5	18.0585
	0.50	16.8688	7	16.0872	3	16.4782	5	16.0872	3	15.6922
	0.75	15.7185	7	14.3883	2	15.0540	3	14.3883	2	14.6376
	1.00	15.1606	7	13.3094	6	14.2365	1	13.3094	6	14.1516
$\psi = 45°$	0.01	23.6677	0.1	23.6667	0.1	23.6672	0.1	23.6667	0.1	23.6405
	0.10	21.2200	4	21.1435	4	21.1817	4	21.1403	4	20.3550
	0.25	18.6350	7	18.2798	5	18.4574	6	18.2650	5	17.4212
	0.50	16.4710	8	15.5285	2	16.0000	5	15.4898	2	15.2127
	0.75	15.5321	9	14.0201	2	14.7772	4	13.9590	2	14.2711
	1.00	15.1344	9	13.1213	5	14.1305	2	13.0410	6	13.8223
$\psi = 30°$	0.01	23.5348	0.5	23.5330	0.5	23.5339	0.5	23.5329	0.5	23.4211
	0.10	20.3985	6	20.2619	5	20.3302	5	20.2543	5	19.3092
	0.25	17.6413	8	17.0784	5	17.3599	7	17.0473	5	16.2724
	0.50	15.8261	10	14.5117	1	15.1698	6	14.4402	1	14.3466
	0.75	15.2491	12	13.3087	2	14.2821	5	13.2045	3	13.6029
	1.00	15.1011	14	12.6674	4	13.8911	5	12.5381	5	13.2552
$\psi = 15°$	0.01	23.1283	1	23.1216	1	23.1249	1	23.1212	1	22.7967
	0.10	18.5085	9	18.1293	6	18.3189	7	18.1063	6	17.0889
	0.25	16.0089	12	14.7984	3	15.4045	7	14.7258	3	14.3548
	0.50	15.2016	16	12.9625	1	14.0881	8	12.8304	2	13.0962
	0.75	15.2249	20	12.3470	3	13.8010	9	12.1788	4	12.6943
	1.00	15.3754	23	12.0920	3	13.7589	10	11.9011	5	12.5201
$\psi = 1°$	0.01	17.3469	10	16.7007	6	17.0239	8	16.6607	6	15.7693
	0.10	15.6402	28	11.9973	2	13.8601	14	11.7794	3	12.1891
	0.25	16.0752	34	11.9489	1	14.0864	17	11.7014	3	12.0326
	0.50	16.1949	35	11.9768	0.2	14.1702	18	11.7233	2	12.0051
	0.75	16.2221	35	11.9873	0.1	14.1912	18	11.7327	2	12.0026
	1.00	16.2323	35	11.9919	0.1	14.1993	18	11.7370	2	12.0120

Table 3. Friction-factor in hexagonal duct.

		case 1		case 2		case 3		case 4		
	K	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	f Re	$\varepsilon(\%)$	GITT
$\psi = 85°$	0.01	23.5471	1	23.5452	1	23.5462	1	24.0562	1	23.8210
	0.05	21.9317	3	21.8879	3	21.9098	3	24.3043	8	22.5700
	0.10	20.2911	5	20.1321	6	20.2116	5	24.6516	16	21.3110
	0.25	17.1591	7	16.3958	11	16.7776	9	25.7676	39	18.5020
	0.50	15.2358	4	13.1472	17	14.1924	11	27.3331	72	15.9030
	1.00	15.6000	8	11.0461	24	13.3284	8	28.6500	98	14.4770
$\psi = 75°$	0.01	23.5699	1	23.5678	1	23.5689	1	23.7443	0	23.8450
	0.05	22.0433	3	21.9971	3	22.0202	3	22.8135	1	22.6840
	0.10	20.5066	5	20.3399	5	20.4233	5	21.8281	1	21.5200
	0.25	17.6193	7	16.8335	11	17.2265	9	19.7219	4	18.9230
	0.50	15.8800	4	13.7826	16	14.8319	10	17.8267	8	16.4840
	1.00	16.1176	8	11.6881	22	13.9054	7	16.3655	10	14.9390
$\psi = 60°$	0.01	23.5788	1	23.5763	1	23.5776	1	23.6777	1	23.8590
	0.05	22.1098	3	22.0534	3	22.0816	3	22.5135	1	22.7630
	0.10	20.6763	5	20.4761	6	20.5762	5	21.2963	2	21.6690
	0.25	18.1459	6	17.2424	10	17.6943	8	18.7451	3	19.2610
	0.50	16.8307	1	14.5399	14	15.6857	8	16.4988	3	17.0020
	1.00	17.2473	12	12.6915	18	14.9701	3	14.7815	4	15.3990
$\psi = 45°$	0.01	23.5474	1	23.5437	1	23.5455	1	23.6308	1	23.8380
	0.05	22.0248	3	21.9430	3	21.9839	3	22.3273	2	22.6990
	0.10	20.6338	4	20.3522	6	20.4930	5	21.0156	3	21.5710
	0.25	18.4976	4	17.3148	10	17.9063	7	18.4399	4	19.1760
	0.50	17.8017	5	15.0296	12	16.4158	3	16.3787	4	17.0070
	1.00	18.7002	21	13.6283	12	16.1647	5	14.9594	3	15.4070
$\psi = 30°$	0.01	23.4337	1	23.4264	1	23.4300	1	23.5255	1	23.7510
	0.05	21.6729	3	21.5204	4	21.5967	3	21.9336	2	22.3670
	0.10	20.2842	4	19.7892	6	20.0367	5	20.4608	3	21.0440
	0.25	18.7659	2	16.9400	8	17.8531	3	17.9391	3	18.4390
	0.50	18.9949	17	15.2260	7	17.1106	5	16.2892	0.1	16.2990
	1.00	20.5379	39	14.4446	2	17.4947	18	15.4082	4	14.7870
$\psi = 15°$	0.01	23.0326	2	23.0065	2	23.0196	2	23.1704	1	23.4360
	0.05	20.6590	3	20.1941	5	20.4266	4	20.7780	2	21.2240
	0.10	19.5118	1	18.2146	6	18.8633	3	19.0308	2	19.3650
	0.25	19.5798	19	15.9761	3	17.7781	8	16.9049	3	16.4010
	0.50	21.1435	46	15.2428	5	18.1968	25	16.0786	11	14.5230
	1.00	23.0054	71	15.2048	13	19.1277	42	15.9264	19	13.4280
$\psi = 1°$	0.01	19.4274	7	17.3171	4	18.3723	2	18.1837	1	18.0730
	0.05	22.5146	67	15.4601	15	18.9997	41	16.1875	20	13.4760
	0.10	24.1119	92	15.6306	24	19.9119	58	16.2728	29	12.5800
	0.25	25.0735	107	15.8767	31	20.5475	69	16.4830	36	12.1320
	0.50	25.2841	110	15.9577	33	20.7018	72	16.5599	38	12.0370
	1.00	25.3452	111	15.9863	33	20.7490	73	16.5880	38	12.0100

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9. REFERENCES

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