

USE BOUNDARY ELEMENT METHOD FOR ANALYSIS 3D OF ELASTIC PROBLEMS

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Abstract. *In the present work, it is presented already consecrated direct formularization of the Boundary Element Method (BEM), for analysis of three-dimensional elastic problems. The elaborated computational code from MatLab software, considers the composed domain of some elastic sub-regions associates. The used language allows the graphical exploration of the results in simple way to the user, beyond if adapting well to protocol data-communication MPI used for parallel computation. The procedures for obtain parallel code allow to the use of some microcomputers processing at the same time parted tasks. This allows the possibility of better discretization, influencing in the final computational time as well as in the desired precision. The basic solution of Kelvin is used, having as elements of discretization of boundary the triangular plans with discontinuous linear functional approach. Processed applications are presented evidencing the precision of the method and adequacy and limitations of the used fundamental solution*.

Keywords. *Boundary Element Method, 3D Analysis, MatLab Platform, Parallel Processing*

1. Introduction

Boundary Element Method (BEM) is the more recent numerical method of the point of view of computational applications, amongst those most used. This denomination is due Brebbia (1978). The BEM consists of getting the solution of the differential equations, which describe the behavior of a body in its domain, through the solution of integral equations on the boundary. Such procedure can reduce of a unit the dimensions of the analyzed problems, what it results in lesser amounts of input data and, consequently, system of algebraic equations lesser. On the other hand, the matrix of the system is generally full and not symmetrical.

To get boundary integral equation, that makes possible the analysis of the problem, BEM needs the concept of fundamental solution. It represents the reply in a point of the infinite domain due to application of unit force in another point of the same domain. The use of a fundamental solution, that generically can be classified as a disadvantage, in the truth provides to versatility and precision to the method (Becker, 1992).

BEM has emerged as a stronger alternative mainly in the problems whose domain are extended to the infinite space (or the half-infinite). In these cases, the mesh of elements used by BEM in the boundary discretization needs shape only part this, so that the used fundamental solution in the method already contemplates the influence of the infinite (or the half-infinite). Other methods, as for example the Finite Element Method (FEM) - Cook et al. (1989), use fictitious boundaries to represent the infinite. This can cause serious errors in the numerical results, in the elastodinâmica, where waves can be reflected by such boundary.

The application of the MEC for the study of three-dimensional problems has as precursory Cruse (1968) and Lachat (1975). This subject also is boarded in Nakaguma (1979), Curotto (1981), Sá & Telles (1986), Silva (1989), Barbirato (1991), Coda (1993), among others. In Nakaguma (1979) and Barbirato (1991 and 1999) are used formulations of the BEM for three-dimensional analysis with the basic solutions of Kelvin and Mindlin.

The three-dimensional analysis of engineering problems has been each time more used due to the great advance of the electronic industry, that produces fast computers and with large capacity of storage. Still thus, some problems must to be processed in mainframes computers, nor always accessible. On the other hand, the technique of the computational parallel processing (Foster, 1995) uses *clusters*, constituted of microcomputers, whose investment is well lesser that the previous equipment. The results gotten with the parallel processing are so positive (Barra et al., 1999), disclosing a new boarding for the numerical simulations. Therefore, discretizations with a more intense mesh of elements can be employees, including themselves new elaborated implementations, as the dynamic analysis, the possibility of fractures into structure, amongst others.

2. Basic Formulation

The formulation developed in this work is based on the integral representation of displacements, for an elastic body of domain Ω and boundary Γ.

2.1. Domain points

The integral representation of displacements for domain points of the three-dimensional solid can be obtain by the use of the weighed residues technique, also used in other numeric methods, or of the Betti theorem and the Static Reciprocity, according to Barbirato (1999). The BEM portability, for coupling to other methods, can be obtained when one uses the technique of the weighed residues, distributing the error in all the domain of body. But it was using Betti theorem that Somigliana (1886) arrived at the displacement integral representation, known also as Somigliana identity, described of the following form:

$$u_i(s) = -\int_{\Gamma} p_{ij}^*(s,Q)u_j(Q)d\Gamma(Q) + \int_{\Gamma} u_{ij}^*(s,Q)p_j(Q)d\Gamma(Q) + \\ + \int_{\Omega} u_{ij}^*(s,q)b_j(q)d\Omega(q) \tag{1}$$

The Eq. (1) represents the displacement in point s of the domain in cartesian direction i, from the values of displacements and tractions in point Q of the boundary and, considering body forces, the components b_j of point q into domain. Next, Eq. (2) represents the values of the tensions in internal point s, from the values of displacements and tractions of point Q of the boundary, increased of the relative parcel the body forces (b_k), in point q into domain, when considered.

$$\sigma_{ij}(s) = -\int_{\Gamma} S_{ijk}^*(s,Q)u_k(Q)d\Gamma(Q) + \int_{\Gamma} D_{ijk}^*(s,Q)p_k(Q)d\Gamma(Q) + \\ + \int_{\Omega} D_{ijk}^*(s,q)b_k(q)d\Omega(q) \tag{2}$$

where the tensors S_{ijk}^* and D_{ijk}^* that in appear are determined through the derivation of the tensors of displacements and tractions, respectively, defined for the fundamental solution. Therefore, these components are given by the inherent expressions to each chosen fundamental solution. The Kelvin fundamental solution was chosen in this paper, therefore the three-dimensional expressions are as follows:

$$S_{ijk}^* = \frac{G}{4\pi(1-\nu)r^3}\left\{3r_{,n}\left[(1-2\nu)\delta_{ij}r_{,k} + \nu\left(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}\right) - 5r_{,i}r_{,j}r_{,k}\right] + \right. \\ \left. + 3\nu\left(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}\right) + (1-2\nu)\left(3n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}\right) - (1-4\nu)n_k\delta_{ij}\right\} \tag{3}$$

$$D_{ijk}^* = \frac{1}{8\pi(1-\nu)r^2}\left\{(1-2\nu)\left(\delta_{ki}r_{,j} + \delta_{kj}r_{,i} - \delta_{ij}r_{,k}\right) + 3r_{,i}r_{,j}r_{,k}\right\}. \tag{4}$$

2.2. Boundary points

For points into domain the Somigliana identity (Eq. 1) is valid. But, Boundary Element Method needs equation that allow the parameterization of point belongs of the boundary or outside that, allowing the agreement with the Eq. (1). For this an artifice is used, making the transformation of the boundary point (S) in one of domain (s), where it's possible apply the Somigliana identity. Then, adding to the domain one it has left infinitesimal complementary, one can characterize the boundary point (S) as a point of the interior, now being part of the domain. Effects the limits that appear in the integral equation, results in:

$$c_{ij}(S)u_j(S) = -\int_\Gamma p_{ij}^*(S,Q)u_j(Q)d\Gamma(Q) + \int_\Gamma u_{ij}^*(S,Q)p_j(Q)d\Gamma(Q) + \\ + \int_\Omega u_{ij}^*(S,q)b_j(q)d\Omega(q). \tag{5}$$

where

$$c_{ij} = \quad \mathbf{I}\begin{cases} 0, \text{ for external points to the domain } \Omega \\ \dfrac{1}{2}, \text{ for boundary } (\Gamma) \text{ points} \\ 1, \text{ for internal points to the domain } \Omega \end{cases} \tag{6}$$

2.3. Formulation of Boundary Element Method

Until this point, a formulation was presented that allows the three-dimensional elastic, isotropic and homogeneous solid analysis, through the boundary integral equations. However the analytical solution of these equations practically becomes inapplicable, being difficult its resolution. Then, the use of numerical approach is necessary, by the discretization of the boundary of the body, capable to result in a system of linear algebraic equations of simple resolution. In this section the known numerical procedure as Boundary Element Method (BEM) will be detailed.

2.3.1. Discretization

Consider the body of investigation and make a mapping in its boundary through a finite number of elements: triangular or quadrangular, plans or not. According to Rodriguez (1986), the approach is made considering the boundary (Γ) of domain (Ω) formed for a finite number of segments or boundary elements Γ_j, in which nodal values of displacements and tractions are defined. One can determine geometrically the elements by the Cartesian coordinates of its nodal values $\mathbf{X}^n$. Considering the interpolation functions $\mathbf{\Psi}^\mathbf{T}$, the coordinates of a nodal point any of the elements could be gotten of the following relation:

$$\mathbf{X} = \Psi^T \mathbf{X}^n \tag{7}$$

Physical variables of the problem, as the displacements and tractions, are also approached by functions of interpolation using the respective nodal values, functional values of point. As the choice of the functions of interpolation, boundary elements receive a classification from the following form: constants, linear, quadratic, or superior order. Making use of these approach expressions previously shown, it is more convenient to present the Eq. (5) in the matrix form, where its substitution becomes immediate. The matrices will be shown in bold type, as follows:

$$\mathbf{c}(S)\mathbf{u}(S) = -\int_\Gamma \mathbf{p}^*(S,Q)\mathbf{u}(Q)d\Gamma(Q) + \int_\Gamma \mathbf{u}^*(S,Q)\mathbf{p}(Q)d\Gamma(Q) + \\ + \int_\Omega \mathbf{u}^*(S,q)\mathbf{b}(q)d\Omega(q) \tag{8}$$

Taking in account the previous considerations, and approaching the boundary of the solid in "J" elements, with "N" nodal points and its domain in "M" cells, the integral equation for displacements, Eq.(8), results of the following form:

$$
\begin{aligned}
\mathbf{c}(S)\mathbf{u}(S) = &-\sum_{j=1}^{J}\left[\int_{\Gamma_j}\mathbf{p}^*(S,Q)\phi^T(Q)d\Gamma(Q)\right]\mathbf{U^n} + \\
&+\sum_{j=1}^{J}\left[\int_{\Gamma_j}\mathbf{u}^*(S,Q)\phi^T(Q)d\Gamma(Q)\right]\mathbf{P^n} + \\
&+\sum_{m=1}^{M}\left[\int_{\Omega_m}\mathbf{u}^*(S,q)\phi_c(q)d\Omega(q)\right]\mathbf{B^n}
\end{aligned}
\tag{9}
$$

Effects computation of the integrals of the Eq. (9) through numerical procedures, as shown in Barbirato (1999), and writing this equation to the "N" nodal points of boundary, it is arrived in the following expression:

$$
\bar{\mathbf{c}}\mathbf{U} + \hat{\mathbf{H}}\mathbf{U} = \mathbf{GP} + \mathbf{DB} \tag{10}
$$

where the matrices $\hat{\mathbf{H}}, \mathbf{G}$ e $\mathbf{D}$ result of the add of the integrals on each element "J", defined in the Eq. (9). One can notice that in the Eq. (10), it's possible to group the matrices that multiply the vector of the nodal values of displacements ($\mathbf{H} = \bar{\mathbf{c}} + \hat{\mathbf{H}}$), resulting in the equation:

$$
\mathbf{HU} = \mathbf{GP} + \mathbf{DB} \tag{11}
$$

With relation the matrices **H, G** e **D**, the formulation leads to a mixing process of calculation, that is, in particular part of the boundary the displacements is unknown and known tractions, while that in other parts complementary of the boundary, the tractions appear with values unknown (not prescribed) and are prescribed displacements. Therefore, for the resolution of the system it is necessary to make the exchange of columns between these matrices, so that all the unknown are all in a side of equation and the values known in another side. Then, using the above described artifice a system of algebraic equation can be assembled as shows the following equation:

$$
\mathbf{AX} = \mathbf{F} \tag{12}
$$

where:

A is a matrix with 3Nx3N order, that it above contains array elements **H** and **G** duly changed (exchange of columns) as commented;

X is the vector of the unknowns, displacements and tractions, in accordance with the established conditions of boundary; and

F is the independent vector formed by the multiplication of the coefficients due matrices **H** and **G** to the prescribed components of displacements and tractions, adding itself still values of the parcel of the body forces.

The solution of the system of linear equations is trivial, being able itself to use any of the consecrated methods.

3. Computational Implementation

The computational implementation of the Boundary Element Method formulation presented in the previous sections was carried through in the *MatLab* platform. Such choice was motivated by the easiness of use of graphical routines, already available in the package, that allow the visualization of the geometry of the problem (graphical of pre- and post-processing), guaranteeing the integrity of the input data. Also the inquiry of the calculated values of displacements and tractions, on boundary, and displacements and tensions, in points of the interior of the body, sufficiently are facilitated with the form of presentation in commanded table form.

The parallelization strategy was developed aiming at to reduce the time of processing of the stage most significant of the method: the assembly of the matrices that stage concurs with the solution of the system of equations. However, for using of vectorial programming the *MatLab* it obtains to decide systems of equations in skillful time. The implementation was developed using the Cornell Multitask Toolbox will be *MatLab* of the Cornell Theory Center that was conceived under the concept of message passing (MPI) and executed in cluster of distributed memory. The adopted methodology was that each processing one mounts n/np lines of matrices **G** and **H**, where n is the number of elements and np the number of processors. The input data are made use in all the machines, then the process of assembly of the matrices is initiated. Each processing one mounts, local, the matrices that are collected and organized in the host node (to master). This process is made using the function *mm_reduce('G', 'mpi_sum ')*, whose second argument is an operator who adds the matrices of each processor

4. Numerical Applications

4.1. Case 1

In this application, a load uniformly distributed will be considered acting on a free area of surface of the half-infinite, as Fig. 1. The area is a rectangle of 18.30m x 9.15m, with a load distributed of compression, longitudinal modulus of elasticity (characteristic of the material, in the case the ground) and described Poisson coefficient below.

$$q = 95.6\, N/m^2$$

$$E = 44.42\, kN/m^2$$

$$\nu = 0.3$$

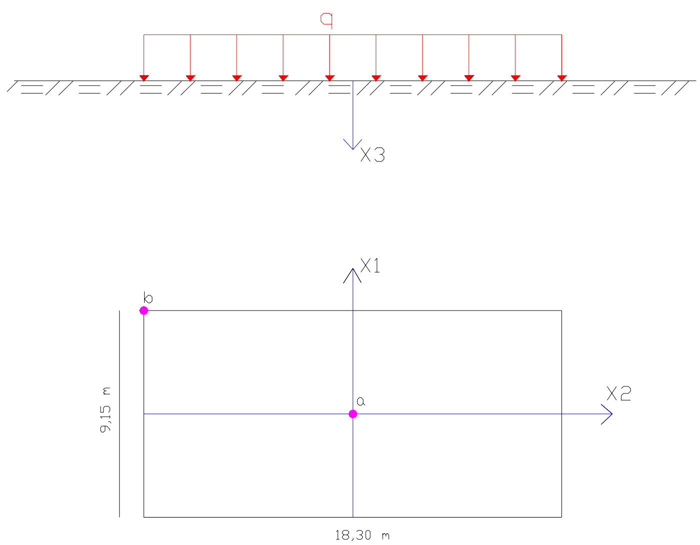


FIGURE 1 – Load distributed on a free tensile rectangular area in the half-infinite.

This application was presented in Sá and Telles (1986), through the fundamental solutions of Kelvin and Mindlin, however, in this paper will be boarded only the Kelvin solution. The discretization of the problem was made with 64 elements in the area to be analyzed. For Kelvin solution is necessary to make discretization around one definitive region of the analysis area, therefore it approaches of the infinite domain. For this, a width of 10.0 m between its sides was considered, this new region was discretized in 92 elements, and no load in this new region was considered. These discretization can be visualized better in Fig. 2. They had been placed colon of analysis of displacements, one located in the center of the area (point) and another one in one of its extremities (point b), in accordance with Fig. 2.

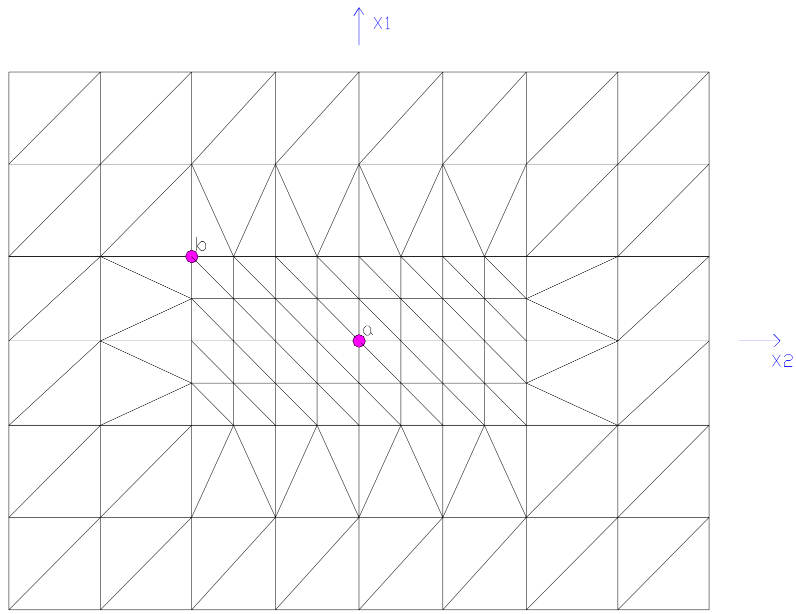


FIGURE 2 – Discretization of the free surface in 64 elements in the area of analysis and 92 elements skirting it.

Applying the routine of output the program drew the object in study, with its respective nodes in the geometric, as Fig. 3. In the central part it is acting the distributed load, where its visualization can be seen better, and is understood between nodes 1 and 45, as Fig. 3. The points of analyses of the displacements are nodes 23 (point) and 1 (point 1), shown in the Fig. 2. In same Fig. 3 it is imagined area of analysis, shown with the three axles, is observed that the area is acting in the equal coordinate x_3 (z) the zero. Where some analyses with relation to the caused deformations can be made had distributed loads.

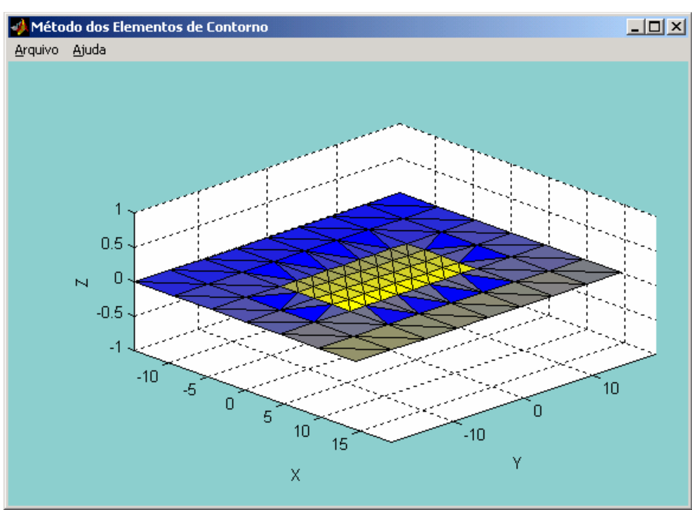


FIGURE 3 – Drawing of the object of analysis with the three coordinate axles.

Making a soon analysis of the importance of this application, one consider the case that this distributed load were an foundation structure of a building, or another structure that had some load on the area, and under this area it had or it was to be executed some workmanship, as, pipes of gas, galleries of water, for example.

As the displacements of each point are found, it becomes simpler to make the analysis of the caused efforts, and to also find the tensions provoked, verifying with this if the workmanship can support or not these efforts. Also being able, to be made this analysis to one determined height in relation the area studies, resulting in a depth in relation the performance of the distributed load. The results of the analysis is presented in Tab. 1.

TABLE 1 – Vertical displacement on points a e b (cm).

Point	Displacement (cm)	
	This Paper	SÁ and TELLES (1986) (using Mindlin solution)
a	2,74	2,83
b	1,34	

Only one discretization was made, therefore it was not seen necessary to make another discretizations, because already he had been verified with a simpler case that the result was converging to the accurate value of the problem. However, the discretization made with 64 elements in the loaded area (study), already supplies a good discretization related with the result that if it desires, and to compare with the ones of Barbirato (1999), that two discretizations had been made, one with 16 elements and another one with 64 elements. The joined results are similar to the found ones in the cited reference.

4.2. Case 2

In this application, a excavation pipe to a depth of 1.0m below of the loaded surface was considered, as Fig. 4, the data is the same of application 1, remaining it discretization of the surface, with the same number of triangular elements, added therefore only the elements of the hollowing of the duct. It was still considered that in this application only the presence of the described load in the cited figure.

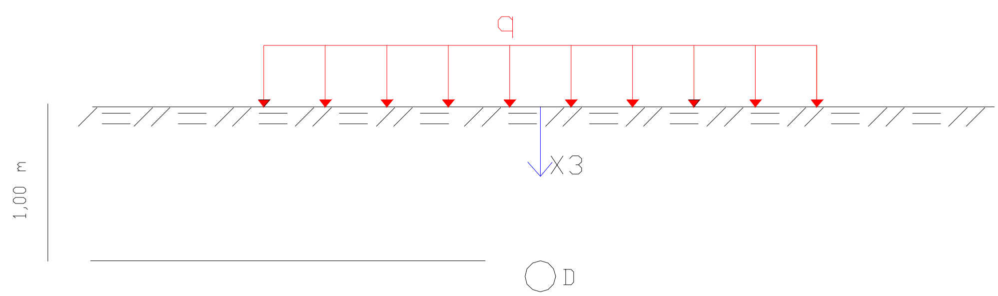


FIGURE 4 – Load distributed in the free surface, above of the excavation of the pipe.

The circumference of the excavation of the pipe was generated through 8 straight lines, from a routine made in the *MatLab* which supplies the point coordinations. The discretization was made with 160 triangular elements and 88 geometric nodes, as it shows Fig. 5.

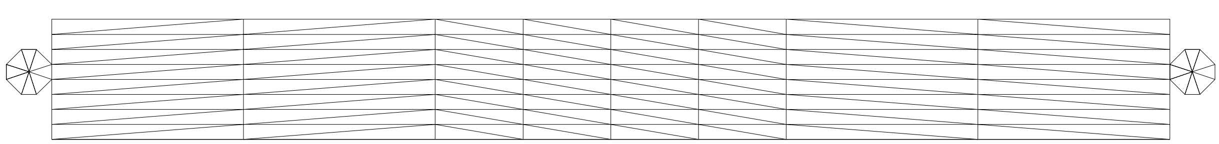
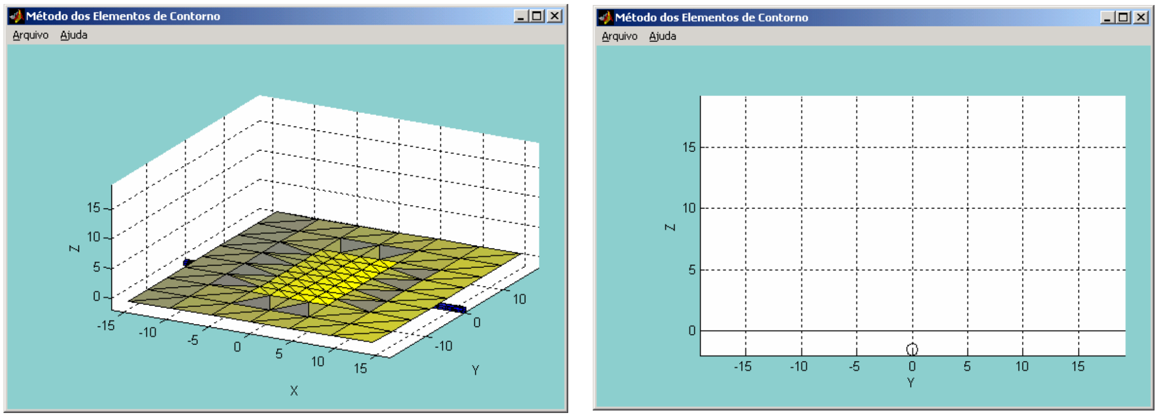


FIGURE 5 – Discretization of the excavation of the pipe.

The routine made in the software *MatLab* drew the application, with its respective geometric nodes, where an analysis of the displacements suffered for the ground next to the excavation of the pipe will be made. The Fig. 6a (and the 6b) had shown another visualization of the problem.



(a) (b)

FIGURE 6 – Drawing of the analysis object - excavation of the pipe: (a) superior and lateral view and (b) frontal view.

The displacements suffered in the excavation of the pipe, in the generator lines upper/lower, are represented in Fig. 7.

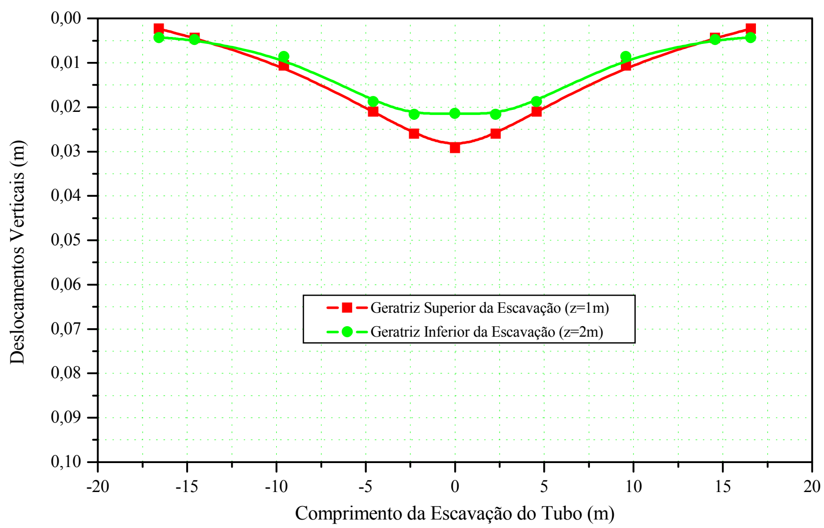


FIGURE 7 – Vertical displacements of the generator lines upper/lower of the excavation.

It can be perceived that a biggest displacement in the center of the hollowing of the pipe, resulted occurred in a value next to the one to the previous application. This application was carried through in a microcomputer with the following characteristics: Pentium IV, 2 GHz of microprocessor and 1 Gb RAM. The necessary computational time for the attainment of the results was of 32 hours and 7 minutes. With the parallel processing, using the **Cluster Typhon** of the UFAL, with 4 available nodes, 9 minutes had been expenses. The nodes of this cluster are formed by Athlons 1.0GHz of CPU.

5. Conclusion

The Boundary Element Method employed, as it was hope, showed adequate for the analysis of three-dimensional elastic bodies. The use of the *MatLab* platform allowed a more efficient programming of the point of view of graphical representation (for pre- and post-processing). Still, with its vocation as language oriented to object, more allowed to the simple code elaboration, using few commands. The job of the parallel processing revealed sufficiently positive, front to the result of measured computational time in presented case 2. The use of clusters of microcomputers (in this in case that the **Cluster Typhon** of the UFAL), can be a technology in substitution to the mainframe computers. The *MatLab* platform allowed the use of the parallel processing in immediate way, needing for its use, of our concepts of computational programming.

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