

NON-LOCAL FINITE STRAIN ELASTICITY MODELS FOR TRUSSES

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Abstract. *The present work addresses the study of strain localization phenomena in hyperelastic media undergoing finite deformation. In this context, localization is a consequence of geometric instabilities. A regularization technique, called non-local integral, is used to restore the well-posedness of the boundary value problem and to eliminate spurious mesh dependence of the numerical results. In the non-local model here proposed for truss structures, the strain at a point is assumed to depend on weighted average of the stretch over a certain neighborhood, and not only on its local value. Numerical implications of this choice are considered. The effectiveness of proposed model in finite element analysis is tested with reference to a bar under quasi-static loading.*

Key Words. *Localization, Large Strains, Non-local Models.*

1. Introduction

Classical local continuum theories are based on the assumption that the material behavior at a point only depends on the value of a set of state and internal variables at that point. In other words, classical continuum theories do not incorporate a material length scale. These theories are able to interpret the material behavior in a large number of applications. However, when the material response becomes highly non homogeneous (localized) due to, e.g., strain softening or very large strains, local theories become inadequate. With local models, the width of the localization zones tends to zero, with the nowadays well known numerical consequences in terms of pathological mesh dependence. Mathematically, the boundary value problem becomes ill-posed - Rice (1976) Benallal et al. (1988); Rizzi (1995) and the mathematical description no longer represents the physical reality. This problem is well reported in the literature when material nonlinearity, such as damage or plasticity, is considered.

The ill-posedness of a problem can be overcome by using regularization techniques, which provide accurate numerical solutions. Non-local gradient models (Frémond and Nedjar (1996); de Borst et al. (1995); Comi and Driemeier (1998); Comi (1999)), non-local integral models (Bazant et al. (1984); Bazant (1987); Pijaudier-Cabot and Bazant (1987)Pijaudier-Cabot and Bazant (1987); Comi (2001)) and micropolar models(de Borst and Muhlhaus (1991); Steinmann and Stein (1994)) have been formulated and effectively used.

Extension of these inelastic models to the large strain context has also been studied in the literature - Benallal and Tvergaard (1995); Steinmann (1996, 1999); Brunig et al. (2001) among others, but always the regularization is performed on the inelastic response. Thus, as pointed out in Benallal and Tvergaard (1995), the boundary value problem may still become ill-posed due to geometric effects at very large strains.

In this work we focus on localization phenomena due to purely geometrical nonlinearity in an hyperelastic medium. A non-local hyperelastic model for truss structures is proposed in which the stretch is treated as a non-local variable, defined as the weighted average of the local stretch over a certain neighborhood. The effectiveness of this model in finite element analysis is tested with reference to a bar under quasi-static loading. The formulation is based on the non-local elastic model proposed by Eringen (1972a,b) in the small strains context. Physical motivations for using non-local elasticity are presented in References Eringen (1992); Pan (1996); Povstenko (1995).

2. Hyperelastic local model

When a bar is loaded, its configuration changes from a given initial state, at $t = t_0$, with cross section area A, volume V and length L, to the current state, at $t = t_1$, with cross section area a, volume v and length ℓ, from which a stretch ratio λ can be defined as

$$\lambda = \frac{\ell}{L}. \tag{1}$$

where $\lambda = 1$ indicates an unstrained state.

There are many possibilities to define a strain measure ε from the stretch $\varepsilon = f(\lambda)$; here the following deformation family is considered

$$\varepsilon_m = \begin{cases} \frac{1}{m}(\lambda^m - 1) & \text{if } m \neq 0, \\ \ln \lambda & \text{if } m = 0 \end{cases} \tag{2}$$

Noteworthy members of this family are the logarithmic strain ($m = 0$), the linear strain ($m = 1$) and the Green strain ($m = 2$).

In structural analysis, one can choose any of the strain measures defined in order to describe kinematics. However, an immediate question is what stress measure σ should be used associated with the chosen strain.

According to Hill (1978), regardless of the strain and stress measure adopted, the internal power per unit of volume,

$$\bar{U}^i = \sigma \dot{\varepsilon}, \tag{3}$$

should always be the same. The expression *work–conjugate* is used to imply that a set of strain–stress definitions is physically consistent.

When the strains become large, it is common to adopt the specific logarithmic strain ($\varepsilon = \ln \lambda$) and its conjugate stress ($\sigma = \frac{N}{a} \frac{v}{V}$), called *true* or *Cauchy stress*(Driemeier et al., submitted): these strain and stress measures will be used in the next sections.

If a liner elastic behavior is considered, the following relation between initial A and current a area for the cross section (Crisfield, 1997) holds

$$a = A\lambda^{-2\nu} \tag{4}$$

ν being the Poisson coefficient. Denoting by $V = AL$ and $v = a\ell$ the initial and current volume, respectively, one has

$$v = V\lambda^{1-2\nu} \tag{5}$$

For $\nu = 0$ the area of the cross section is preserved for any value of stretching, while for $\nu = 0.5$ the volume of the structure is constant. For $0. < \nu < 0.5$, area and volume are modified by the deformation. When $\nu = 0.5$, the stress measure is given by $\sigma = \frac{F}{a}$, i.e., the applied force divided by the current area.

3. Bar in tension: local model

3.1. Numerical results

The classical analysis of a simple bar loaded in tension allows to demonstrate the shortcomings of a local model in the presence of very large strains. References Peerlings et al. (2001); Comi (1999) highlight these shortcomings when damage is considered, while References Bazant and Belytschko (1984); de Borst and Muhlhaus (1991); Comi and Corigliano (1996); Nilsson (1997) explore the same problem for plasticity. As already mentioned, the traditional focus is on the material non-linearity.

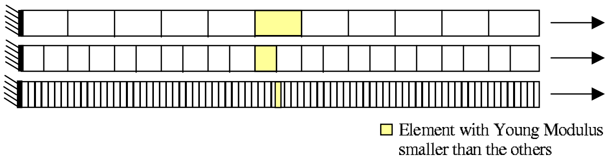

Figure 1. Uniaxial bar in tension with different discretizations. Initial length $L = 10\ mm$.

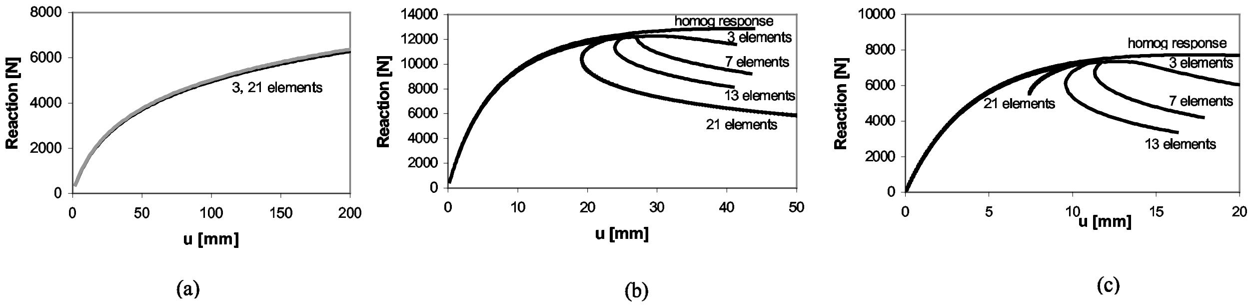

Figure 2. Axial displacement versus reaction force for different discretizations and poisson coefficient: (a)$\nu = 0$, (b)$\nu = 0.3$ and (c)$\nu = 0.5$.

Figure 1 shows a bar uniformly loaded and divided into n elements. The central element is endowed with a slightly different Young Modulus in order to induce the localized response.

Finite element analysis were performed in a *house* code, changing the Poisson parameter and the mesh. The same results were obtained with the commercial program *Abaqus*.

Figure 2a shows the results in terms of reaction versus axial displacement curves obtained with $\nu = 0$, i.e. when transversal area is maintained constant. No localization occurs in this case and a mesh-independent solution is found.

Figures 2b and 2c show the response for $\nu = 0.3$ and $\nu = 0.5$, respectively. The localization occurs first for the case of $\nu = 0.5$. In both cases, the configuration update leads to a nonconstant stress along the bar.

For the cases illustrated in Figures 2b and 2c, it is clear that, by refining the mesh, the width of the zone where localization occurs tends to zero and no convergence is observed.

3.2. Localization problem for a two elements bar in tension

In order to understand the localization phenomena for the hyperelastic case illustrated in the previous section, let us consider the bar of Figure 3 divided into two elements. The constitutive law for the material is taken to be

$$\sigma = E\varepsilon \tag{6}$$

where

$$\varepsilon = \ln(\lambda) \tag{7}$$

and E is the Young modulus.

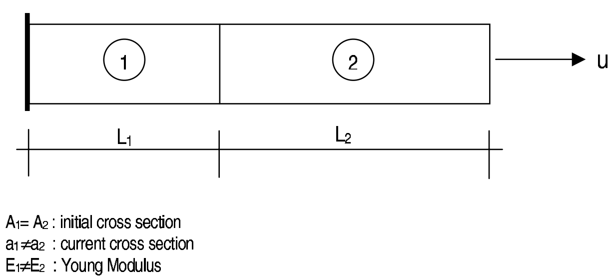

Figure 3. Uniaxial bar in tension subjected to an imposed displacement.

The equilibrium equation is given by

$$\sigma_1 a_1 - \sigma_2 a_2 = 0 \tag{8}$$

where σ_1, a_1 and σ_1, a_2 are the stress and current cross section for elements 1 and 2, respectively.

The compatibility equation can be written in rate form as,

$$\dot{u} = \ell_1 \dot{\varepsilon}_1 + \ell_2 \dot{\varepsilon}_2 \tag{9}$$

where u is the axial displacement at the end of the bar.

Substitution of Equations (4) and (6) into Equation (8) results in

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{\alpha}{\beta}\left(\frac{\ell_1}{\ell_2}\right)^{2\nu} \tag{10}$$

where

$$\alpha = \frac{E_2}{E_1} \qquad \beta = \left(\frac{L_1}{L_2}\right)^{2\nu}. \tag{11}$$

The derivative of Equation (10) results in

$$\dot{\varepsilon}_1 \ell_2^{2\nu} + \dot{\ell}_2 \varepsilon_1 2\nu \ell_2^{2\nu-1} = \frac{\alpha}{\beta}\left(\dot{\varepsilon}_2 \ell_1^{2\nu} + \dot{\ell}_1 \varepsilon_2 2\nu \ell_1^{2\nu-1}\right) \tag{12}$$

From the definition (7) one has

$$\dot{\varepsilon}_1 = \frac{\dot{\ell}_1}{\ell_1} \qquad \dot{\varepsilon}_2 = \frac{\dot{\ell}_2}{\ell_2} \tag{13}$$

Substituting Equation (13) into Equation (12), and after some algebraic manipulations, one obtains

$$\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2} = \frac{\varepsilon_1}{\varepsilon_2}\frac{(1-2\nu\varepsilon_2)}{(1-2\nu\varepsilon_1)} \tag{14}$$

From Equation (14) one can conclude that, when the strain reaches the value $\frac{1}{2\nu}$ at one of the elements, the strain rate in the other element vanishes. For strain values higher than $\frac{1}{2\nu}$ in one element, the ratio $\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2}$ becomes negative and further increase of strain is localized. It is worth noting that for $\nu = 0$ the critical value of deformation tends to infinite and no localization occurs, as already numerically found.

The element in which localization first occurs can be obtained by analyzing Equation (10), which can be rewritten as,

$$\varepsilon_1 = \frac{E_2}{E_1}\left(\frac{\ell_1}{L_1}\frac{L_2}{\ell_2}\right)^{2\nu}\varepsilon_2 \tag{15}$$

Considering that before localization the initial and current length preserves the proportionality, the response will be defined by the ratio of Young modulus. Accordingly, if $E_1 > E_2$, localization occurs in element 2, and if $E_2 > E_1$, localization occurs in element 1.

Figure 4 shows the response in terms of reaction versus axial displacement for the different cases listed in Table 1, all with $\nu = 0.5$.

Analysis	E_1 (N/mm^2)	E_2 (N/mm^2)	L_1 (mm)	L_2 (mm)	α	β
01	20000	21000	0.6	0.4	1.05	1.50
02	20000	21000	0.1	0.9	1.05	0.11
03	25000	21000	0.6	0.4	1.25	1.50
04	20000	18000	0.6	0.4	0.90	1.50

Table 1. Data for different analysis of the bar in tension, for $\nu = 0.5$.

From Equation (9), $\dot{u} = 0$ when

$$\ell_1 \dot{\varepsilon}_1 = -\ell_2 \dot{\varepsilon}_2 \tag{16}$$

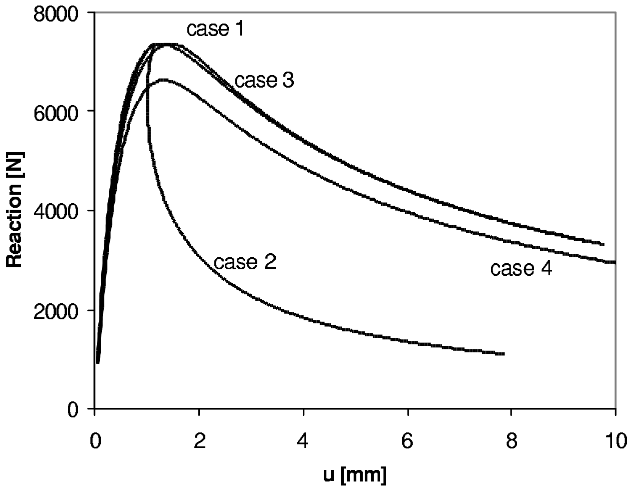

Figure 4. Axial Displacement x Reaction curve for the bar in tension.

which is possible only when $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ have different signs.

When the element which unloads is larger than the one in loading (case 2), the curve presents *snap-back* ($\dot{u} < 0$). Using Equations (9) and (13), the *snap-back* condition reads

$$-\dot{\ell}_1 > \dot{\ell}_2 \tag{17}$$

This means that *snap-back* phenomenon is observed when the element which unloads shrinks faster than the other element is stretched.

Figure 5a shows the strain localization, evidenced by unloading in bar 2, starting exactly at $\varepsilon = 1$. At this stage (for the same value of imposed displacement u) the strain rate ratio changes its sign, as shown in Figure 5b.

Note that localization for this elastic case can only be appreciate if the finite strain analysis is carried on until very large strains are attained.

4. Non-local model

According to Eringen (1972b,a), the non-local elastic model can be defined introducing the non-local strain, defined as a weighted average of the corresponding local measure taken over the neighboring material points of the body, i.e.

$$\bar{\varepsilon}(\mathbf{x}) = \int_v W(\mathbf{x} - \mathbf{s})\, \varepsilon(\mathbf{s})\, \mathrm{d}v \tag{18}$$

where $\bar{\varepsilon}$ is the non-local strain variable at point $\mathbf{x}$ and $W(\mathbf{x} - \mathbf{s})$ is a weight function. A possible choice for $W(\mathbf{x} - \mathbf{s})$ is the gaussian weight function, as proposed in Pijaudier-Cabot and Bazant (1987) Comi (2001),

$$W(\mathbf{x} - \mathbf{s}) = \frac{1}{W_0(\mathbf{x})} \exp\left(\frac{-\|\mathbf{x} - \mathbf{s}\|^2}{2\vartheta^2}\right) \quad \text{with} \quad W_0(\mathbf{x}) = \int_v \exp\left(\frac{-\|\mathbf{x} - \mathbf{s}\|^2}{2\vartheta^2}\right) \mathrm{d}v \tag{19}$$

The parameter ϑ represents an internal scale, which is considered a material characteristic with dimension of a length. This parameter defines the dimension of the neighborhood that affects the non-local function.

When the deformations become large, it is also necessary to update geometrical characteristics as a function of the non-local strain consideration. To this end, it is easier to consider as the non-local variable the stretching λ instead of the strain ε.

For the static case and vanishing body force the boundary value problem for an isotropic, non-local hyper-elastic three-dimensional bar, reads

$$\begin{aligned} &\frac{\partial \sigma}{\partial \mathbf{x}} = 0 \\ &\sigma = E\varepsilon \\ &\varepsilon = \ln \bar{\lambda} \end{aligned} \tag{20}$$

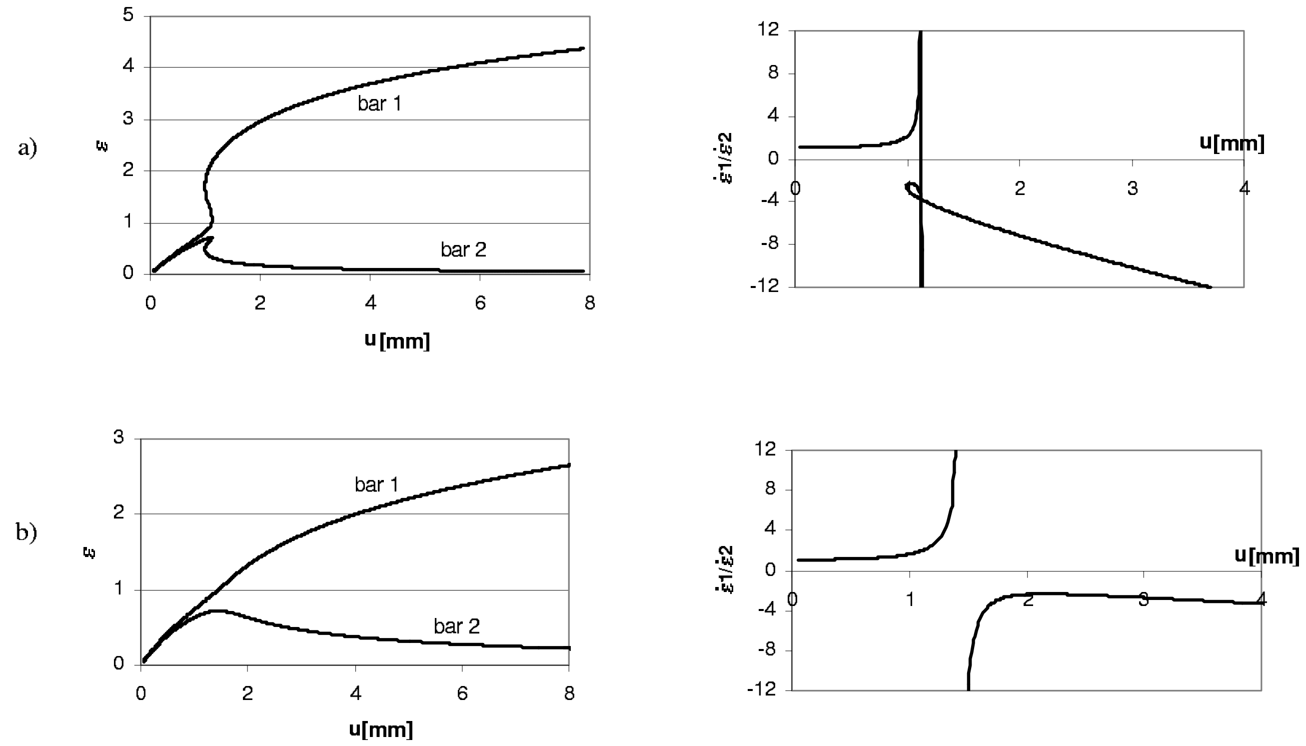

Figure 5. Strain in each element and relative strain rate evolution during the analysis, for cases (a)1 and (b)2.

as the governing equations. $\bar{\lambda}$ is the non-local stretching ratio variable.

The non-local model was implemented in the Non-Linear Finite Element code described in References Driemeier et al. (submitted), whose results are shown next.

5. Example

The same problem considered in Section 3.1 and shown in Figure 1 is analyzed with different discretizations (7, 13, 21 and 61 elements). The elastic constants are $E = 21000\ N/mm^2$, $\nu = 0.5$ and the bar length is $L = 10mm$. Again, the central element is defined with a different Young modulus in order to induce the localized response. For the non-local model, an internal length $\vartheta = 1.0\ mm$ was selected in order to balance the loaded and unloaded elements after localization.

Figure 6 shows the response in terms of reaction versus displacement. As a result of the non-local regularization, no pathological mesh dependence of the response occurs. In contrast to what is found with the local model (see Figure 2c), as the mesh is refined, the response converges to a unique, physically acceptable solution.

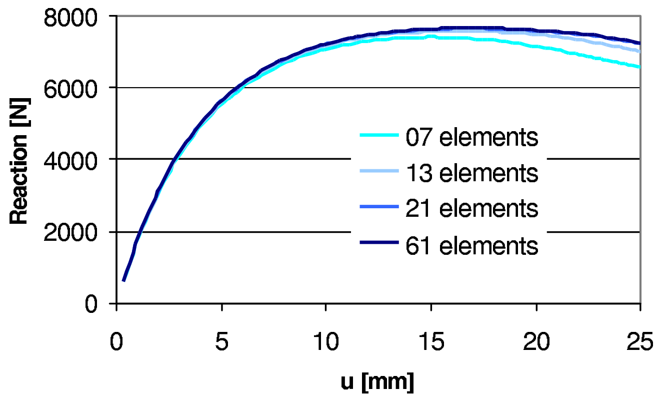

Figure 6. Axial displacement x Reaction curve for different meshes.

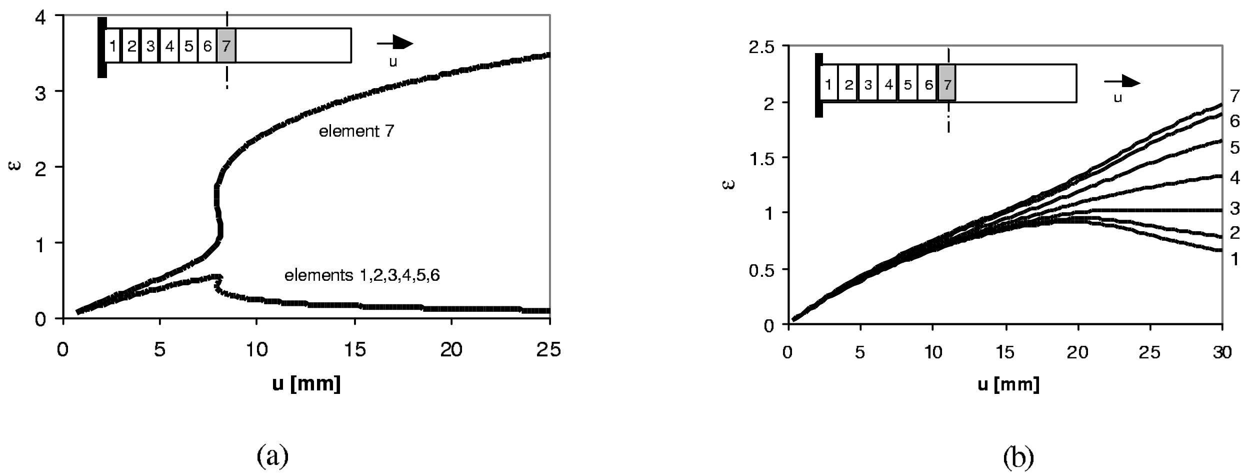

Figure 7. Strain evolution in the elements for (a) local and (b) non-local analysis.

Figures 7a,b compare the results of the local and non-local analysis with 13 elements. From Figure 7a one can see that, starting from a certain point, with the local model the strain becomes localized and increases in one element only. On the contrary, for the non-local analysis (Figure 7b), the strain continues to increase within a band, whose length is defined by the material internal length of the model ϑ, which for the considered mesh comprises 4 elements.

6. Conclusions

The present work discusses the localization phenomenon in hyperelastic truss structures undergoing very large deformations. In this case, although the material behavior is linear, the nonlinear geometry can induce a localization not fully explored yet.

The non-local elasticity model for small strains proposed by Eringen (1972b,a) is used here to restore the well-posedness of the problem, and some corrections to take into account finite strains are made considering the non-local stretching ratio $\bar{\lambda}$.

A simple example, often used in the literature to check regularization properties of the proposed approaches, shows that the spurious mesh dependence, consequence of the localization in finite element analysis, is eliminated when using the non-local model.

The material length ϑ is related to the width of the strain localization zone and, thus, it is an important parameter in governing the softening branch of the response. Accordingly, when the finite element is larger than the localization zone, the mesh dependence can still be observed. As the mesh is refined, the solution converges to a unique response, as shown in Figure 6.

The example shows localization only for $u > 5\ mm$ of axial displacement, which corresponds to approximately $\varepsilon = 50\%$. Although at this strain level, most engineering materials have being yielded, the objective of this work is to highlight the geometrical mesh dependence. The approach can be extended in order to embrace both nonlinearities: geometric and material, through a plastic and/or damage model. When the elastic strains are small, the authors believe, the regularization of the internal plastic and/or damage variable may eliminate the mesh dependence. But, for materials as rubber, only the material regularization may not eliminate the problem.

7. Acknowledgments

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