THE COMPOSITE ELEMENT METHOD WITH MIXED INTERPOLATION APPLIED TO THE THIN PLATE VIBRATION PROBLEM

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Abstract. The development of the Composite Element Method (CEM), which combines the Finite Element Method (FEM) approach and analytical solutions obtained from the Classical Theory (CT), is a novel technique to solve free vibration problems. Some of the elements proposed and modelled have shown that the free vibration problem of prismatic bars and Timoshenko's beam could be solved in a more efficient and accurate way using CEM. However, certain elements are not capable of solving free vibration plate, mainly with those that are considered thin. In this paper a new element of CEM using the MITC (mixed interpolation of tensorial components) formulation from a 4-node plate element has been proposed. The displacement field is expanded, merging the nodal values from FEM with the analytical functions of the classical solutions. The classical solution functions must satisfy certain specific boundary conditions in such a way that they do not change the nodal values of FEM. These functions must also be the solutions of the frequency equation. The objective of the present work is to apply the CEM on the thin and thick plate model. Examples are included to illustrate the efficiency and accuracy of the method and the results are compared between usual MITC element and CEM element. The locking effect related to the thin plates is also discussed.

Keywords. *Composite Element Method, Free Vibration, Finite Element Method, Thin Plate*

1. Introduction

To understand the structural behavior very often it is required to find their natural frequency and modal vibration. In this way, several numerical techniques have been developed, being the Finite Element Method one of the most successful.

Aiming at improving the accuracy of the results for modal vibration definition, using the same finite element mesh, the Composite Element Method (CEM) has been proposed by Zeng (1998a, 1998b, 1998c). The CEM does not change the stiffness and mass matrices because the analytical functions increase them without the need matricial rearranging. Then the CEM can be considerate a hierarchical method.

Carvalho et all (2002), using this technique, solved the free vibration problem applied to thick plate problem based on Reissner's model. It has been noticed that the results depend on the performance of the finite element adopted.

Depending on the finite element used to solve the free vibration problem applied to thin plate, the phenomenon called transverse shear locking effect appears. To deal with these cases, several papers have been proposed (e.g. Bathe and Dvorkin, 1985; Zienkiewics, 1971; Zienkiewics and Taylor, 1977).

Hereafter, to examine the CEM applied to solve free vibration of thin plates, it will be developed a 4-node plate element using the MITC (*m*ixed *i*nterpolation of *t*ensorial *c*omponents) formulation. Aiming at better results, two types of refinements can be employed: h and c. The first one corresponds to the increase of the number of elements in the finite element mesh. The other one is related to the increment of the number of analytical functions on the CEM displacement field.

2. The Composite Element Method

A linear combination of the FEM shape functions and the analytical solutions from the Classical Theory (CT) have determined the displacement field of the CEM. Then,

$$u(x,y) = u_{FEM}(x,y) + u_{CT}(x,y) \tag{1}$$

where u_{FEM} is the usual displacement field of the FEM, which is based on a local coordinate system, and u_{CT} is the corresponding displacement field from the CT.

The FEM displacement field can be approximated by interpolation of the nodal values, using H as the shape function vector and q the nodal displacement vector. Considering *(x,y)* as coordinates of a generic point inside the element; results:

$$u_{FEM}(x,y) = H^T(x,y)\ q \tag{2}$$

Similarly, considering *Ø* as the set of analytical functions, and *c* the composite functions coordinate vector, which is relative to the *c*-degree of freedom (dof), the CT displacement field is given by:

$$u_{CT}(x,y) = Ø^T(x,y)\, c \tag{3}$$

Replacing Eq. (2) and (3) into Eq. (1), results in:

$$u(x,y) = H^T q + Ø^T c \tag{4}$$

The next step is to develop the interpolation functions matrices and the nodal values vectors for the finite element plate model and for the CT.

3. The classical theory of free vibration plate

In order to obtain the *c*-displacement field of CEM, the free vibration plate theory has to be observed. Considering *w* as the plate's transverse displacement, *q=q(x,y,t)* the surface loads, *D* the flexural stiffness and *ρ* the specific weight, then, by Hamilton's Principle, the differential equation of movement is given by:

$$D\nabla^4 w + \rho h \ddot{w} = q \tag{5}$$

For the case of vibration problems, the surface loads are null and the Eq. (5) becomes:

$$D\nabla^4 w + \rho h \ddot{w} = 0 \tag{6}$$

To solve the previous equation, it is convenient to separate the variables, resulting:

$$w(x,y,t) = W(x,y)T(t) \tag{7}$$

where,

$$\begin{aligned} &\frac{\ddot{T}}{T} = -\omega^2 \\ &\frac{D}{\rho h}\frac{\nabla^4 W}{W} = \omega^2 \end{aligned} \tag{8}$$

In this case, ω^2 is the link for the two variables (*W* and *T*). Physically, *ω* is also known as the natural frequency. The problem is completely solved when the function *W(x,y)* is determined. One way to solve this differential equation system is using the infinity series approximation (Szilard, 1974), considering w_{mn} as coefficients to be determined, and *θ(x,y)*, as any functions that obey boundary conditions:

$$W(x,y) = \sum_m \sum_n w_{mn}\phi_{mn}(x,y) \tag{9}$$

From Eq. (9), it is possible to write *q(x,y)* as the product of two other functions, which both depend on one argument. Then,

$$\phi_{mn} = X_m(x)\, Y_n(y) \tag{10}$$

The functions *Xm(x)* and *Yn(y)* are linearly independent and each set of functions must satisfy the boundary conditions in its own direction. It is possible to use the same type of functions in both directions. The problem is solved by two sets of linearly independent equations and, for each direction, the classical beam solution can be applied. For the particular case of the *X*-direction, the classical beam solution is:

$$\frac{\partial^4 W(x,t)}{\partial x^4} = -\frac{m}{EI}\frac{\partial^2 W(x,t)}{\partial t^2} \tag{11}$$

where, m is the mass per length unit, E is the Young's Modulus, and I is the flexural inertia moment.

Supposing a harmonic solution, it is possible to write:

$$\frac{d^4 X_m(x)}{dx^4} = \frac{\lambda_m{}^4}{l^4} X_m(x) \tag{12}$$

where, λ_m is the shape parameter and l is the span length, giving:

$$\frac{\lambda_m{}^4}{l^4} = \frac{m\omega^2}{EI} \tag{13}$$

The general solution of Eq. (12) is:

$$X_m(x) = C_1 \sin\frac{\lambda_m x}{l} + C_2 \cos\frac{\lambda_m x}{l} + C_3 \sinh\frac{\lambda_m x}{l} + C_4 \cosh\frac{\lambda_m x}{l} \tag{14}$$

The constants C_1, C_2, C_3 e C_4 are obtained from setting the boundary conditions. The parameter λ_m is the root of the characteristic equation. As in the case of the beam element, considering the clamped-clamped boundary conditions, it can be shown that:

$$\begin{gathered} X_m(0) = X_m(l) = 0 \\ \left.\frac{dX_m}{dx}\right|_{x=0} = \left.\frac{dX_m}{dx}\right|_{x=l} = 0 \end{gathered} \tag{15}$$

Replacing the boundary conditions (15) into Eq. (14), a set of homogeneous equations can be derived:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos(\lambda_m) & \sin(\lambda_m) & \cosh(\lambda_m) & \sinh(\lambda_m) \\ 0 & 1 & 0 & 1 \\ -\sin(\lambda_m) & \cos(\lambda_m) & \sinh(\lambda_m) & \cosh(\lambda_m) \end{bmatrix} . \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

The frequency equation is obtained by imposing the system determinant of Eq. (16) equal to zero.

$$f = \cos\lambda_m . \cosh\lambda_m - 1 = 0 \tag{17}$$

Table 1 shows the roots of the frequency equation. The parameter λ_m is the m-th eigenvalue, while the corresponding vibration modes are obtained from Eqs. (16) and (17).

Table 1. The frequency equation roots.

λ_m	VALUE
λ_1	4,7300
λ_2	7,8532
λ_3	10,9956
λ_4	14,1372
λ_5	17,2788
λ_6	20,4204
λ_7	23,5619
λ_8	26,7035
λ_9	29,8451

The X_m solution is:

$$X_m(x) = a_2.\left\{\sin\left(\frac{\lambda_m x}{l}\right) - \sinh\left(\frac{\lambda_m x}{l}\right) - \frac{\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m - \cosh\lambda_m}.\left[\cos\left(\frac{\lambda_m x}{l}\right) - \cosh\left(\frac{\lambda_m x}{l}\right)\right]\right\} \tag{18}$$

Applying the same procedure for the Y-direction, it follows that, for a square plate, the result is:

$$Y(y) = b_2.\left\{\sin\left(\frac{\lambda_n y}{l}\right) - \sinh\left(\frac{\lambda_n y}{l}\right) - \frac{\sin\lambda_n - \sinh\lambda_n}{\cos\lambda_n - \cosh\lambda_n}.\left[\cos\left(\frac{\lambda_n y}{l}\right) - \cosh\left(\frac{\lambda_n y}{l}\right)\right]\right\} \tag{19}$$

Here, the sub-indices *m* and *n* are relative to the *X* or *Y* directions and *l* is the span of the plate. For indices above to nine, the λ_m values can be obtained by:

$$\lambda_m = (2m+1)\pi/2 \tag{20}$$

4. The finite element approximation

The finite element plate model is based on the nodal interpolation functions. In this case, the plate variables can be determined by the following expressions:

$$w(x,y) = \sum_{i=1}^{n} h_i U^i \qquad \beta_x = -\sum_{i=1}^{n} h_i \theta_y^i \qquad \beta_y = \sum_{i=1}^{n} h_i \theta_x^i \tag{21}$$

where *w(x,y)*, θ_x and θ_y are de nodal degrees of freedom relative to the generic coordinate system. Figure (1) contains the representation of nodal value of displacement of a four node plate element considering constant thickness and its medium plane.

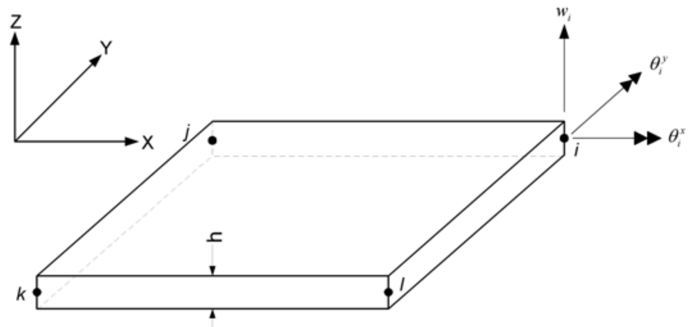

Figure 1. Nodal displacement for four node plate element.

As in the usual FEM methodology, the following stiffness matrix is obtained:

$$[K] = \int_A \int_{-h/2}^{h/2} [B_f]^T [D_f][B_f] dz\, dA + k \int_A \int_{-h/2}^{h/2} [B_c]^T [D_c][B_c] dz\, dA \tag{22}$$

where, $[B_f]$ and $[B_c]$ are the interpolation matrices for bending and shear; $[D_f]$ is the constitutive matrix for the flexural behavior of the plate; $[D_c]$ is the constitutive matrix for shear and k is the shear correction factor.

The $[B_c]$ matrix, which is associated to shear deformations, and considering the *x, y* coordinates can be written by:

$$[B_c] = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{23}$$

Now, considering the MITC4 element that has been proposed by Bathe and Dvorkin (1985), the isoparametric coordinate system could be equaled to the ξ, η, resulting:

$$\gamma_{\xi z} = \frac{1}{2}(1+\eta)\left(\frac{w_1 - w_2}{2} + \frac{\theta_y^1 + \theta_y^2}{2}\right) + \frac{1}{2}(1-\eta)\left(\frac{w_4 - w_3}{2} + \frac{\theta_y^4 + \theta_y^3}{2}\right)$$

$$\gamma_{\eta z} = \frac{1}{2}(1+\xi)\left(\frac{w_1 - w_4}{2} - \frac{\theta_x^1 + \theta_x^4}{2}\right) + \frac{1}{2}(1-\xi)\left(\frac{w_2 - w_3}{2} - \frac{\theta_x^2 + \theta_x^3}{2}\right) \quad (24)$$

where *w* and θ are the transverse nodal displacement and rotation, respectively. The numeric-indices are related to the nodal number of the element as indicated in Fig. (1).

The γ_{xz} and γ_{yz} shear strains can be written by

$$\gamma_{xz} = \gamma_{\xi z} \sin\beta - \gamma_{\eta z} \sin\alpha$$

$$\gamma_{xz} = -\gamma_{\xi z} \cos\beta + \gamma_{\eta z} \cos\alpha \quad (25)$$

where α and β are the angles between the ξ and *x* axes and η and *x* axes, respectively. More details can be found in Bathe and Dvorkin (1985).

Figure (2) contains the representation of the isoparametric element and presents the global and local coordinate system.

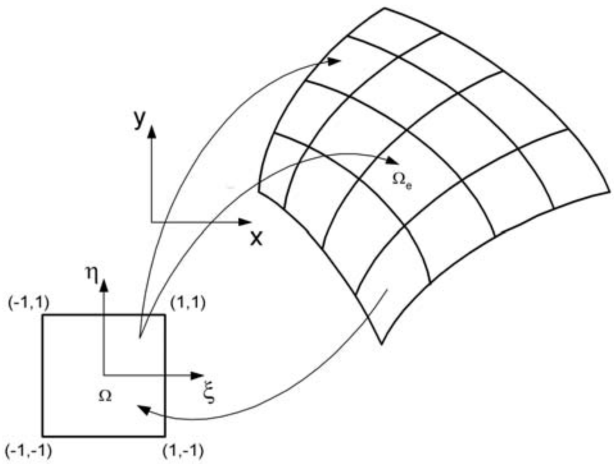

Figure 2. Representation of the isoparametric element.

The mass matrix [*M*] is evaluated by**:**

$$[M] = \int_A \int_{-h/2}^{h/2} \rho\{u\}\{\delta u\}\, dz\, dA \quad (26)$$

5. The CT approximation

Defining α_x and α_y as rotation section and *c* as the transverse displacement associated to the CT, it is possible to write:

$$c = \sum_{i=1}^{n} \psi_i c_i$$

$$\alpha_x = -\sum_{i=1}^{n} \psi_i \theta_y^{ci} \quad (27)$$

$$\alpha_y = \sum_{i=1}^{n} \psi_i \theta_x^{ci}$$

CT flexural deformations are determined in a similar way. The strain deformation relations are:

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} = -z\begin{bmatrix}\dfrac{\partial\alpha_x}{\partial x}\\ \dfrac{\partial\alpha_y}{\partial y}\\ \dfrac{\partial\alpha_x}{\partial y}+\dfrac{\partial\alpha_x}{\partial y}\end{bmatrix} \tag{28}$$

and also, the shear deformations are:

$$\begin{bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{bmatrix} = \begin{bmatrix}\dfrac{\partial c}{\partial x}-\alpha_x\\ \dfrac{\partial c}{\partial y}-\alpha_y\end{bmatrix} \tag{29}$$

The matrix of the shape functions for the CT is associated to the *c*-dof. This matrix is:

$$[\phi]^T = \begin{bmatrix} 0 & 0 & 0 & \cdots & \vdots & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \cdots & \phi_m & 0 & 0\\ 0 & 0 & 0 & \cdots & \vdots & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & \cdots & 0 & \phi_m & 0\\ 0 & 0 & 0 & \cdots & \vdots & 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & \cdots & 0 & 0 & \phi_m \end{bmatrix} \tag{30}$$

The *c*-dof vector for the CT is:

$$\{C\}^T = \{0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad c_1 \quad \theta_x^{c1} \quad \theta_y^{c1} \quad c_2 \quad \theta_x^{c2} \quad \theta_y^{c2} \quad \cdots \quad c_m \quad \theta_x^{cm} \quad \theta_y^{cm}\} \tag{31}$$

The upper-indices of the rotation variable θ are related to the *c*-shape functions. As in the FEM, the nodal displacements may be expressed by the CT nodal values. Using the strain-displacement relations, the matrices $[BMC]_f$ and $[BMC]_c$, (respectively the flexural and the shear transformation matrices for the CT) can be obtained. Applying them in the Virtual Work Principle, results:

$$[K] = \int_A \int_{-h/2}^{h/2} [BMC]_f^T [D_f][BMC]_f \, dz\, dA + k\int_A \int_{-h/2}^{h/2} [BMC]_c^T [D_c][BMC]_c \, dz\, dA \tag{32}$$

6. The CEM approximation

The previous formulation can be arranged in a common matrix:

$$[W] = [H] + [\Phi] \tag{33}$$

where $[H]$ is the associated matrix of the FEM shape functions, and $[\,\Phi\,]$ is associated to the CT.
In this way, the transformation matrix is:

$$W = \begin{bmatrix} h_1 & 0 & 0 & h_2 & 0 & 0 & h_3 & 0 & 0 & h_4 & 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 & 0 & 0\\ 0 & h_1 & 0 & 0 & h_2 & 0 & 0 & h_3 & 0 & 0 & h_4 & 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 & 0\\ 0 & 0 & h_1 & 0 & 0 & h_2 & 0 & 0 & h_3 & 0 & 0 & h_4 & 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & \phi_4 \end{bmatrix} \tag{34}$$

The mass matrix $[M]$ can be evaluated by:

$$[M] = \int_A \rho [W]^T \begin{bmatrix} h & 0 & 0\\ 0 & \dfrac{h^3}{12} & 0\\ 0 & 0 & \dfrac{h^3}{12}\end{bmatrix} [W]\, dA \tag{35}$$

7. Examples

Based on the CEM theory shown above, a Finite Element Code has been developed. The software can solve the free vibration problem using the FEM solution and the CEM from one up to four functions of CT, named by CEM-1F to CEM-4F, as described. For the computation of the element stiffness matrix, 2x2 Gaussian integration is used to FEM terms and 4x4 Gaussian integration is used to CT terms. The results shown in these exemples are restricted to regular geometries. Other examples applied to irregular geometries can be found at Carvalho (2002).

7.1. Free vibration of a simple supported square plate with thickness change

In this example, a simple supported plate is analyzed using the enhanced element with thickness change. The results are presented in a dimensionless form, λ, whose expression is:

$$\lambda = \omega\left(\frac{\rho h b^4}{D}\right)^{1/2} \tag{36}$$

where, ω is the natural frequency; ρ is the volumetric density; h is the thickness; b is the span of the plate and D is the flexural rigidity. Assuming the Poisson's ratio ν=0.3, the Young's Modulus is 2.1 10^{11} MPa and the volumetric density is 7800 kg/m3. The results for the first frequency are shown in Fig. (3), which contrasts the relative difference from the CEM solutions and FEM solution. The FEM solution has been determined by using four nodes quadrangular elements MITC, according Bathe and Dvorkin (1985). There are two points showing the exact solution to the relation span/thickness (*b/h*) equal 10 and 100. It can be noted that the CEM solution tend to be more accurate than FEM solution. Both cases were obtained using 10x10 mesh. The exact solution to *b/h*=10 has been determined by Srinivas et all (1970) and to *b/h*=100 can be found in Leissa (1973).

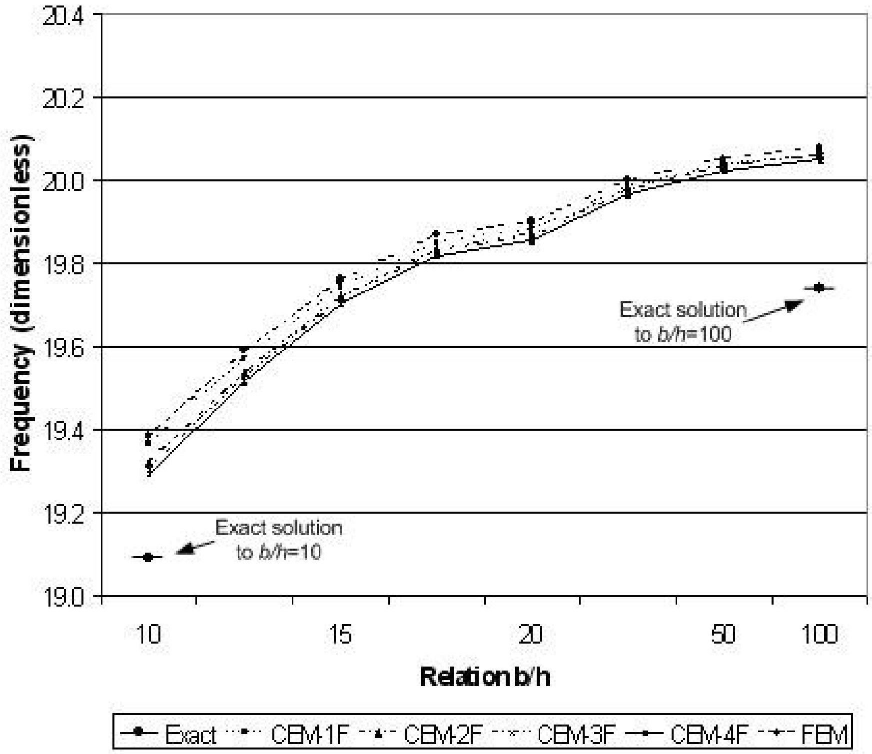

Figure 3. Results for the first frequency for FEM and CEM.

7.2. Free vibration of a simple supported square plate with mesh refinement

In this case, it is possible to compare the results using refinement h and c. The ratio *b/h* is equal to 10 and the Poisson's ratio ν=0.3, the Young's Modulus is 2.1 10^{11} MPa and the volumetric density is 7800 kg/m3.

To check the convergence of the CEM, the number of elements has been increased. Meshes with (3x3), (4x4), until (10x10) elements have been considered. Table 2 shows the results to the third frequency. The exact solution for the third dimensionless frequency is 70.089. This value has been obtained from Srinivas et all (1970).

Table 2. Third frequency dimensionless obtained from FEM and CEM

	MESH 3X3	MESH 4X4	MESH 5X5	MESH 6X6	MESH 7X7	MESH 8X8	MESH 9X9	MESH 10X10
FEM	109.14	89.25	81.54	77.69	75.49	74.10	73.17	72.51
CEM-1F	106.32	87.48	80.33	76.92	74.94	73.66	72.83	72.24
CEM-2F	105.92	87.15	80.26	76.89	74.91	73.65	72.82	72.23
CEM-3F	105.25	86.76	80.18	76.85	74.86	73.63	72.81	72.22
CEM-4F	105.12	86.74	80.17	76.84	74.86	73.63	72.81	72.22

8. Conclusions

The new formulation of the CEM using the MITC is an effective numerical method to solve the free vibration problem of thin plates without presenting transverse shear locking. A computational program to the four node plate element using one to four enrichment functions of CT has been developed.

Although the CEM solutions are always better than those from the FEM (because the finite element space solution is enriched by the analytical shape functions), their computational costs are increased. This should be compensated by some advantages. An advantage of CEM is to produce better results to high frequencies with coarse meshes. Another one is the hierarchical construction of the stiffness and mass matrices.

The CEM results depend on the performance of the finite element adopted. The knowledge about the behaviour of the global matrices of CEM should also be further studied and using the CEM to determine local modal of vibration.

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