ROBUST REGULATION OF METRO LINES USING TIME-VARIANT CONTROL LAW

Wânderson de Oliveira Assis
IMT - Instituto Mauá de Tecnologia, EEM/DEL, Praça Mauá 1, 09580-900 São Caetano do Sul, SP, Brasil.
wanderson.assis@maua.br

Basílio E. A. Milani
UNICAMP, Faculdade de Engenharia Elétrica e de Computação
Cx. P. 6101 - 13081-970 - Campinas, SP, Brasil
basilio@dt.fee.unicamp.br

Abstract. This paper presents a new formulation for robust traffic regulation of metro lines. The approach uses a linear time-variant state feedback control law computed in real-time. The formulation is based on the stability analysis using the eigenvalues of the feedback system and consider the traffic model uncertainties and the constraints on its state and control variables. It is assumed the constraints on model variables and uncertain parameter domain defined by convex compact polyhedra. The simplicity and computational efficiency of this formulation makes it applicable to real-time regulation of nowadays metro lines and presents better performance than the obtained using already known robust regulation approaches.

Keywords. *Robust regulation, Metro lines, Linear programming, Optimization, Stability analysis.*

1. Introduction

The traffic of trains in high-frequency metro lines is known to be naturally unstable. Consider, for example, a train delayed with respect to the nominal time schedule of the line (Cury et all, 1980), (Assis et all, 2002). Due to this delay, its time interval relative to the preceding train is increased, more passengers will be waiting at the platforms to get on the train, resulting in increased delay (Campion et all, 1985). Thus, traffic control is necessary to keep the departure time at the stations, the time interval between successive trains as close as possible to their nominal values. The control actions consist of instructions given by the controller to the trains at the stations, increasing or decreasing their speed during the running time to the next station and/or their waiting time at the platform. The control is naturally constrained by the speed range allowed to the trains, minimum waiting time at the stations and traffic security rules. The time interval between successive trains is also constrained by the traffic security rules and the maximum train occupancy (Van Breusegem et all, 1991). Milani et all (1997) proposed a formulation based on linear programming for robust constrained regulation of metro lines, considering explicitly the constraints on the state and control variables, the parameter uncertainties and random operational disturbances. The constraints on model variables and uncertain parameter domain were defined by convex compact polyhedra. Exploring structural properties of the traffic model, the method gives a linear time-invariant control law, with bidiagonal structure, obtained solving independent linear programming problems, related one to one with the platforms of the line. This paper presents a new formulation for robust traffic regulation of metro lines using a possibly nonlinear time-variant state feedback control law computed in real time. The formulation is based on eigenvalues assignement of the closed-loop matrix. The simplicity of the formulation and the computational efficiency makes it applicable to practical application. Moreover, using a time-variant control law, the formulation can presents better regulation performance than robust regulation approaches, where we have time-invariant control.

2. Traffic Modeling

Consider n trains ($i = 1, \cdots, n$) in an open metro line with N platforms ($k = 1, 2, \cdots, N$) (Fig. (1)).

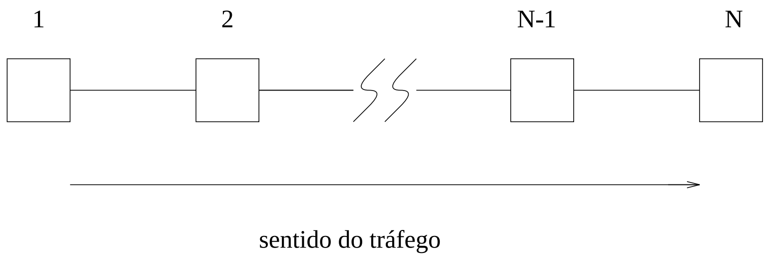

Figure 1. Open metro line with N platforms

Defining $y_i(k)$ as the deviation of the actual departure time of train i in platform k with respect to the nominal time schedule results the basic equation for the traffic of trains (Milani et all, 1997) e (Corrêa et all, 2001):

$$(1 - c_i(k+1))y_i(k+1) = y_i(k) - c_i(k+1)y_{i-1}(k+1) + u_i(k+1) + v_i(k+1) \tag{1}$$

where:
- The parameter $c(k)$ is related to passengers demand, inherently uncertain;
- $v_i(k)$ is the random disturbance;
- $u_i(k)$ is the control action applied to train i between the platforms $k-1$ and k in order to increase ($u_i(k) > 0$) or to decrease ($u_i(k) < 0$) the running time.

Throughout this paper: for two real matrices nxm, $A = (a_{i,j})$ and $B = (b_{i,j})$, $A \leq B$ is equivalent to $a_{i,j} \leq b_{i,j}$ for all i, j such that $1 \leq i \leq n$ and $1 \leq j \leq m$. $A \geq 1$ are equivalent to $a_{i,j} \geq 0$, $a_{i,j} \geq 1$, respectively. $|A| = (|a_{i,j}|)$.

a) Stations Sequential Model (SSM)

The equations (1) can be put in matrix form as the following station sequential model (SSM), suitable for stability analysis (Campion et all,1985):

$$Y_{K+1} = A_K(Y_K + U_{K+1} + V_{K+1}) \tag{2}$$

$$\begin{aligned}
Y_K &\triangleq \begin{bmatrix} y_1(k) & y_2(k) & \cdots & y_n(k) \end{bmatrix}^T; && \forall k = 1, 2, \cdots, N \\
U_K &\triangleq \begin{bmatrix} u_1(k) & u_2(k) & \cdots & u_n(k) \end{bmatrix}^T; && \forall k = 1, 2, \cdots, N \\
V_K &\triangleq \begin{bmatrix} v_1(k) & v_2(k) & \cdots & v_n(k) \end{bmatrix}^T; && \forall k = 1, 2, \cdots, N
\end{aligned}$$

$A_K = (C_{K+1})^{-1}$ where C_{K+1} is the following bidiagonal $\Re^{n \times n}$ matrix:

$$C_{K+1} \triangleq \begin{bmatrix} 1-c(k+1) & 0 & \cdots & 0 \\ c(k+1) & 1-c(k+1) & \cdots & 0 \\ 0 & c(k+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-c(k+1) \end{bmatrix} \tag{3}$$

The order of the state space representation is the number of trens. The terms c(k+1) presents a uncertain degree assumed equivalent to all trains: $c_i(k+1) = c(k+1)$, $\forall i$.

b) Real Time Model (RTM)

The real time model (RTM) (Campion et all,1985), (Van Breusegem et all, 1991) presents a state-space formulation suitable for robust control design:

$$Y_{j+1} = A(C)Y_j + B(C)[U_j + V_j] \tag{4}$$

$$\begin{aligned}
Y_j &\triangleq \begin{bmatrix} y_{j-1}(1) & y_{j-2}(2) & \cdots & y_{j-N}(N) \end{bmatrix}^T; && \forall i = j-1, j-2, \cdots, j-k; && \forall k = 1, 2, \cdots, N \\
U_j &\triangleq \begin{bmatrix} u_j(1) & u_{j-1}(2) & \cdots & u_{j-N+1}(N) \end{bmatrix}^T; && \forall i = j-1, j-2, \cdots, j-k; && \forall k = 1, 2, \cdots, N \\
V_j &\triangleq \begin{bmatrix} v_j(1) & v_{j-1}(2) & \cdots & v_{j-N+1}(N) \end{bmatrix}^T; && \forall i = j-1, j-2, \cdots, j-k; && \forall k = 1, 2, \cdots, N
\end{aligned}$$

$$C \triangleq \begin{bmatrix} c(1) & c(2) & \cdots & c(N) \end{bmatrix} \tag{5}$$

where Y_j, U_j and V_j are N dimensional vectors representing the state, the control and the external disturbances of the system, respectively; C is the vector of uncertain parameters; $A(C)$ and $B(C)$ are given by:

$$A(C) \triangleq \begin{bmatrix} \frac{-c(1)}{1-c(1)} & 0 & \cdots & 0 & 0 \\ \frac{1}{1-c(2)} & \frac{-c(2)}{1-c(2)} & \cdots & 0 & 0 \\ 0 & \frac{1}{1-c(3)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-c(N-1)}{1-c(N-1)} & 0 \\ 0 & 0 & \cdots & \frac{1}{1-c(N)} & \frac{-c(N)}{1-c(N)} \end{bmatrix} \tag{6}$$

$$B(C) \triangleq \begin{bmatrix} \frac{1}{1-c(1)} & 0 & \cdots & 0 \\ 0 & \frac{1}{1-c(2)} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{1-c(k)} \end{bmatrix} \tag{7}$$

The equation (4) presents the term $y_i(k)$ multiplicated by the factor $\frac{1}{1-c(k+1)}$ where tipically $0 \leq c(k+1) < 1$. This means that, if the deviation of the train i-1 at the station $k+1$ is equal to zero, without control at the station $k+1$, the deviation at the train i at the station $k+1$ increase with respect to the deviation at the station k: this is the well-known intrinsically unstable behaviour of a high density public transportation system. Previous robust regulation formulations based on Real Time Model present a time-invariant control law for the solution of the traffic regulation (Milani et all, 1997), (Corrêa et all, 2001).

3. Constrained Robust Regulation Problem (CRRP)

Consider an open metro line with N platforms and equations (4)-(7). Consider also the uncertain parameters C, the disturbances V_j and the control U_j, restricted to the polyhedrons:

$$C^L \leq C \leq C^U \tag{8}$$

$$-d_v \leq V_j \leq d_v \tag{9}$$

$$-d_u \leq U_j \leq d_u \tag{10}$$

where $C^L, C^U \in \Re^N, 0 < C^L \leq C^U < 1$, $d_v \in \Re^N, d_v \geq 0$ and $d_u \in \Re^N, d_u \geq 0$.

Train occupancy requirements and security rules impose limits on the admissible variation of the time interval between two successive trains $(Y_{j+1} - Y_j)$, which will be considered restricted to the polyhedron:

$$-d_h \leq Y_{j+1} - Y_j \leq d_h \tag{11}$$

where $d_h \in \Re^N, d_h \geq 0$.

Consider system equations (4)-(7), constraints (8)-(11) and a linear time-invariant state feedback control law:

$$U_j = FY_j \tag{12}$$

In order to guarantee the stability in the sense of Lyapunov of the traffic of trains in the metro line, considering the state feedback control law (12), the states Y_j, along all trajectories of system (4) will be considered restricted to the polyhedron:

$$-d_y \leq Y_j \leq d_y \tag{13}$$

where $d_y \in \Re^N, d_y \geq 0$. This corresponds to assume that (13) is positively invariant with respect to system (4), (Bitsoris et all, 1988); (Milani et all, 1997) which is equivalent to require that:

$$-d_y \leq Y_{j+1} \leq d_y \tag{14}$$

must be satisfied for all Y_j in (13).

Definition 3.1: Constrained Robust Regulation Problem (CRRP): find a matrix $F \in \Re^{N\times N}$ such that for all C (8), V_j (9), Y_j (13), $j \geq 0$, the following constraints are jointly satisfied (Milani et all, 1997):

$$Y_{j+1} = [A(C) + B(C)F]Y_j + B(C)V_j \tag{15}$$

$$\begin{array}{c} -d_y \leq Y_{j+1} \leq d_y \\ -d_u \leq U_j \leq d_u \\ -d_h \leq Y_{j+1} - Y_j \leq d_h \\ -d_v \leq V_j \leq d_v \end{array} \tag{16}$$

It can be verified that the problem CRRP is solbable iff all the following N independent sets of constraints have a feasible solution:

Constraint set 1:

$$\begin{array}{c} |f_{1,1} - c(1)^U|d_{y1} - (1-c(1)^U)d_{y1}\gamma_{y1} \leq -d_{v1} \\ |f_{1,1} - c(1)^L|d_{y1} - (1-c(1)^L)d_{y1}\gamma_{y1} \leq -d_{v1} \\ |f_{1,1} - 1|d_{y1} - (1-c(1)^U)d_{h1}\gamma_{h1} \leq -d_{v1} \\ |f_{1,1} - 1|d_{y1} - (1-c(1)^L)d_{h1}\gamma_{h1} \leq -d_{v1} \\ |f_{1,1}|d_{y1} - d_{u1}\gamma_{u1} \leq 0 \\ 0 \leq \gamma_{y1}, \gamma_{h1}, \gamma_{u1} \leq 1 \end{array} \tag{17}$$

Constraint sets k; $k = 2:N$:

$$\begin{array}{c} |f_{k,k} - c(k)^U|d_{yk} + |f_{k,k-1} + 1|d_{yk-1} - (1-c(k)^U)d_{yk}\gamma_{yk} \leq -d_{vk} \\ |f_{k,k} - c(k)^L|d_{yk} + |f_{k,k-1} + 1|d_{yk-1} - (1-c(k)^L)d_{yk}\gamma_{yk} \leq -d_{vk} \\ |f_{k,k} - 1|d_{yk} + |f_{k,k-1} + 1|d_{yk-1} - (1-c(k)^U)d_{hk}\gamma_{hk} \leq -d_{vk} \\ |f_{k,k} - 1|d_{yk} + |f_{k,k-1} + 1|d_{yk-1} - (1-c(k)^L)d_{hk}\gamma_{hk} \leq -d_{vk} \\ |f_{k,k}|d_{yk} + |f_{k,k-1}|d_{yk-1} - d_{uk}\gamma_{uk} \leq 0 \\ 0 \leq \gamma_y, \gamma_h, \gamma_u \leq 1 \end{array} \tag{18}$$

The solution of the CRRP can be obtained by N reduced order independent mathematical programming problems, using the constraints (17), (18) and a performance index given by:

$$min J_k = d_{yk}\gamma_{yk} + p d_{hk}\gamma_{hk} + q d_{uk}\gamma_{uk} \tag{19}$$

where $p \geq 0$ and $q \geq 0$ are scalars used for relative weighting of γ_{yk}, γ_{hk} and γ_{uk}.

Using routine algebraic manipulations and linear programming properties, the mathematical programming problems (17) and (18) with the performance index (19) can be solvable by independent linear programming problems (Milani et all, 1997).

4. Stability Analysis

Consider the SSM (2) with $v_i(k) = 0$, $\forall i, j$:

$$Y_{K+1} = A_K Y_K + A_K U_{K+1} \tag{20}$$

where:

$$A_K \triangleq \begin{bmatrix} \frac{1}{1-c(k+1)} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-c(k+1)}{(1-c(k+1))^2} & \frac{1}{1-c(k+1)} & 0 & \cdots & 0 & 0 \\ \frac{(c(k+1))^2}{(1-c(k+1))^3} & \frac{-c(k+1)}{(1-c(k+1))^2} & \frac{1}{1-c(k+1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^n (c(k+1))^{n-2}}{(1-c(k+1))^{n-1}} & \frac{(-1)^{n-1}(c(k+1))^{n-3}}{(1-c(k+1))^{n-2}} & \frac{(-1)^n (c(k+1))^{n-4}}{(1-c(k+1))^{n-3}} & \cdots & \frac{1}{1-c(k+1)} & 0 \\ \frac{(-1)^{n-1}(c(k+1))^{n-1}}{(1-c(k+1))^n} & \frac{(-1)^n (c(k+1))^{n-2}}{(1-c(k+1))^{n-1}} & \frac{(-1)^{n-1}(c(k+1))^{n-3}}{(1-c(k+1))^{n-2}} & \cdots & \frac{-c(k+1)}{(1-c(k+1))^2} & \frac{1}{1-c(k+1)} \end{bmatrix} \tag{21}$$

Matrix A_K is lower triangular with eigenvalues equal to $1/(1-c(k+1))$. Since $0 \leq c(k+1) < 1$, and assumming $U_{K+1} = V_{K+1} = 0$, the eigenvalues of A_K are outside the unity circle, showing the unstable behavior of traffic in metro lines in the abscence of control. Moreover, since the system not stationary, all the eigenvalues inside the unity circle is not sufficient to assure asymptotic stability (Chen, 1984).

Considering the state feedback control law $U_{K+1} = F_K Y_K + A_K^{-1} G_K Y_{K+1}$, the equation (20) can be rewritten as:

$$(I - G_K)Y_{K+1} = A_K(I + F_K)Y_K \tag{22}$$

where I is the identity matrix and:

$$F_K \triangleq \begin{bmatrix} f_{k_{1,1}} & f_{k_{1,2}} & \cdots & f_{k_{1,j}} \\ f_{k_{2,1}} & f_{k_{2,2}} & \cdots & f_{k_{2,j}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k_{i,1}} & f_{k_{i,2}} & \cdots & f_{k_{i,j}} \end{bmatrix} \tag{23}$$

$$G_K \triangleq \begin{bmatrix} g_{k_{1,1}} & g_{k_{1,2}} & \cdots & g_{k_{1,j}} \\ g_{k_{2,1}} & g_{k_{2,2}} & \cdots & g_{k_{2,j}} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k_{i,1}} & g_{k_{i,2}} & \cdots & g_{k_{i,j}} \end{bmatrix} \tag{24}$$

Defining $A_{K_f}(k) = A_K(I_N + F_K)$ and $G_{K_f} = I_N - G_K$, after some algebraic manipulations in (22), one has:

$$Y_{K+1} = (G_{K_f}^{-1} A_{K_f})Y_K \tag{25}$$

where:

$$A_{K_f} = \begin{bmatrix} \frac{1}{1-c(k+1)} & 0 & \cdots & 0 \\ \frac{-c(k+1)}{(1-c(k+1))^2} & \frac{1}{1-c(k+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^n (c(k+1))^{n-2}}{(1-c(k+1))^{n-1}} & \frac{(-1)^{n-1}(c(k+1))^{n-3}}{(1-c(k+1))^{n-2}} & \cdots & 0 \\ \frac{(-1)^{n-1}(c(k+1))^{n-1}}{(1-c(k+1))^n} & \frac{(-1)^n (c(k+1))^{n-2}}{(1-c(k+1))^{n-1}} & \cdots & \frac{1}{1-c(k+1)} \end{bmatrix} \begin{bmatrix} f_{k_{1,1}}+1 & f_{k_{1,2}} & \cdots & f_{k_{1,j}} \\ f_{k_{2,1}} & f_{k_{2,2}}+1 & \cdots & f_{k_{2,j}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k_{i,1}} & f_{k_{i,2}} & \cdots & f_{k_{i,j}}+1 \end{bmatrix}$$

$$G_{K_f} = \begin{bmatrix} 1-g_{k_{1,1}} & -g_{k_{1,2}} & \cdots & -g_{k_{1,j}} \\ -g_{k_{2,1}} & 1-g_{k_{2,2}} & \cdots & -g_{k_{2,j}} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{k_{i,1}} & -g_{k_{i,2}} & \cdots & 1-g_{k_{i,j}} \end{bmatrix}$$

Proposition 4.1: The time-variant system $G_{Kf}(k)Y_{K+1} = A_{K_f}(k)Y_K$, where $c(k+1)$ is known, is asymptoticaly stable if the following constraints are satisfied:

$$F_K \triangleq \begin{bmatrix} f_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & f_{k_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{k_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & f_{k_n} \end{bmatrix} \tag{26}$$

$$G_K \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ g_{k_1} & 0 & \cdots & 0 & 0 \\ -g_{k_1}g_{k_2} & g_{k_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-1}g_{k_1}g_{k_2}\cdots g_{k_{n-2}} & (-1)^n g_{k_2}g_{k_3}\cdots g_{k_{n-2}} & \cdots & 0 & 0 \\ (-1)^n g_{k_1}g_{k_2}\cdots g_{k_{n-1}} & (-1)^{n-1}g_{k_2}g_{k_3}\cdots g_{k_{n-1}} & \cdots & gk_{n-1} & 0 \end{bmatrix} \tag{27}$$

$$-\lambda_{k_i} \leq \frac{f_{k_i}+1}{1-c(k+1)} \leq \lambda_{k_i} \tag{28}$$

$$0 \leq \lambda_{k_i} < \lambda_m \leq 1 \tag{29}$$

$$g_{k_i} = \frac{c(k+1)}{1-c(k+1)} \tag{30}$$

$$\forall k = 0, 1, \cdots, N-1 \qquad (stations)$$
$$\forall i = 1, 2, \cdots, n \qquad (trains)$$

where λ_m is an scalar that represents the desired stability degree and λ_{k_i} are weighting variables that we have to consider at the performance index.

Proof: It can be verified, if the constraints (26) and (27) are satisfied, substituting in (25) gives:

$$G_{K_f}^{-1}A_{K_f} = \begin{bmatrix} \frac{f_{k_1}+1}{1-c(k+1)} & 0 & \cdots & 0 \\ \frac{g_{k_1}(f_{k_1}+1)}{1-c(k+1)} - \frac{c(k+1)(f_{k_1}+1)}{(1-c(k+1))^2} & \frac{f_{k_2}+1}{1-c(k+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n \frac{g_{k_{n-1}}(c(k+1))^{n-2}(f_{k_1}+1)}{(1-c(k+1))^{n-1}} + (-1)^{n-1}\frac{(c(k+1))^{n-1}(f_{k_1}+1)}{(1-c(k+1))^n} & (-1)^{n-1}\frac{g_{k_{n-1}}(c(k+1))^{n-3}(f_{k_2}+1)}{(1-c(k+1))n-2} + (-1)^n \frac{(c(k+1))^{n-2}(f_{k_2}+1)}{(1-c(k+1))^{n-1}} & \cdots & \frac{f_{k_n}+1}{1-c(k+1)} \end{bmatrix} \tag{31}$$

Substituting (30) in (31):

$$G_{K_f}^{-1}A_{K_f} = \begin{bmatrix} \frac{f_{k_1}+1}{1-c(k+1)} & 0 & \cdots & 0 \\ 0 & \frac{f_{k_2}+1}{1-c(k+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{f_{k_n}+1}{1-c(k+1)} \end{bmatrix} \tag{32}$$

It is easy to verify that $G_{K_f}^{-1}A_{K_f}$ is a diagonal matrix with elements $\frac{f_{k_i}+1}{1-c(k+1)}$.
It can be verified in (28):

$$\lambda_{k_i}(c(k+1)-1) - 1 \leq f_{k_i} \leq \lambda_{k_i}(1-c(k+1)) - 1; \quad \forall i = 1, 2, \cdots, n \ ; \quad \forall k = 0, 1, \cdots, N-1 \tag{33}$$

Using the worst case in (29):

$$c(k+1) - 2 \leq f_{k_i} \leq -c(k+1); \quad \forall i = 1, 2, \cdots, n \ ; \ \forall k = 0, 1, \cdots, N-1 \tag{34}$$

Substituting in (32), it can be verified that all elements of $G_{K_f}^{-1}A_{K_f}$ presents magnitudes less than 1, which concludes the proof. □

The formulation proposed in (26) - (30), presents the control g_{k_i} fixed and related to the traffic constant $(c(k+1))$. Nevertheless, considering the uncertainties in $c(k)$ (5), (8), the problem with time-variant control law proposed can not be aplicable.

In this case, consider the following proposition:

Proposition 4.2: The time-variant system $G_{Kf}(k)Y_{K+1} = A_{K_f}(k)Y_K$, uncertain $c(k)$ is asymptoticaly stable if the constraints (26), (27), (29) and the following ones are satisfied:

$$-\lambda_{k_i} \leq \frac{f_{k_i}+1}{1-c(k+1)^U} \leq \lambda_{k_i} \tag{35}$$

$$-\lambda_{k_i} \leq \frac{f_{k_i}+1}{1-c(k+1)^L} \leq \lambda_{k_i} \tag{36}$$

$$g_{k_i} = \frac{c(k+1)^U}{1-c(k+1)^U} \tag{37}$$

$$\begin{aligned}
&\frac{(1-c(k+1)^U)(1-c(k+1)^L)^{n-1}}{c(k+1)^U - c(k+1)^L} + (1-c(k+1)^L)^{n-2} + (1-c(k+1)^L)^{n-3}c(k+1)^L + \cdots + \\
&\quad +(1-c(k+1)^L)(c(k+1)^L)^{n-3} + (c(k+1)^L)^{n-2} < \frac{(1-c(k+1)^L)^n}{\lambda_m(c(k+1)^U - c(k+1)^L)} \\
&\qquad \forall k = 0,1,\cdots,N-1 \qquad (stations) \\
&\qquad \forall i = 1,2,\cdots,n \qquad (trains)
\end{aligned} \tag{38}$$

where λ_m is an scalar that represents the desired stability degree and λ_{k_i} are weighting variables that we have to consider at the performance index.

Proof: Equation (31) can be rewritten as:

$$G_{K_f}^{-1}A_{K_f} = \begin{bmatrix}
\frac{f_{k_1}+1}{1-c(k+1)}\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right) & 0 & \cdots & 0 \\
\left(\frac{g_{k_1}(f_{k_1}+1)}{1-c(k+1)} - \frac{c(k+1)(f_{k_1}+1)}{(1-c(k+1))^2}\right)\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right) & \frac{f_{k_2}+1}{1-c(k+1)}\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\left((-1)^n \frac{g_{k_{n-1}}(c(k+1))^{n-2}(f_{k_1}+1)}{(1-c(k+1))^{n-1}} + (-1)^{n-1}\frac{(c(k+1))^{n-1}(f_{k_1}+1)}{(1-c(k+1))^n}\right)\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right) & \left((-1)^{n-1}\frac{g_{k_{n-1}}(c(k+1))^{n-3}(f_{k_2}+1)}{(1-c(k+1))n-2} + (-1)^n\frac{(c(k+1))^{n-2}(f_{k_2}+1)}{(1-c(k+1))^{n-1}}\right)\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right) & \cdots & \left(\frac{f_{k_n}+1}{1-c(k+1)}\right)\left(\frac{1-c(k+1)^U}{1-c(k+1)^U}\right)
\end{bmatrix} \tag{39}$$

Considering the worst case for λ_{k_i} in (35) and (36) when $(c(k) = c(k)^U)$

$$\left|\frac{f_{k_i}+1}{1-c(k+1)^U}\right| \leq \lambda_{k_i} \tag{40}$$

Substituting (40) and the control (37) in $G_{K_f}^{-1}A_{K_f}$ (39):

$$G_{K_f}^{-1}A_{K_f} = \begin{bmatrix}
\frac{\lambda_{k_1}(1-c(k+1)^U)}{1-c(k+1)} & 0 & \cdots & 0 \\
\frac{\lambda_{k_1}c(k+1)^U}{1-c(k+1)} - \frac{\lambda_{k_1}c(k+1)(1-c(k+1)^U)}{(1-c(k+1))^2} & \frac{\lambda_{k_2}(1-c(k+1)^U)}{1-c(k+1)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^n\frac{\lambda_{k_1}(c(k+1))^{n-2}c(k+1)^U}{(1-c(k+1))^{n-1}} + (-1)^{n-1}\frac{\lambda_{k_1}(c(k+1))^{n-1}(1-c(k+1)^U)}{(1-c(k+1))^n} & (-1)^{n-1}\frac{\lambda_{k_2}(c(k+1))^{n-3}(c(k+1)^U)}{(1-c(k+1))n-2} + (-1)^n\frac{\lambda_{k_2}(c(k+1))^{n-2}(1-c(k+1)^U)}{(1-c(k+1))^{n-1}} & \cdots & \frac{\lambda_{k_n}(1-c(k+1)^U)}{1-c(k+1)}
\end{bmatrix} \tag{41}$$

where $c(k+1)$ is the traffic constant for the station k+1 during the period, restrited to the limits $c(k+1)^U$ and $c(k+1)^L$.

It can be verified that if $c(k+1) = c(k+1)^U$ (32) is obtained. Nevertheless, in the worst case, $c(k+1) = c(k+1)^L$ and $\lambda_{k_i} = \lambda_m$ in (29) for all $i = 1,2,\cdots,n$, (41) becomes:

$$G_{K_f}^{-1}A_{K_f} = \begin{bmatrix}
\frac{\lambda_m(1-c(k+1)^U)}{1-c(k+1)^L} \\
\frac{\lambda_m c(k+1)^U}{1-c(k+1)^L} - \frac{\lambda_m c(k+1)^L(1-c(k+1)^U)}{(1-c(k+1)^L)^2} \\
\vdots \\
(-1)^n\frac{\lambda_m(c(k+1)^L)^{n-2}c(k+1)^U}{(1-c(k+1)^L)^{n-1}} + (-1)^{n-1}\frac{\lambda_m(c(k+1)^L)^{n-1}(1-c(k+1)^U)}{(1-c(k+1)^L)^n}
\end{bmatrix}$$

$$\left.\begin{matrix} 0 & \cdots & 0 \\ \frac{\lambda_{k_2}(1-c(k+1)^U)}{1-c(k+1)^L} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (-1)^{n-1}\frac{\lambda_m(c(k+1)^L)^{n-3}(c(k+1)^U)}{(1-c(k+1)^L)n-2} + (-1)^n\frac{\lambda_m(c(k+1)^L)^{n-2}(1-c(k+1)^U)}{(1-c(k+1)^L)^{n-1}} & \cdots & \frac{\lambda_m(1-c(k+1)^U)}{1-c(k+1)^L} \end{matrix}\right] \tag{42}$$

Now, considering the worst case for Y_K in the SSM (25) and analysing matrix (42) it can be noted that the system is asymptoticaly stable if the sum of the absolute value of the terms related to train n is less than 1. After some algebraic manipulations it can be verified that it is equivalent to satisfy the constraint (38), which concludes the proof. □

It can be verified that (25) considering (26) - (30) can be rewritten as:

$$Y_{K+1} = \begin{bmatrix} \frac{f_{k_1}+1}{1-c(k+1)} & 0 & \cdots & 0 \\ 0 & \frac{f_{k_2}+1}{1-c(k+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{f_{k_n}+1}{1-c(k+1)} \end{bmatrix} Y_K \tag{43}$$

Analysing SSM (2), it can be verified that:

$$y_i(k+1) = \frac{1}{1-c(k+1)}y_i(k) + \frac{f_{k_i}}{1-c(k+1)}y_i(k) \tag{44}$$

Note that if the control is:

$$u_i(k+1) = f_{k_i}y_i(k) + c(k+1)y_{i-1}(k+1) \tag{45}$$

this equation is equivalent to the basic equation for the traffic of trains described in (1) assuming $v_i(k) = 0$. This also can be obtained from the state feedback equation proposed:

$$U_{K+1} = F_K Y_K + A_K^{-1} G_K Y_{K+1}$$

Assuming F_K and G_K obtained by the constraints (26) - (30), SSM (2) and the matrix A_K (21) it is easy to obtain (45). Analysing (15) and the formulation proposed by Corrêa et all (1997), it can be verified that the terms f_{k_i} for $i = 1, 2, \cdots, n$ described in SSM are equivalent to the terms $f_{k,k-1}$ presented in RTM. Similarly, if $f_{k,k} = c(k+1)$ in RTM, without uncertainties, the SSM will have the control $u_i(k+1)$ described by (45). Thus it can be concluded that the sufficient condition for the RTM system to be asymptoticaly stable corresponds to:

$$Y_{j+1} = [A(C) + B(C)F]Y_j + B(C)V_j \tag{46}$$

$$F = \begin{bmatrix} f_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ f_{2,1} & f_{2,2} & 0 & \cdots & 0 & 0 \\ 0 & f_{3,2} & f_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f_{N-1,N-1} & 0 \\ 0 & 0 & 0 & \cdots & f_{N,N-1} & f_{N,N} \end{bmatrix} \tag{47}$$

$$-\lambda_{k,k-1} \leq \frac{f_{k,k-1}+1}{1-c(k)^U} \leq \lambda_{k,k-1} \quad \forall k = 2, 3, \cdots, N \tag{48}$$

$$-\lambda_{k,k-1} \leq \frac{f_{k,k-1}+1}{1-c(k)^L} \leq \lambda_{k,k-1} \quad \forall k = 2, 3, \cdots, N \tag{49}$$

$$f_{k,k} = c(k)^U \qquad \forall k = 1, 2, \cdots, N \tag{50}$$

$$0 \leq \lambda_{k,k-1} < \lambda_m \leq 1 \qquad \forall k = 2, 3, \cdots, N \tag{51}$$

$$\frac{(1-c(k+1)^U)(1-c(k+1)^L)^{n-1}}{c(k+1)^U - c(k+1)^L} + (1-c(k+1)^L)^{n-2} + (1-c(k+1)^L)^{n-3}c(k+1)^L + \cdots + \\ +1 - c(k+1)^L(c(k+1)^L)^{n-3} + (c(k+1)^L)^{n-2} < \frac{(1-c(k+1)^L)^n}{\lambda_m(c(k+1)^U - c(k+1)^L)} \tag{52}$$

Note at platform $k = 1$ the control is constant given by $c(k)^U$. The designer can adjust the stability degree by λ_m, but respecting the conditions in (51) and (52). Nevertheless it is necessary to verify that, depending on the vector Y_K, for lower values in λ_m, the problem can be unfeasible, due the saturation of the control. So, it is necessary to satisfy the control bounds defined by:

$$-\gamma_{uk}d_{uk} \leq f_{k,k-1}y_i(k-1) + c(k)^U y_{i-1}(k) \leq \gamma_{uk}d_{uk}; \qquad 0 \leq \gamma_{uk} \leq 1 \tag{53}$$

5. Real-Time Robust Regulation

The real-time robust regulation of metro lines can be formulated as independent reduced order linear programming problems related one to one with platforms of the line, initiating at the 2^{nd} platform. The problem consider all the operational constraints related to safety traffic in the time deviation $y_j(k)$, time-variant control $u_j(k)$, and variation of the time interval between two successive trains $(y_j(k) - y_{j-1}(k))$.

In the first platform the control is constant $f_{1,1} = c(k)^U$. Thus the state $y_i(1)$ can be obtained directly by:

$$y_i(1) = \frac{f_{1,1} - c(1)}{1 - c(1)} y_{i-1}(1) + \frac{1}{1 - c(1)} v_i(1) \tag{54}$$

where $v_i(1)$ is the random disturbance and $y_{i-1}(1)$ is known by the initial condition.

For the others $(N - 1)$ platforms, the following formulation is proposed:

P.L. k; $k = 2 : N$:

$$min(w\lambda_{k,k-1} + zd_{yk}\gamma_{yk} + pd_{hk}\gamma_{hk} + qd_{uk}\gamma_{uk})$$

$$\begin{gathered}
y_i(k) = \frac{f_{k,k-1} + 1}{1 - c(k)} y_i(k-1) + \frac{f_{k,k} - c(k)}{1 - c(k)} y_{i-1}(k) + \frac{1}{1 - c(k)} dvk \\
-\gamma_{yk} d_{yk} \leq y_i(k) \leq \gamma_{yk} d_{yk} \\
-\gamma_{uk} d_{uk} \leq f_{k,k-1} y_i(k-1) + c(k)^U y_{i-1}(k) \leq \gamma_{uk} d_{uk} \\
-\lambda_{k,k-1} \leq \frac{f_{k,k-1} + 1}{1 - c(k)} \leq \lambda_{k,k-1} \\
f_{k,k} = c(k)^U \\
0 \leq \lambda_{k,k-1} < \lambda_m \leq 1 \\
-\gamma_{hk} d_{hk} \leq y_i(k) - y_{i-1}(k) \leq \gamma_{hk} d_{hk} \\
\frac{(1 - c(k)^U)(1 - c(k)^L)^{N-1}}{c(k)^U - c(k)^L} + (1 - c(k)^L)^{N-2} + (1 - c(k)^L)^{N-3} c(k)^L + \cdots + \\
+(1 - c(k)^L)(c(k)^L)^{N-3} + (c(k)^L)^{N-2} < \frac{(1 - c(k)^L)^N}{\lambda_m (c(k)^U - c(k)^L)}
\end{gathered} \tag{55}$$

where $w \geq 0$, $z \geq 0$, $q \geq 0$ and $p \geq 0$ are scalars used for relative weighting of $\lambda_{k,k-1}$, γ_{yk}, γ_{hk} and γ_{uk}; λ_m can be considered as variable in the problem or adjusted by the designer to obtain the desired stability degree; $c(k)^U$ is the upper bound constant related to the passenger demand; d_{uk}, d_{yk} and d_{hk} are the limits imposed to the control, state and time interval between two consecutive trains; $v_i(k)$ is the random disturbance. The delays (states $y_i(k-1)$ and $y_{i-1}(k)$) are known or estimated on-line nearly simultaneously when the trains arrive at the platforms. The formulation have to consider the worst case of uncertainties and random perturbance. Thus, (55) must be feasible for $c(k) = c(k)^U$ and $c(k) = c(k)^L$, and similarly for $d_{vk} = d_{vk}^U$ and $d_{vk} = d_{vk}^L$.

6. Numerical Example

Consider a metro line with $N = 10$ plataforms and passenger demand parameters bounded by:

$$\begin{aligned}
C^U &= [\ .200 \quad .210 \quad .250 \quad .200 \quad .120 \quad .150 \quad .250 \quad .170 \quad .180 \quad .125\ ] \\
C^L &= [\ .180 \quad .189 \quad .225 \quad .180 \quad .108 \quad .135 \quad .225 \quad .153 \quad .162 \quad .112\ ]
\end{aligned} \tag{56}$$

Also consider the state vector Y_j, the interval $(Y_{j+1} - Y_j)$, the control U_j and the disturbance V_j bounded by:

$$d_y = [\ 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30 \quad 30\ ] \tag{57}$$

$$d_h = [\ 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60\ ] \tag{58}$$

$$d_u = [\ 20 \quad 20 \quad 20 \quad 20 \quad 20 \quad 20 \quad 20 \quad 20 \quad 20 \quad 20\ ] \tag{59}$$

$$d_v = [\ 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2\ ] \tag{60}$$

To ilustrate the performance obtained using the real-time robust regulation comparing with the time-invariant control law proposed in CRRP (Milani et all, 1997), are presented a set of 100 simulated trajectories for the delay (state $y_j(k)$), interval $(y_j(k) - y_{j-1}(k))$ and control $(u_j(k))$ for a train j along the platforms, starting from the initial condition:

$$Y_0 = [\ 30 \quad -30 \quad 0 \quad -30 \quad 30 \quad 0 \quad 0 \quad 30 \quad -30 \quad 0\ ]$$

The simulations consider random C and V_j bounded by (8), (56), (9) and (60).

Fig. (2) presents the results obtained using the formulation CRRP and considering the following weighting parameters in the performance criterium:

$$p = 1 \qquad ; \qquad q = \frac{1}{5}$$

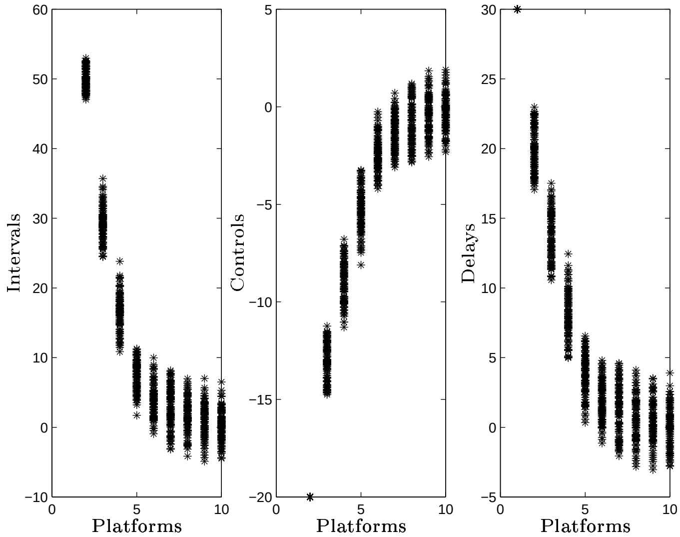

Figure 2. Constrained Robust Regulation (CRRP)

The robust control law $U_j = FY_j$ presented the following bi-diagonal state feedback matrix:

$$F = \begin{bmatrix} 0.1909 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.6163 & 0.0504 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6083 & 0.0583 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.6183 & 0.0484 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6361 & 0.0306 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.6290 & 0.0377 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.6083 & 0.0583 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.6246 & 0.0420 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6225 & 0.0442 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6349 & 0.0318 \end{bmatrix}$$

Fig. (3) presents the results obtained using the real-time robust regulation, considering $\lambda_m = 0,3$ and weighting parameters:

$$z = 1 \qquad ; \qquad q = \frac{1}{5} \qquad ; \qquad p = 1$$

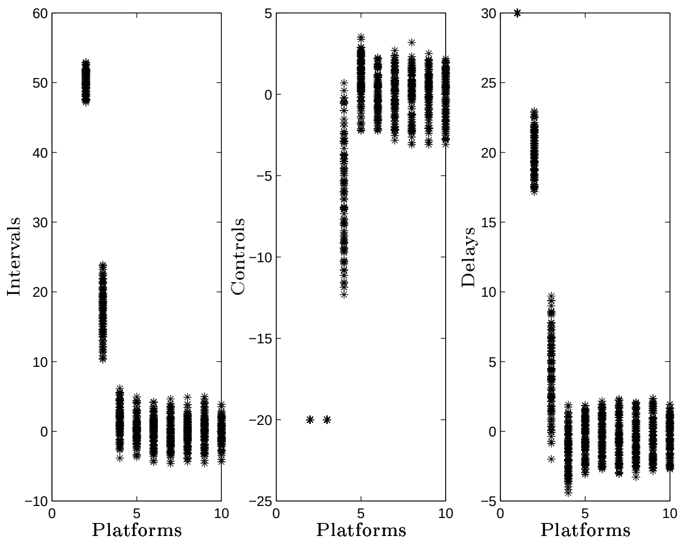

Figure 3. Real-Time Robust Regulation

Comparing the results of the real-time robust regulation, where the feedback control matrix F is time-variant, with the results presented in CRRP, the better performance of the real-time regulation approach is evident.

It can be verified the proposed approach uses the maximum possible control to eliminate the delays and intervals deviations.

7. Conclusion

A new approach for robust regulation of metro lines with nonlinear time-variant state feedback control law was presented. The control law is computed in real-time, assuring asymptotic stability in the presence of traffic model uncertainties and constraints on its state and control variables. The constraints on model variables, the disturbances and uncertain parameter domain were defined by convex compact polyhedra. A numerical example ilustrates the efficiency of the proposed approach which presents better regulation performance than known robust regulation approach using time-invariant feedback control law. The computational efficiency of the proposed approach makes it applicable to real-time regulation of nowadays metro lines.

8. References

Assis, W.O. e Milani, B.E.A., 2002,"Generation of Optimal Schedules for Metro Lines Using Model Predictive Control" - International Federation of Automatic Control - 15^{th} IFAC World Congress, Barcelona, Spain.

Bitsoris, G., 1988, "Positively Invariant Polyhedral Sets of Discrete-Time Linear Systems", Int. J. Control, Vol. 47, No. 6, pp.1713-1726.

Campion, G., Van Breusegem, V., Pinson, P. and Bastin, G., 1985, "Traffic Regulation of an Underground Railway Transportation System By State 385-402.

Chen, C., 1984, "Linear System Theory and Design", Holt, Rinehart and Winston, Inc., Orlando.

Corrêa, S.S. e Milani, B.E.A., 2001, "Regulação Robusta de Tráfego em Linhas de Metrô", Revista Controle & Automação, Vol.12, no. 2.

Cury, J.E., Gomide, F.A. and Mendes, M.J., 1980, "A Methodology for Generation of Optimal Schedules for an Undeground Railway System", IEEE Transactions on Automatic Control, vol. 25, n.2.

Milani, B.E.A. e Corrêa, S.S., 1997, "Decentralized Robust Regulation of Metro Lines" Proc. American Control Conference, pp. 218, 219.

Van Breusegem, V., Campion, G. and Bastin, G., 1991, "Traffic Modeling Automatic