ANALYSIS OF A CONTINUUM DAMAGE MODEL OF BRITTLE MATERIALS

João S. Santos
Universidade Federal Fluminense, LMTA/PGMEC/UFF,

Angela C. Souza
Universidade Federal Fluminense, LMTA/PGMEC/UFF,
angela@mec.uff.br

Fernando P. Duda
Universidade Federal do Rio de Janeiro, PEM/COPPE/UFRJ,
fduda@ufrj.br

Abstract. *This paper is concerned with an one-dimensional version of the continuum theory of brittle materials with microstructure presented by Duda and Souza (2001). The ability of the theory to deal with rate and gradient effects as well as different behavior under traction and compressiom is investigated.*

Keywords. *continuum damage mechanics, microstructure, brittle damage.*

1. Introduction

It is well known that in many circumstances the macroscopic behavior of materials is strongly influenced by microscale processes. In the case of elastic-brittle solids, macroscale loads acting on a material body promote nucleation, growth and coalescence of microcracks, which affect the macroscopic response by means of degradation of material properties.

Continuum models for problems involving degradation have been obtained within the framework of continuum damage mechanics, where new variables (damage variables) are introduced in order to represent the microstructural damage in a continuum sense. The damage variables are considered either as internal state variables or as internal degrees of freedom.

By considering the damage variables as internal degrees of freedom, Duda and Souza (2001) developed a finite strain continuum theory for the behavior of elastic-brittle solids which accounts for microscale processes as well as rate and gradient effects. Microscale processes were considered by introducing a scalar microstructural field and the corresponding force system, presumed consistent with its own balance. Rate and nonlocal effects were taken into account by including constitutive dependences on the rate and the gradient of the microstructural field. The microstructural field was chosen in order to represent the cohesion state within the material: it varies from 1 (pristine material) to 0 (cracked-up material).

In this paper we present and analyse an one-dimensional and small-strain version of the theory developed by Duda and Souza (2001). The main ingredients of this theory are : i) basic laws: the standard force and moment balances; the microforce balance; the dissipation inequality that includes, via the microforces, the power expended during microstructural changes. The basic laws are postulated following the framework developed by Fried and Gurtin (1994); and Gurtin (1996). ii) constitutive theory: constitutive equations consistent with the dissipation inequality, that include both rate and gradient dependences of the microstructural field. The microforce balance, augmented with suitable constitutive information, yields: a kinetic equation for the microstructural field; criteria for both cohesion decreasing (damage initiation and growth) and cohesion increasing (damage healing); the notion of elastic range; a criterion for damage healing impossibility. This model can also describe distinct behavior under traction and compression.

This modelling approach is similar to the dynamic fracture modelling approaches presented by Paas et all (1993) and Peerlings (1999). But in contradistinction to their works, this approach is based on a clear separation of basic balance laws from constitutive equations, including dissipation. In this respect, this theory is similar to the isotropic damage theory presented by Costa-Mattos and Sampaio (1995); Domingues (1996); and Frémond and Nedjar (1996).

The paper is organized as follows. Section 2 introduces the basic notions of a continuum theory for a linear elastic material with microstructure. A general constitutive theory is presented in Section 3, where constitutive assumptions are introduced. Section 4 presents the theory for the case in which the microstructural field is the cohesion descriptor. The analysis of the model is given in Section 5.

2. Basic Notions

Let us consider a bar in a given reference configuration, a fixed interval $[0, L]$, whose points are denoted by x. The "macro" motion of a given point is described by

$$y = y(x, t), \tag{1}$$

where y denotes the position of the material point x at time t. The microstructure of a given point is described by

$$\alpha = \alpha(x, t), \tag{2}$$

where α is the cohesion of the material point x at time t. Quantities of interest are the strain

$$\varepsilon(x, t) = \frac{\partial y}{\partial x}(x, t) - 1, \tag{3}$$

and $p(x, t) = \dfrac{\partial \alpha}{\partial x}(x, t)$.

Two force systems are introduced: the macroforce system, which expends power during the macromotion; the microforce system, which expends power during microstructural changes. The macroforce system is described by (σ, b), where σ is the macrostress and b is the external body force per unit reference lenght. The microforce system is described by (ξ, π, μ), where ξ is the microstress, π and μ are, respectively, the internal and external microbody force per unit length. The macroforce system is presumed consistent with its own balance, which in local form gives

$$\frac{\partial \sigma}{\partial x} + b = 0, \tag{4}$$

whereas the microforce system is constrained by the microforce balance

$$\frac{\partial \xi}{\partial x} - \pi + \mu = 0. \tag{5}$$

Within the present purely mechanical context, the Second Law, or Dissipation Inequality, in local form is given by:

$$\dot{\psi} - \sigma \dot{\varepsilon} - \xi \dot{p} - \pi \dot{\alpha} \leq 0, \tag{6}$$

where ψ is the free energy per unit referential length.

3. Constitutive Theory

We consider constitutive equations by giving ψ, σ, ξ and π at any given point and time when:

$$(\delta, \dot{\alpha}) := (\varepsilon, \alpha, p, \dot{\alpha}) \tag{7}$$

are known at that point and time, i.e.:

$$\psi = \hat{\psi}(\delta, \dot{\alpha}), \quad \sigma = \hat{\sigma}(\delta, \dot{\alpha}), \quad \xi = \hat{\xi}(\delta, \dot{\alpha}), \quad \pi = \hat{\pi}(\delta, \dot{\alpha}). \tag{8}$$

Constitutive dependences on p and $\dot{\alpha}$ are introduced in order to obtain a weakly nonlocal theory and to account for dissipative effects taking place during microstructural changes, respectively. Since the term $\pi\dot{\alpha}$ appears in the dissipation inequality (6), $\hat{\pi}$ must be well-defined only when $\dot{\alpha} \neq 0$. We allow $\hat{\pi}$ to be singular at $\dot{\alpha} = 0$, and for this reason, π is taken to be constitutively indeterminate and defined by (5) at $\dot{\alpha} = 0$. Thus, we consider that $\hat{\psi}$, $\hat{\sigma}$ and $\hat{\xi}$ are smooth functions whereas $\hat{\pi}$ has a singularity at $\dot{\alpha} = 0$ and is smooth otherwise.

By inserting (8) into the dissipation inequality (6) and using the Coleman-Noll procedure, it follows that:

$$\hat{\sigma} = \frac{\partial \hat{\psi}}{\partial \varepsilon}, \quad \hat{\xi} = \frac{\partial \hat{\psi}}{\partial p} \quad \text{and} \quad \frac{\partial \hat{\psi}}{\partial \dot{\alpha}} = 0 \tag{9}$$

and the response functions $\hat{\pi}$ must comply with the residual inequality:

$$\hat{\pi}_d(\delta, \dot{\alpha})\, \dot{\alpha} := \left(\hat{\pi}(\delta, \dot{\alpha}) - \frac{\partial \hat{\psi}(\delta)}{\partial \alpha} \right) \dot{\alpha} \geq 0. \tag{10}$$

It is convenient to decompose the dissipative response function $\hat{\pi}_d$ as

$$\hat{\pi}_d := a^{\pm}(\delta) + b^{\pm}(\delta, \dot{\alpha}) \tag{11}$$

where

$$a^{\pm}(\delta) := \lim_{\epsilon \longrightarrow 0} \hat{\pi}_d(\delta, \pm\epsilon) \quad \text{and} \quad b^{\pm}(\delta, \dot{\alpha}) := \hat{\pi}_d(\delta, \dot{\alpha}) - a^{\pm}(\delta), \tag{12}$$

are, respectively, the rate-independent and the rate-dependent parts of the dissipative response function $\hat{\pi}_d$, and in order to satisfy (10) we assume from now on that

$$a^{+} \geq 0, \quad a^{-} \leq 0, \quad b^{+} \geq 0 \quad \text{and} \quad b^{-} \leq 0. \tag{13}$$

4. Special Theory

We interpret the microstructural field α as the cohesion variable: it varies from 1 (pristine material) to 0 (cracked-up material). In the continuum damage mechanics literature $(1 - \alpha)$ is called the damage variable. If $\dot{\alpha} > 0$ ($\dot{\alpha} < 0$) in a given material point, its cohesion is undergoing a positive (negative) growth. Now, we specialize the theory presented before by supposing the following free energy:

$$\hat{\psi}(\varepsilon, \alpha, p) = \frac{\alpha}{2} E\,\varepsilon^2 + f(\alpha) + g(p), \tag{14}$$

where E is the elasticity modulus, $f(\alpha)$ is the defect energy and $g(p) = \dfrac{\kappa}{2} p^2$, with $\kappa \geq 0$, is the gradient energy. We define the damage energy release rate as

$$\tau(\varepsilon) = \frac{\partial}{\partial \alpha}\left(\frac{\alpha}{2} E\,\varepsilon^2\right) = \frac{1}{2} E\,\varepsilon^2. \tag{15}$$

With these assumptions, the microforce balance is written as:

$$\begin{cases} b^{+}(\delta, \dot{\alpha}) = r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) - \tau(\varepsilon) & \text{if} \quad \dot{\alpha} > 0, \\[2ex] b^{-}(\delta, \dot{\alpha}) = r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) - \tau(\varepsilon) & \text{if} \quad \dot{\alpha} < 0, \end{cases} \tag{16}$$

where:

$$r^{\pm}\left(\delta, \frac{\partial^2 \alpha}{\partial x^2}\right) := \kappa \frac{\partial^2 \alpha}{\partial x^2} - f'(\alpha) - a^{\pm}(\delta). \tag{17}$$

Equation $(16)_1$ implies that $\tau(\varepsilon) < r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right)$ is a necessary condition for positive cohesion growth, whereas equation $(16)_2$ implies that $\tau(\varepsilon) > r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right)$ is a necessary condition for negative cohesion growth. Now, we assume that both are also sufficient conditions for microstructural changes. Thus, as $r^{+} \leq r^{-}$, we have: $r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) \leq \tau(\varepsilon) \leq r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right)$ if and only if $\dot{\alpha} = 0$. Therefore, from (16), we write the kinetic law for α as:

$$\begin{cases} b^{+}(\delta, \dot{\alpha}) = r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) - \tau(\varepsilon) & \text{if} \quad \tau(\varepsilon) < r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right), \\[2ex] \dot{\alpha} = 0 & \text{if} \quad r^{+}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) \leq \tau(\varepsilon) \leq r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right), \\[2ex] b^{-}(\delta, \dot{\alpha}) = r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right) - \tau(\varepsilon) & \text{if} \quad \tau(\varepsilon) > r^{-}\left(\delta, \dfrac{\partial^2 \alpha}{\partial x^2}\right). \end{cases} \tag{18}$$

Equations $(18)_1$ and $(18)_3$ correspond to positive $(\dot{\alpha} > 0)$ and negative $(\dot{\alpha} < 0)$ cohesion growth, respectively. Equation $(18)_2$ corresponds to situations where the cohesion state is frozen in, i.e., no microstructural changes occur and the material behaves elastically.

We interpret

$$r^{-}\left(\delta, \frac{\partial^2 \alpha}{\partial x^2}\right) = \kappa \frac{\partial^2 \alpha}{\partial x^2} - f'(\alpha) - a^{-}(\delta) \tag{19}$$

as the resistance against decohesion, or the damage resistance function. Its dependence on ε allows the present theory to account for unequal tensile and compressive responses.

Notice that if $r^{+} < 0$ damage healing is impossible even if $\tau = 0$. Therefore we conclude that in order to preclude damage healing it is enough to assign a large enough value to a^{+}, which means that the damage is irreversible. Thus, in order to preclude damage healing it is enough to assign a large value to a^{+}. In this case $r^{+} < 0$ and the only threshold is $r := r^{-}$. Then, from (18), we have:

$$\begin{cases} \dot{\alpha} = 0 & \text{if} \quad \tau(\varepsilon) \le r\left(\varepsilon, \alpha, \dfrac{\partial \alpha}{\partial x}, \dfrac{\partial^2 \alpha}{\partial x^2}\right), \\ b^{-}(\varepsilon, \alpha, \dfrac{\partial \alpha}{\partial x}, \dot{\alpha}) = r\left(\varepsilon, \alpha, \dfrac{\partial \alpha}{\partial x}, \dfrac{\partial^2 \alpha}{\partial x^2}\right) - \tau(\varepsilon) & \text{if} \quad \tau(\varepsilon) > r\left(\varepsilon, \alpha, \dfrac{\partial \alpha}{\partial x}, \dfrac{\partial^2 \alpha}{\partial x^2}\right). \end{cases} \tag{20}$$

If we assume that $b^{-}(\sigma, \dot{\alpha}) = \beta\, \dot{\alpha}$, where $\beta > 0$ is the kinetic modulus, that external body forces are neglected, and healing is impossible, the governing equations of the theory ((3), (4), $(9)_1$, (14) and (20)) simplify to

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\alpha E \dfrac{\partial u}{\partial x}\right) = 0, \\ \beta \dot{\alpha} = -\langle \dfrac{1}{2} E \left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\kappa \dfrac{\partial^2 \alpha}{\partial x^2} - f'(\alpha) + A\left(\varepsilon, \alpha, \dfrac{\partial \alpha}{\partial x}\right)\right)\rangle \end{cases} \tag{21}$$

where $\langle x \rangle := \begin{cases} x & \text{if} \quad x \ge 0, \\ 0 & \text{otherwise,} \end{cases}$

5. Examples

In this section we provide simulations using (21). We consider the simplest form to model different tensile and compressive responses:

$$a(\varepsilon) := \begin{cases} a_T & \text{if} \quad \varepsilon \ge 0, \\ a_C & \text{otherwise,} \end{cases} \tag{22}$$

where a_T and a_C are positive parameters that represent different resistances to traction and compression. We are going to present the results of three models concerned with the following functions $f(\alpha)$:

$$f_1(\alpha) = \omega(1-\alpha), \quad f_2(\alpha) = \omega\alpha(1-\alpha) \quad \text{and} \quad f_3(\alpha) = \frac{\lambda}{2}\alpha^2 - \omega\alpha + c, \tag{23}$$

where all constants are positives and $\omega > \lambda$.

We consider two kinds of solutions for a bar of lenght L: i) Homogeneous solution, where there is not the diffusive term $\left(\kappa \dfrac{\partial^2 \alpha}{\partial x^2} = 0\right)$ and ii) Non-homogeneous solution, where this term is considered $\left(\kappa \dfrac{\partial^2 \alpha}{\partial x^2} \neq 0\right)$. The parameter for all solutions are: $E = 27000$ MPa, $\lambda = 0.5 \times 10^{-5}$ MPa, $\omega = 0.5 \times 10^{-4}$ MPa, $a_T = 0.1 \times 10^{-6}$ MPa, $a_C = 0.026$ MPa and $L = 1000$ mm.

The quantity $w + a_T$ represents the strain energy of the material necessary to begin the rupture process, under tensile load. This tensile test must be performed at slow loading velocity to remain in a quasi-static situation. The quantity $w + a_C$ is similar but under compressive load. The viscosity parameter of damage β can be identified by performing experiments at different loading velocities. The parameter κ measures the influence of the damage at a material point on the damage of its neighborhood, and can be valued by performing non-homogeneous loading tests, like bending tests for instance (Fr/e mond and Nedjar (1996)).

We consider initially the homogeneous problem under prescribed axial elongation (ΔL), with strain $\varepsilon = \dfrac{\Delta L}{L}$. Figures 1 and 2 show the results for monotonous loading $(\Delta L = at)$: figure 1 shows the influence of loading velocity a and figure 2 shows the Strain $\times$ Stress curves for different values of kinetic modulus β. These are classical results (see for example Domingues (1996)).

We show in figure 4 differents tensile and compressive responses under the loading history shown in figure 3. They are in agreement with the usual values obtained from experiments (see, for instance, Frémond and Nedjar (1996), p.1093)

In figure 5 we can observe differents curves for differents functions $f(\alpha)$ (equation (23)). The function $f_2(\alpha)$ is apropriate to describe brittle materials whereas $f_1(\alpha)$ and $f_3(\alpha)$ are used to describe quasi brittle materials.

Now we consider the non-homogeneous problem under prescibed axial displacement:

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = u_L(t), \tag{24}$$

and under prescribed α:

$$\alpha(0,t) = 1 \quad \text{and} \quad \alpha(L,t) = 1, \tag{25}$$

with the initial condition: $\alpha(x,0) = 1$. The numerical results were obtained by using Finite Element Method, with 11 bar elements and with monotonous loading $u_L(t) = at$, $a = 2.15$ mm/s.

Figure 6 shows the evolution of damage in a bar at this increasing loading and the figure 7 shows the influence of parameter k. We can see in this result the localization phenomena which is decreasing under increasing k, as we expected.

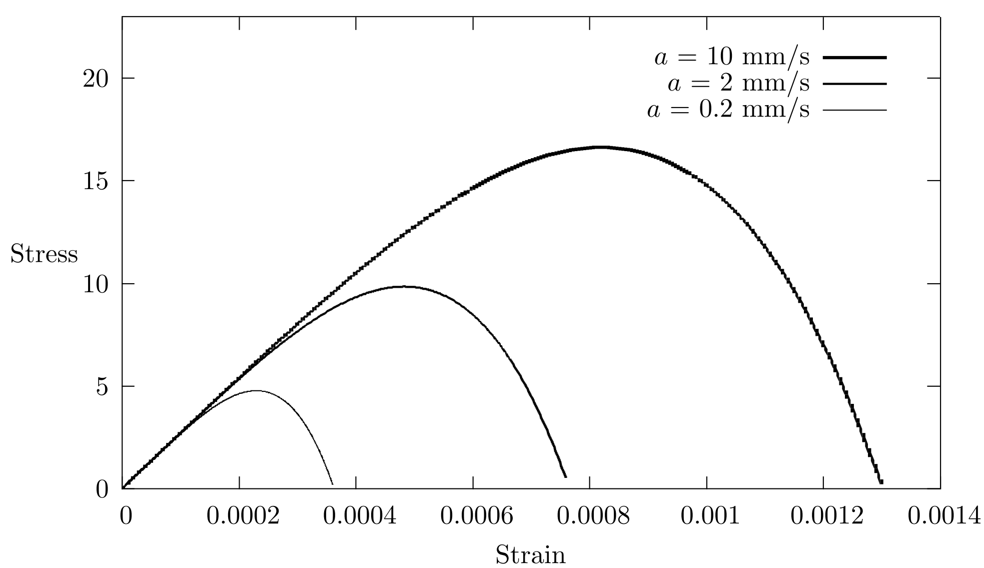

Figure 1: Influence of parameter a

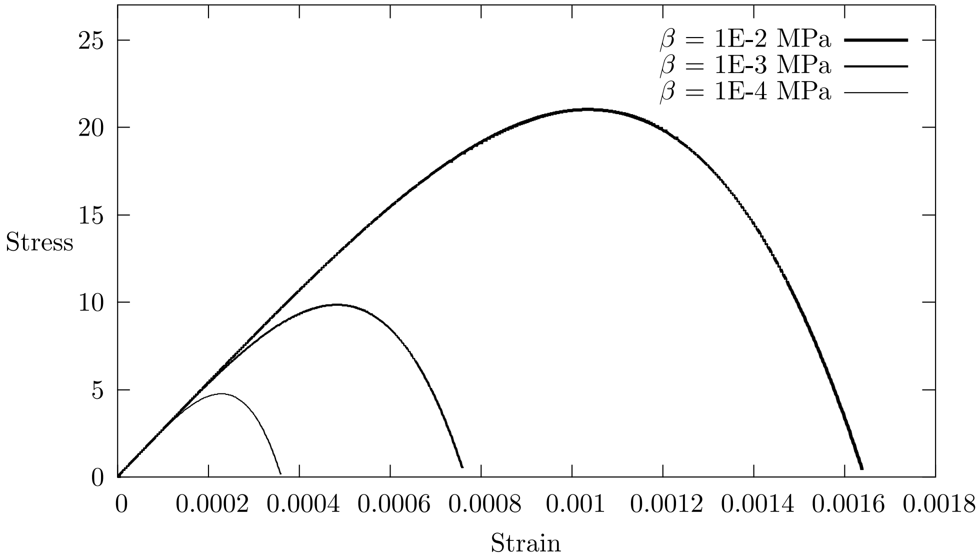

Figure 2: Influence of parameter β

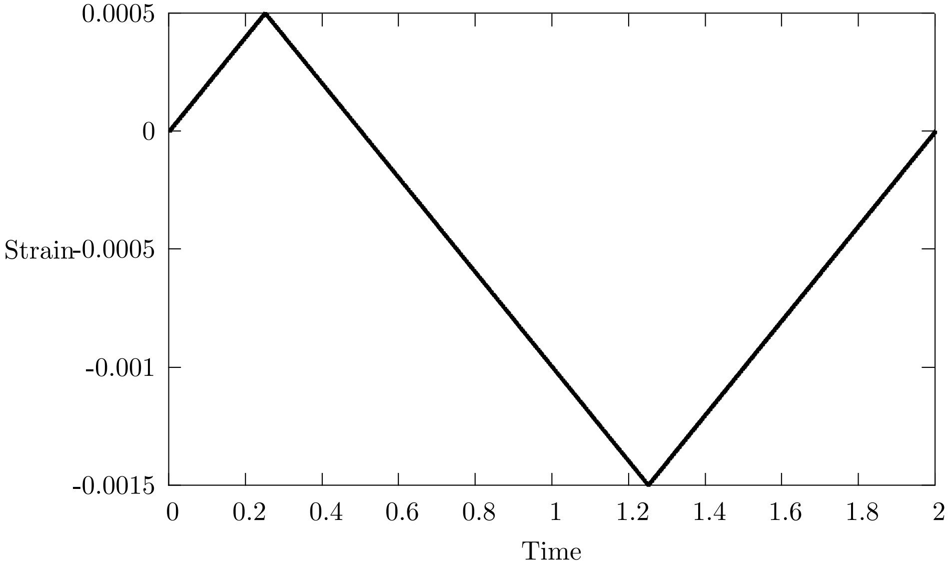

Figure 3: Tensile and compressive loading

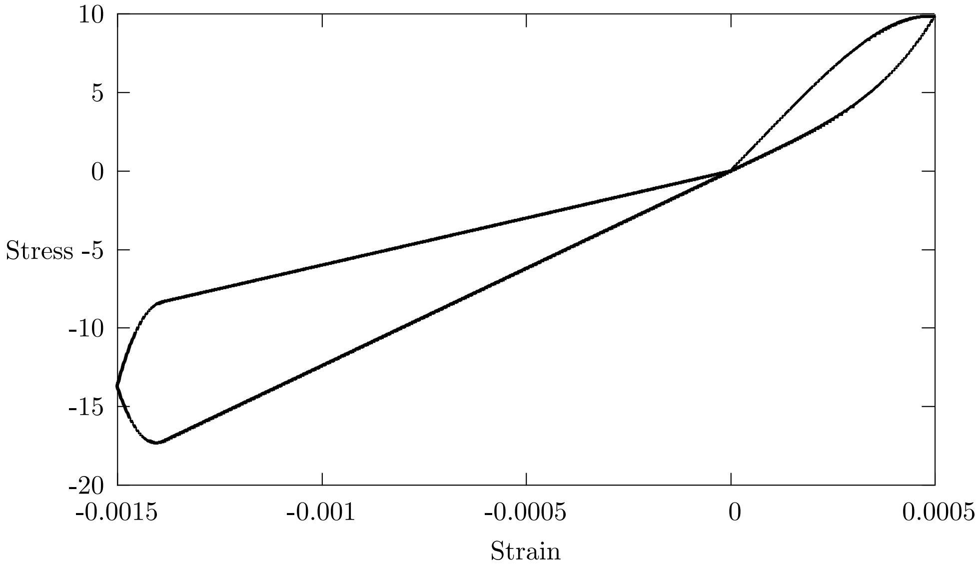

Figure 4: Differents tensile and compressive responses

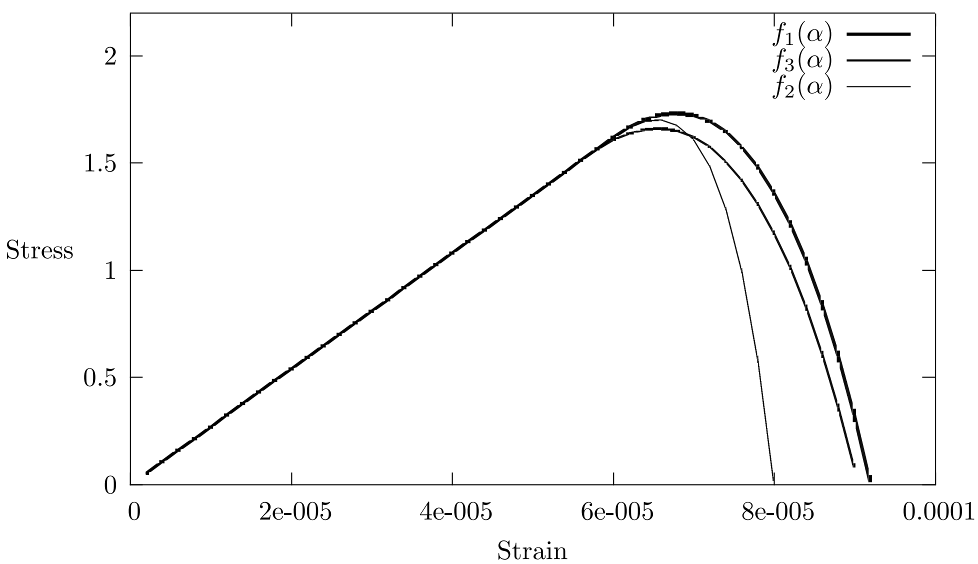

Figure 5: Influence of function $f(\alpha)$

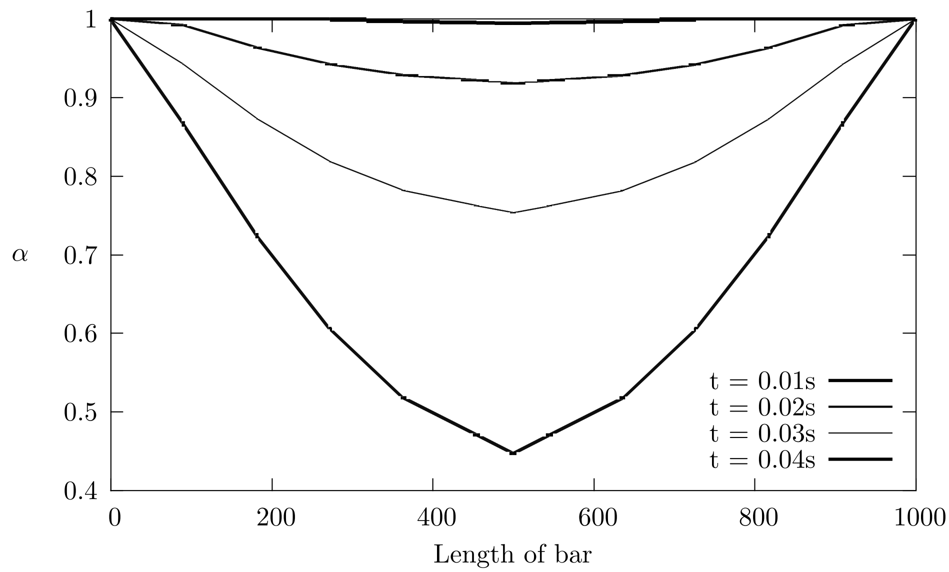

Figure 6: Damage evolution

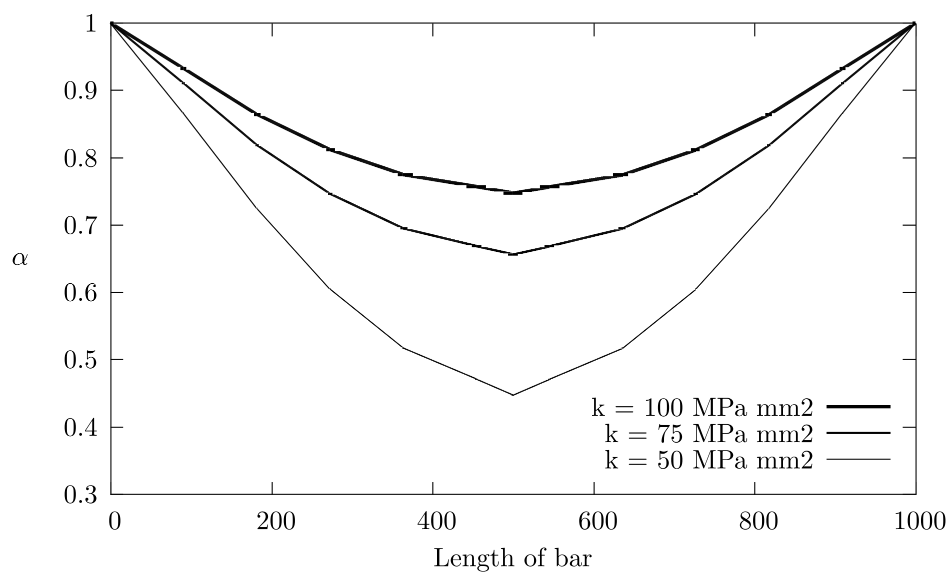

Figure 7: Localization phenomena

6. Conclusions

Following the framework of continuum mechanics, an one-dimensional theory for the macroscopic behavior of brittle solids was formulated, which accounted for: microscale processes by introducing a scalar microstructural field and the corresponding microforce system; rate and nonlocal effects, by including constitutive dependences on the rate and the gradient of the microstructural field; different responses under traction and compression. The microstructural field was interpreted as the cohesion state within the material: it varied from 1 (pristine material) to 0 (cracked-up material).

The simulations provided good qualitative results. Thus, at least in the range of the simulations performed, the ability of the theory to deal with important features of the behavior of brittle and quasi- brittle materials was confimed.

7. Acknowledgements

The support of this research by the CNPq and CTPETRO/CNPq (466146/00) is gratefully acknowledged.

References

Costa Mattos H. and Sampaio R., 1995, "Analysis of the fracture of brittle elastic materials using a continuum damage model", Structural Eng. Mech., Vol.3, pp.411-427.

Domingues S.M.P., 1996, "Análise de Materiais Elásticos Frágeis através de um Modelo de Dano Contínuo", in Portuguese, Doctorate Thesis at Pontificia Universidade Católica - PUC.

Duda, F.P., Souza A.C., 2001, "On a Continuum Theory for Crack Propagation in Brittle Materials", Proc 16th Int. Conference on Struc. Mech. Reactor Tech. - SMiRT 16, CD-ROM, Washington.

Frémond M. and Nedjar B., 1996, "Damange, gradient of damage and principles of virtual power", Int. J. Solids Structures, Vol.33(8), pp.1083-1103.

Fried E. and Gurtin M. E., 1994, "Dynamic solid-solid transition with phase characterized by an order parameter", Physica D, Vol.72, pp.287-308.

Gurtin M. E., 1996, "Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance", Physica D, Vol.92, pp.178-192.

Paas, M.W., Schreurs P.G. and Brekelelmans W.M., 1993, "A Continuum Approach to Brittle and Fatigue Damage: Theory and Numerical Procedures", Int. J. Solids Structures, Vol.30, pp.579-599.

Peerlings, R.J., 1999, "Enhanced Damage Modelling for Fracture ans Fatigue", Ph.D. Thesis at Technische Universiteit Eindhoven.