

STABILIZED FEM METHODS FOR GENERALIZED NEWTONIAN FLOWS

José Karam Filho
jkfi@lncc.br

João N. C. Guerreiro
joao@lncc.br

Regina C. Almeida
rcca@lncc.br

Laboratório Nacional de Computação Científica - MCT
Av Getúlio Vargas 333, Petrópolis, RJ
CEP 25651-071 Brazil

Abstract. *In this paper we present a Petrov-Galerkin finite element method to approach generalized Navier-Stokes-like equations which model non-Newtonian fluid flows having pseudoplasticity and mild convection effects. This formulation yields a method with two iteration levels. The first one deals with the nonlinearity that comes from the constitutive equation and the other is in charge of solving the nonlinearity of the convection feature of these problems.*

Keywords. *non-Newtonian flows, power law, stabilized finite element, mild convection.*

1. Introduction

In the context of pure pseudoplasticity working with an apparent fluidity instead of the viscosity, modelled by generalized Stokes equations, the main numerical difficulties are due to the nonlinearity arising from the power-law constitutive model and the incompressibility constraint. These conditions prevails the use of Newton-like methods for very large or very small power indices, see Glowinski (1984); Glowinsky and Marrocco (1975). Some iterative algorithms have been studied by Carey et al. (1989); Fortin and Glowinsky (1983); Gartling (1986); Saad and Schultz (1986) even for pure elliptic non linear cases. In Karam Filho et al. (1998), a stabilized mixed finite element formulation in four variables was developed to overcome these numerical difficulties. The solution is then obtained by a convergent Uzawa algorithm that decouples the discrete equations and allows the use of Newton-like methods for the resulting nonlinear equations. The resulting formulation presents good accuracy for velocity as well as for the stress and yields stable results even for very large range of the power index.

When considering nonlinear pseudoplastic effects combined with advection an additional numerical difficulty appears. The advective term, with a hyperbolic character, generates an extra nonlinearity that leads to instability problems for large Reynolds numbers.

Without loosing the good properties of the formulation proposed in Karam Filho et al. (1998), in the present work we analyse the problem of non-Newtonian fluid flows in the presence of mildly convective terms, resulting in a method with two iteration levels. The first one deals with the nonlinearity that comes from the constitutive equation and the other is in charge of solving the nonlinearity of the convection feature of these problems. Despite of not being the focus of the present work, it should be mentioned that in the presence of very high gradient solutions an extra stabilization term is required which can be provided by discontinuity capturing terms like the one proposed in Almeida and Silva (1997).

The outline of this work is as follows. Section 2 presents the necessary mathematical definitions that shall be used throughout the paper. Section 3 defines the problem as well as the constitutive equations. The variational formulation is presented in Section 4 together with the algorithm developed to drive the solution. The results are conducted in Section 5 and conclusions are drawn in Section 6.

2. Preliminary Definitions

Some preliminary definitions are required in order to introduce the variational formulation. First of all, let $\Omega \in \Re^2$ be a bounded domain, with boundary Γ. Let $C^m(\Omega)$ be the set of all real functions defined in Ω with continuous derivatives until the order m, $C_0^m(\Omega)$ the subset of functions of $C^m(\Omega)$ with compact support in Ω and $D(\Omega) = C_0^\infty$ the space of all test functions to which the dual, $D^*(\Omega)$, is the space of linear funcionals defined in $D(\Omega)$ (distribution space). $< \cdot, \cdot >$ is the duality product between $D(\Omega)$ and $D^*(\Omega)$, such that given $f \in D^*(\Omega)$,

$$< f, \phi >= \int_\Omega f \ \phi \ d\mathbf{\Omega}, \qquad \forall \phi \in D(\Omega). \tag{1}$$

Given $|\alpha| = \alpha_1 + \alpha_2$, $\forall \alpha_i$ integer, the $\alpha-$derivative of f in the distributional sense is the element $\partial^\alpha f \in D^*(\Omega)$ such that

$$< \partial^\alpha f, \phi >= (-1)^{|\alpha|} < f, \partial^\alpha \phi > \qquad \text{with} \quad \partial^{\alpha(\cdot)} = \frac{\partial^{|\alpha|}(\cdot)}{\partial x_1^{\alpha_1} + \partial x_2^{\alpha_2}}. \tag{2}$$

Let $L^n(\Omega)$ be the Banach space of functions g with modulus n-integrable, $1 \leq n < \infty$, and respective norm

$$L^n(\Omega) = \{g : \Omega \longrightarrow \mathcal{R}; \int_\Omega |g(x)|^n \, d\Omega < \infty\}, \quad \|g\|_{0,n} = (\int_\Omega |g|^n \, d\Omega)^{1/n}, \tag{3}$$

and its dual $L^s(\Omega)$, with the duality pair between $g \in L^n(\Omega)$ and $f \in L^s(\Omega)$ given by

$$(g, f) = \int_\Omega g f \, d\Omega \qquad \text{with} \quad \frac{1}{n} + \frac{1}{s} = 1, \tag{4}$$

and the subspace with zero mean

$$L_0^n(\Omega) = \{g \in L^n(\Omega); \int_\Omega g \, d\Omega = 0\}. \tag{5}$$

For $L^2(\Omega)$ we shall use $\|g\| = \|g\|_{0,2}$. The Sobolev space of order m, subspace of $L^n(\Omega)$, is defined as

$$W^{m,n}(\Omega) = \{g :\in L^n(\Omega); \partial^\alpha g \in L^n(\Omega), 0 \leq |\alpha| \leq m\}, \tag{6}$$

with the usual norm and associated seminorm

$$\|g\|_{m,n} = (\sum_{|\alpha| \leq m} \int_\Omega |\partial^\alpha g|^n \, d\Omega)^{1/n}, \qquad |g|_{m,n} = (\int_\Omega |\partial^m g| \, d\Omega^n)^{1/n} \tag{7}$$

and $W_0^{m,n}(\Omega)$ given by

$$W_0^{m,n}(\Omega) = \{g \in W^{m,n}(\Omega); g = 0 \text{ in } \Gamma\}. \tag{8}$$

Consider the usual finite element discretization in which $\Omega = \cup_{e=1}^{Ne} \Omega^e$, with Ne the number of elements and the mesh parameter $h = \max\{h_e\}$. Let $P_h^k(\Omega)$ be the finite element space of continuous polynomials of order k and class C^0 and $Q_h^l(\Omega)$ be the finite element space of discontinuous polynomials of order l and class C^{-1}. Then,

$$\begin{aligned}
&U_h = U_h^l = \{\boldsymbol{\sigma}_h \in (Q_h^l(\Omega) \cap L_0^2(\Omega))^2\} \subset U = \{\boldsymbol{\sigma} \in (L_0^2(\Omega))^2\}, \\
&U_{Th} = U_{Th}^l = \{\mathbf{S}_h \in (Q_h^l(\Omega) \cap L^2(\Omega))^2, \text{tr } \mathbf{S}_h = 0\} \subset U_T = \{\mathbf{S} \in (L^2(\Omega))^2, \text{tr } \mathbf{S} = 0\}, \\
&V_h = V_h^k = \{\mathbf{u}_h \in (P_h^k(\Omega) \cap W_0^{1,2}(\Omega))^2\} \subset V = \{\mathbf{u} \in (W_0^{1,2}(\Omega))^2\}, \\
&\vartheta_h = \{\theta_h \in P_h^k(\Omega) \cap W_0^{1,2}(\Omega)\} \subset \vartheta = \{\theta \in W_0^{1,2}(\Omega)\}, \\
&\mathcal{W}_h = \{p_h \in Q_h^l(\Omega) \cap L_0^2(\Omega)\},
\end{aligned} \tag{9}$$

and the product spaces with their respective norms

$$\begin{aligned}
&\overline{U}_h = U_{Th} \times U_h, \qquad \|\{\mathbf{T}_h, \tau_h\}\|_{\overline{U}} = \|\mathbf{T}_h\|_U + \|\tau_h\|_U, \qquad \forall \{\mathbf{T}_h, \tau_h\} \in \overline{U}_h, \\
&\overline{V}_h = U_{Th} \times V_h, \qquad \|\{\lambda_h, \mathbf{v}_h\}\|_{\overline{V}} = \|\lambda\|_U + \|\mathbf{v}_h\|_V, \qquad \forall \{\lambda_h, \mathbf{v}_h\} \in \overline{V}_h.
\end{aligned} \tag{10}$$

Define, also, an equivalent mesh dependent norm,

$$\|\boldsymbol{\sigma}_h\|_{h,U} = \|\boldsymbol{\sigma}_h\|_U + \sup_{\boldsymbol{\tau}_h \in U_h} \frac{(\text{div}\boldsymbol{\sigma}, \text{div}\boldsymbol{\tau}_h)_h}{\|\text{div}\tau_h\|}, \quad \forall \boldsymbol{\tau}_h \in U_h, \quad \text{div}\tau_h \neq 0, \quad \boldsymbol{\sigma} \in U. \tag{11}$$

3. Problem Statement

The stationary Non-Newtonian flow behaviour, that fits the generalized newtonian model, can be modelled by the conservation of momentum equation

$$-\text{div}\boldsymbol{\sigma} + m_e(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \mathbf{f}, \qquad \text{in} \qquad \Omega; \tag{12}$$

with the constitutive equation

$$\boldsymbol{\epsilon}(\mathbf{u}) = \nabla^s \mathbf{u} = A(\mathbf{S})\mathbf{S}, \qquad \text{with} \qquad \mathbf{S} = \boldsymbol{\sigma} - \frac{1}{2} \text{tr } \boldsymbol{\sigma} I \ , \tag{13}$$

where $\boldsymbol{\epsilon}(\mathbf{u})$ is the shear strain tensor and $A(\mathbf{S})$ is a nonlinear function. The incompressibility condition is the constraint

$$\text{div}\mathbf{u}=0 \qquad \text{in} \qquad \Omega. \tag{14}$$

These equations are subject on the boundary to

$$\begin{aligned} \mathbf{u} &= \overline{\mathbf{u}}, && \text{on} && \Gamma_{\text{u}}, \\ \nabla\mathbf{u}\cdot\mathbf{n} &= \mathbf{g}, && \text{on} && \Gamma_{\text{g}}. \end{aligned} \tag{15}$$

The bounded domain Ω has a smooth boundary $\Gamma = \Gamma_u \cup \Gamma_g, \Gamma_u \cap \Gamma_g = \emptyset$, with an outward unit normal $\mathbf{n}$. $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{u}$ is the velocity vector, $\mathbf{f}$ is the body force vector, m_e is the density, ν is the fixed value of the viscosity and $\mathbf{g}$ is a given function.

From the approximation point of view, the basic difficulties come from the internal incompressibility constraint and from the nonlinear constitutive part, since we are considering mild velocity gradient variations. In this kind of problems we have two different natures of nonlinearities. One comes from the power-law equation, adopted here, that prevails the use of Newton-like methods for very large or very small power indices, see Glowinski (1984), even for pure elliptic non linear cases. The other nonlinearity comes from the advective term with a hyperbolic character that have instability problems for large Reynolds numbers.

3.1. Constitutive Equations

For Newtonian fluids the stress tensor $\boldsymbol{\sigma}$ is related to the shear strain tensor $\boldsymbol{\epsilon}(\mathbf{u})$ through a linear equation,

$$\boldsymbol{\sigma} = \mathbf{S} + p\mathbf{I}, \tag{16}$$

where p is the pressure and $\mathbf{S} = -2\nu\boldsymbol{\epsilon}(\mathbf{u}), \quad \text{tr}\ \mathbf{S} = 0$, is the deviatoric part of the stress, which is valid assuming low molecular weights and constant viscosity ν.

For fluids that do not obey this linear law, such as several polymer melts and liquid metals, it is defined an apparent viscosity, $A^{-1}(\boldsymbol{\epsilon}(\mathbf{u}))$ or an apparent fluidity $A(\mathbf{S})$, such that

$$\mathbf{S} = A^{-1}(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\epsilon}(\mathbf{u}) \quad \text{or} \quad \boldsymbol{\epsilon}(\mathbf{u}) = A(\mathbf{S})\mathbf{S}. \tag{17}$$

Several relations had been proposed to model the behaviour of such fluids in the region between the lower and the upper Newtonian limits ν_L and ν_u. The more frequently used is the power-law model (called Ostwald-de-Waele model), especially to model pseudoplastic behaviour. It can be written as

$$\mathbf{S} = A^{-1}(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\epsilon}(\mathbf{u}) = K''|\boldsymbol{\epsilon}(\mathbf{u})|^{s-2}\boldsymbol{\epsilon}(\mathbf{u}) \tag{18}$$

or

$$B\mathbf{u} = \boldsymbol{\epsilon}(\mathbf{u}) = A(\mathbf{S}) = A(\mathbf{S})\mathbf{S} = K|\mathbf{S}|^{n-2}\mathbf{S}. \tag{19}$$

The consistency parameters K and K'' have dimensions that depend on n and s, respectively, where n and s are the power dimensionless indices such that $1/n + 1/s = 1$. Besides

$$|\boldsymbol{\epsilon}(\mathbf{u})| = (\nabla^s\mathbf{u} : \nabla^s\mathbf{u})^{\frac{1}{2}}, \quad |\mathbf{S}| = (\mathbf{S} : \mathbf{S})^{\frac{1}{2}}, \quad \text{with } \nu_L < A(\mathbf{S}) < \nu_u. \tag{20}$$

For $s < 2$ or $n > 2$ these relations model pseudoplastic fluids and for $s > 2$ or $n < 2$ they model dilatant fluids that are very rare. Obviously, for $n = s = 2$, the Newtonian case is recovered.

4. Variational Formulation

Instead of substituting the constitutive equation into the momentum equation, it will enter in the formulation as a separate equation in order to allow a numerical algorithm to solve the nonlinearity for a large range of the power index. Introducing the Lagrangian multiplier, $\boldsymbol{\lambda}$, and considering the divergence free condition, we can generate the following formulation for the problem: Find $\{\mathbf{S}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\lambda}\} \in U_T \times U \times V \times U_T$, such that

$$\begin{aligned} &(\boldsymbol{\sigma}, \boldsymbol{\nabla}^s\mathbf{v}) + (m_e(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{v}) = -(f, \mathbf{v}), \quad \forall\mathbf{v} \in V; \\ &(A(\mathbf{S})\mathbf{S}, \mathbf{T}) + (\boldsymbol{\lambda}, \mathbf{T}) = 0, \quad \forall\mathbf{T} \in U_T; \\ &(\nabla^s\mathbf{u}, \boldsymbol{\tau}) + (\boldsymbol{\lambda}, \boldsymbol{\tau}_D) = \mathbf{0}, \quad \forall\boldsymbol{\tau} \in U; \\ &(\mathbf{S} - \boldsymbol{\sigma}_D, \boldsymbol{\mu}) = 0, \quad \forall\boldsymbol{\mu} \in U_T. \end{aligned} \tag{21}$$

Note that $\boldsymbol{\lambda}$ is equivalent to $\nabla^s\mathbf{u}$. Besides, $\mathbf{S}$ and σ_D represent stress deviators and are independently treated.

To deal with the pure nonlinear character of the constitutive equation and the incompressibility constraint, for mild convection effects, we shall approach the problem by adapting the stabilized formulation presented in Karam Filho et al. (1998), now introducing the convection term and without the thermal dependence. Hence, given f, find $\{\mathbf{S}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\lambda}_h\} \in \overline{U}_h \times \overline{V}_h$, such that

$$
\begin{aligned}
&(\boldsymbol{\sigma}_h, \boldsymbol{\nabla}^s \mathbf{v}_h) + (m_e(\mathbf{u}_h \cdot \boldsymbol{\nabla})\mathbf{u}_h, \mathbf{v}_h) + (f, \mathbf{v}_h) + \\
&(A(\mathbf{S}_h)\mathbf{S}_h, \mathbf{T}_h) + (\boldsymbol{\lambda}_h, \mathbf{T}_h) + (\nabla^s \mathbf{u}_h, \boldsymbol{\tau}_h) + (\boldsymbol{\lambda}_h, \boldsymbol{\tau}_{Dh}) + \\
&(\mathbf{S}_h - \boldsymbol{\sigma}_{D_h}, \boldsymbol{\mu}_h) + \delta_1\ (\mathbf{S}_h - \boldsymbol{\sigma}_{D_h}, \boldsymbol{\tau}_h - \mathbf{T}_h) + \\
&\tfrac{\delta_2 h^2}{\nu}\ (\mathrm{div}\sigma_h + m_e(\mathbf{u}_h \cdot \boldsymbol{\nabla})\mathbf{u}_h + f, \mathrm{div}\boldsymbol{\tau}_h)_h = 0, \quad \forall \{\mathbf{T}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\mu}_h\} \in \overline{U}_h \times \overline{V}_h.
\end{aligned}
\tag{22}
$$

Obviously, if $\delta_1 = \delta_2 = 0$ discrete Galerkin formulation is recovered.

5. Solution Algorithm

To solve formulation (22), we propose a Uzawa-like algorithm, where in the first step a mixed form is solved for $\boldsymbol{\sigma}_h^l$ and $\mathbf{u}_h^l$. The second step solves the constitutive part for the deviator $\mathbf{S}_h^l$ and the last step updates the Lagrangian multiplier. Thus, the solution algorithm is:

- Given $\boldsymbol{\lambda}_h^l$ and $\mathbf{S}_h^{l-1}$, evaluate $\{\boldsymbol{\sigma}_h^l, \mathbf{u}_h^l\} \in U_h \times V_h$, $l \geq 1$, through

$$
\begin{aligned}
\delta_1(\boldsymbol{\sigma}_{Dh}^l, \boldsymbol{\tau}_{Dh}) + \tfrac{\delta_2 h^2}{\nu}(\mathrm{div}\boldsymbol{\sigma}_h^l, \mathrm{div}\boldsymbol{\tau}_h)_h - (\nabla \mathbf{u}_h^l, \boldsymbol{\tau}_h) + (m_e \mathbf{u}_h^{l-1} \nabla \mathbf{u}_h^{l-1}, \mathrm{div}\boldsymbol{\tau}_h) = \\
= \delta_1\ (\mathbf{S}_h^{l-1}, \boldsymbol{\tau}_{Dh}) - (\boldsymbol{\lambda}_h^l, \boldsymbol{\tau}_{Dh}) - \tfrac{\delta_2 h^2}{\nu}\ (\mathbf{f}, \mathrm{div}\boldsymbol{\tau}_h)_h, \quad \forall \boldsymbol{\tau}_h \in U_h;
\end{aligned}
$$

$$
(\boldsymbol{\sigma}_h^l, \nabla \mathbf{v}_h) + (m_e \mathbf{u}_h^{l-1} \nabla \mathbf{u}_h^{l-1}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h;
$$

- Once known $\boldsymbol{\sigma}_{Dh}^l$ and $\mathbf{u}_h^l$, evaluate $\mathbf{S}_h^l \in U_{Th}$ by

$$
(A(\mathbf{S}_h^l)\mathbf{S}_h^l, \mathbf{T}_h) + \delta_1\ (\mathbf{S}_h^l, \mathbf{T}_h) = \delta_1\ (\boldsymbol{\sigma}_{Dh}^l, \mathbf{T}_h) + (\boldsymbol{\lambda}_h^l, \mathbf{T}_h), \quad \forall \mathbf{T}_h \in U_{Th};
$$

- Once known $\boldsymbol{\sigma}_{Dh}^l$, $\mathbf{S}_h^l$ and $\boldsymbol{\lambda}_h^l$, evaluate $\boldsymbol{\lambda}_h^{l+1} \in U_{Th}$ by

$$
\boldsymbol{\lambda}_h^{l+1} = \boldsymbol{\lambda}_h^l + \rho_h(\boldsymbol{\sigma}_{Dh}^l - \mathbf{S}_h^l),
$$

- until that

$$
\|\boldsymbol{\sigma}_{Dh}^{n,l} - \mathbf{S}_h^{n,l}\| \leq tol.
$$

The δ_1-term has introduced a linear part into the nonlinear equation, allowing the use of a Newton or quasi-Newton method for the remaining algebraic equation

$$
A(\mathbf{S}_h^l(x_i))\mathbf{S}_h^l(x_i) + \delta_1 \mathbf{S}_h^l(x_i) = \delta_1 \boldsymbol{\sigma}_{Dh}^l(x_i) + \boldsymbol{\lambda}_h^l(x_i), \tag{23}
$$

where $i = 1, ...,$ is the number of integration point (once we know $\boldsymbol{\sigma}_{Dh}^l$ and λ_h^l). For small variations of $\nabla \mathbf{u}$, $\|\mathbf{S}_h^n\|, \|\mathbf{u}_h^n\|, \|\boldsymbol{\sigma}_h^n\|$ and $\|\boldsymbol{\lambda}_h^n\|$ generate sequences bounded by $\|\mathbf{f}\|$ and, now, also by $\nu_L(n, K)$ and $\nu_u(n, K)$, ensuring convergence of the above algorithm.

Note that when convergence is reached (23) recovers the original variational constitutive equation. Hence,

$$
\|\boldsymbol{\sigma}_{Dh}^{n,l} - \mathbf{S}_h^{n,l}\| \to 0 \implies \boldsymbol{\lambda}_h^{l+1} \to \boldsymbol{\lambda}_h^l \to \boldsymbol{\nabla} \mathbf{u}_h^l, \tag{24}
$$

and consequently

$$
(A(\mathbf{S}_h^l)\mathbf{S}_h^l, \mathbf{T}_h) = (\boldsymbol{\nabla} \mathbf{u}_h^l, \mathbf{T}_h), \quad \forall \mathbf{T}_h \in U_{Th}, \tag{25}
$$

or

$$
A(\mathbf{S}_h^l(x_i))\mathbf{S}_h^l(x_i) = \boldsymbol{\nabla} \mathbf{u}_h^l(x_i). \tag{26}
$$

As we are considering small variations of $\boldsymbol{\nabla}\mathbf{u}$, error estimates can be obtained in a similar manner to the pure nonlinear case developed in Karam Filho et al. (1998), yielding

$$
\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{h,U} \leq C_1(h^{l+1}|\boldsymbol{\sigma}|_{l+1} + h^k|\mathbf{u}|_k + h^{l+1}|\mathbf{S}|_{l+1} + h^{l+1}|\boldsymbol{\lambda}|_{l+1}),\\
&\|\nabla\mathbf{u} - \nabla\mathbf{u}_h\| \leq C_2(h^{l+1}|\boldsymbol{\sigma}|_{l+1} + h^k|\mathbf{u}|_k + h^{l+1}|\mathbf{S}|_{l+1} + h^{l+1}|\boldsymbol{\lambda}|_{l+1}),\\
&\|\mathbf{S} - \mathbf{S}_h\|_U \leq C_3(h^{l+1}|\boldsymbol{\sigma}|_{l+1} + h^k|\mathbf{u}|_k + h^{l+1}|\mathbf{S}|_{l+1} + h^{l+1}|\boldsymbol{\lambda}|_{l+1}),\\
&\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_U \leq C_4(h^{l+1}|\boldsymbol{\sigma}|_k + h^{l+1}|\mathbf{u}|_k + h^{l+1}|\mathbf{S}|_{l+1} + h^{l+1}|\boldsymbol{\lambda}|_{l+1}),
\end{aligned}
\tag{27}
$$

with the difference that the presence of the convective term contributes with the density, m_e, to the constants C. Estimate for $\boldsymbol{\sigma}_h$ is still obtained in the equivalent $\|.\|_{h,U}$-norm.

6. Numerical Illustrative Example

A simple example has been selected to illustrate differences between Newtonian and Non-Newtonian cases in the presence of some small advective effects. It was done here through the classical 4×1 contraction entrance problem, with the length being equal to eight times the entrance radius.

Comparisons are performed for $n = 2$ (Newtonian) and $n = 4$ (Non-Newtonian), both with consistency parameter set to unity. The entrance velocity was prescribed as a uniform 13.5 value.

In Fig. (1) it can be seen the pseudoplastic behaviour developed by the $n = 4$ fluid that can be compared with the Newtonian behaviour in Fig. (2). They show that even the higher velocity magnitudes after the contraction pseudoplastic effects can be observed for this case.

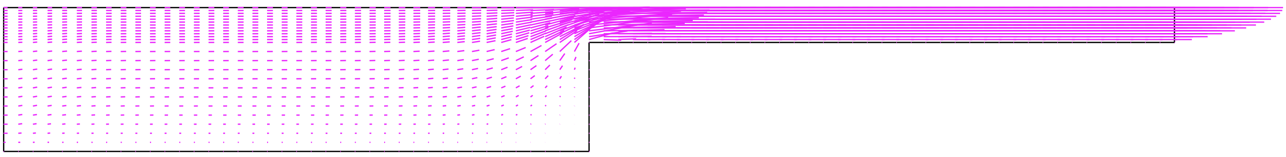

Figure 1: Non-Newtonian Case.

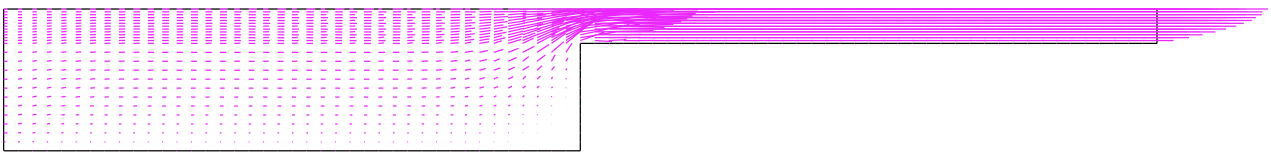

Figure 2: Newtonian Case.

In Fig. (3) and Fig. (4) it can be observed the small circulation effect that is presented only in the Newtonian case, as expected, see Crochet et al. (1985).

Figure 3: Non-Newtonian Case, a zoom.

Figures (5) and (6) show in detail the differences between the fluid flows before the contraction.

Figure 4: Newtonian Case, a zoom.

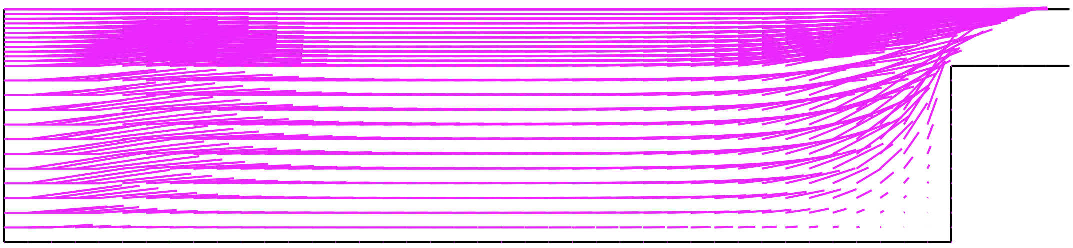

Figure 5: Non-Newtonian Case, a detail.

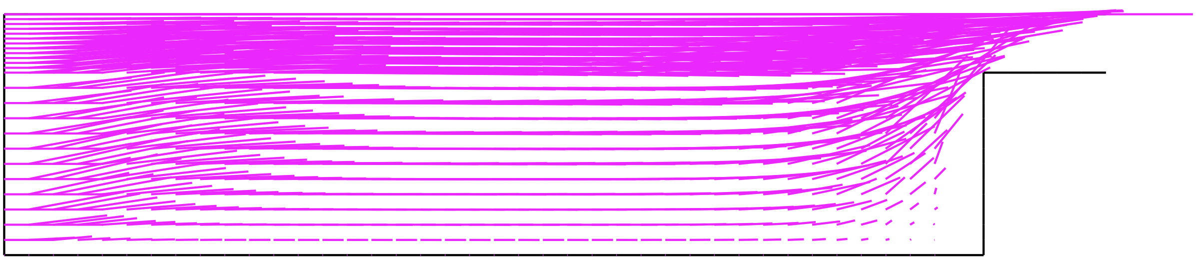

Figure 6: Newtonian Case, a detail.

7. Conclusions

In this work it was presented a mixed stabilized finite element method to approach Non-Newtonian fluid flows. It has been designed by taking advantage of the formulation in Karam Filho et al. (1998) in the sense that it is able to solve the power-law equation in regions where other methods fail by adapting the solution algorithm to a two levels, maintaining the variational constitutive equation uncoupled from the momentum governing equation, even in the presence of the convection term.

From the numerical analises point of view, for problems that do not present high gradients, error estimates are obtained in a manner similar to that of Karam Filho et al. (1998), for no convection, and orders of convergence are preserved too, with differences appearing only in the constants.

From the illustrative classical examples presented, the method distincts between Newtonian and Non-Newtonian behaviours for the same other conditions, what can be seen from the pseudoplastic effect that is developed for $n = 4$ and the small circulation area that appears only for the Newtonian case.

The objective of this work was to study the possibility of introducing the non-linearities of the advective term into the formulation presented before that solved with success for the high non-linear purely elliptic part. The results of this work encourage us with the possibility of adapting other methods that are able to capture higher gradient solutions, like that in Almeida and Silva (1997), to the formulation in Karam Filho et al. (1998), that is able to deal with extreme power indices.

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