SIMULATION OF A FLOW THROUGH AN UNSATURATED NONISOTHERMAL WELLBORE

Maria Laura Martins-Costa‡
Laboratory of Theoretical and Applied Mechanics (LMTA)
Mechanical Engineering Department - Universidade Federal Fluminense
Rua Passo da Pátria, 156, 24210-240 Niterói/RJ, Brazil
laura@mec.uff.br

Rogério M. Saldanha da Gama
Mechanical Engineering Department, Universidade do Estado do Rio de Janeiro
Rua São Francisco Xavier, 524, 20550-013, Rio de Janeiro, RJ, Brazil
rsgama@domain.com.br

Abstract. *This work studies a Newtonian fluid flow through a cylindrical shell rigid porous matrix and the heat transfer associated phenomenon, in order to build a preliminary local description for nonisothermal flows through an unsaturated wellbore. The model is built in by using a mixture theory approach – the mixture consisting of three overlapping continuous constituents, representing the porous matrix (solid constituent), the Newtonian fluid (liquid constituent) and an inert gas included to account for the compressibility of the mixture as a whole. This problem mathematical description is given by a set of four nonlinear partial differential equations. A forced convection hypothesis is assumed and the hydrodynamic part – approximated by means of a Glimm's scheme, combined with an operator splitting technique is used as input for the thermal part – simulated by a finite difference scheme. Some examples illustrate the proposed strategy.*

Keywords*. Unsaturated porous medium forced convection, Mixture Theory, Riemann problem, shock waves, operator splitting.*

1. Introduction

Among the practical applications of transport phenomena in porous media, groundwater flows, enhanced oil recovery processes, contamination of soils by hazardous wastes, storage of nuclear waste material in deep earth rock layers or deep ocean sea beds and pollution movement, could be mentioned. The increasing interest related to such phenomena may be explained by the importance attached to problems that impact the energy self-sufficiency and the environmental state. Transport phenomena in unsaturated porous media are characterized by a strong dependence of the motion on the saturation. Since the media are partially saturated, there is a force (depending on the saturation gradient) giving rise to a fluid flow. According to Tien and Vafai (1990), these phenomena have been studied since the 1920s, emphasizing momentum transport. The drying phenomenon was simulated supposing the fluids motion through the porous medium caused by diffusion only – the balance of linear momentum being substituted by the diffusion equation. In other studies the influence of capillary forces (surface tension) in the modeling of liquid motion has also been considered.

This work studies the dynamics of the filling up of a rigid cylindrical shell porous matrix by a Newtonian fluid and the heat transfer associated phenomenon, in order to build a preliminary local description for nonisothermal flows through a wellbore, using a mixture theory approach in the mechanical modeling. This approach, assuming the mixture composed by superimposed continua, is a convenient framework for modeling multicomponent systems – being supported by a local theory with thermodynamic consistency which generalizes the classical Continuum Mechanics. The mixture – modeling the unsaturated porous medium – consists of three overlapping continuous constituents: a solid (porous medium), a liquid and an inert gas (accounting for the compressibility of the mixture as a whole). Another approach is employed for treating most problems dealing with transport phenomena in porous media – a volume averaging technique – in which concentration and velocity components are described as volumetric averages in order that the momentum transport may be described in a classical continuum mechanics context. This approach has already allowed the analysis of complex problems, among which one could mention, for instance, the multiphase transport process with phase change in unsaturated porous media (Vafai and Whitaker, 1986), or the mixed convection (Aldoss et al., 1996; Chang and Chang, 1996; Chen et al., 1996). A comparison among different models for transport in porous media employing a volume averaging approach is found in Alazmi and Vafai (2000).

In most cases, an accurate mathematical modeling of real problems involving transport phenomena gives rise to nonlinear systems of partial differential equations. Numerical strategies to deal with these problems, such as finite element and finite difference methods, after performing a convenient discretization lead to algebraic systems of equations. In the present work a distinct approach – specifically built to deal with nonlinear hyperbolic systems – is employed.

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‡ All article correspondence should be addressed to this author.

Assuming a one dimensional geometry, a set of four nonlinear partial differential equations describes the problem which, supposing a forced convection flow, is simulated by considering the hydrodynamic problem approximation as an input for the thermal one. The hydrodynamic problem – simulating a radial flow through a porous cylindrical shell, consists of a nonlinear hyperbolic system of two partial differential equations, whose unknowns are the fluid constituent velocity and the saturation – all functions of the position and time. This nonlinear hyperbolic system, which may present discontinuities in addition to classical solutions, is approximated by combining Glimm's scheme – specially designed to deal with discontinuous problems, which consists of performing time evolutions by solving a certain number of associated Riemann problems between each two consecutive steps – and an operator splitting technique to account for the non-homogeneous part of the hyperbolic operator. Once an approximation for the hydrodynamic problem at a given time instant t_{n+1} is known, a finite difference implicit scheme with staggered grids is employed to approximate the thermal problem – namely the determination of the fluid (liquid) and the solid constituents' temperatures at t_{n+1}.

2. Mechanical modeling

Considering a chemically non reacting continuous mixture of a rigid solid constituent at rest, a liquid constituent – from now on denoted as fluid constituent and an inert gas playing the role of the third constituent (accounting for the mixture compressibility), it suffices to solve mass and momentum balance equations for the fluid constituent. It is important to emphasize that although the fluid (a liquid) is assumed incompressible, the fluid constituent is compressible, its compressibility being accounted for by presence of the inert gas coexisting inside the pores with the liquid. The fluid constituent mass balance is given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$\frac{\partial r_F}{\partial t} + \nabla \cdot (r_F \mathbf{v}_F) = 0 \qquad (1)$$

in which r_F stands for the fluid constituent mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture and $\mathbf{v}_F$ is the fluid constituent velocity in the mixture. The balance of linear momentum for the fluid constituent is given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$r_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + r_F \mathbf{b}_F \qquad (2)$$

where $\mathbf{T}_F$ represents the partial stress tensor – analogous to Cauchy stress tensor in Continuum Mechanics – associated with the fluid constituent. The body force is represented by $\mathbf{b}_F$ while $\mathbf{m}_F$ is the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. This momentum source arises from the possible existence of n distinct velocity fields in a mixture of n constituents at each spatial point and any time instant, allowed by the mixture theory, which requires the net momentum supply to the mixture due to all the constituents to be zero: $\sum_{i=1}^{n} \mathbf{m}_i = 0$. The balance of angular momentum is satisfied through an adequate choice of $\mathbf{T}_F$, being automatically fulfilled whenever the partial stress tensor is assumed symmetrical.

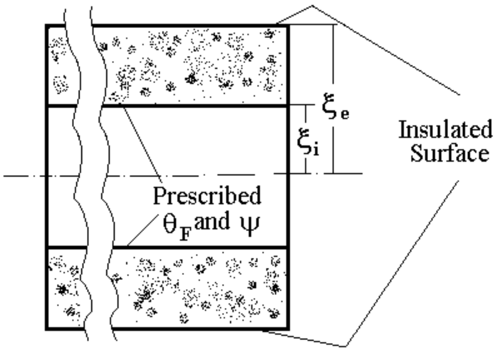

Figure 1. Problem statement: Cylindrical shell porous matrix.

Once thermal nonequilibrium among the constituents is allowed, both the fluid and the solid constituents must satisfy the conservation of energy – only the gas constituent is not required to fulfill the balance equations for being an inert gas. The energy balance is given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$
\begin{aligned}
r_F c_f \left[\frac{\partial T_F}{\partial t} + (\nabla T_F)\mathbf{v}_F \right] &= -\nabla \cdot \mathbf{q}_F + \mathbf{T}_F \cdot \mathbf{D}_F + r_F + \Pi_F \\
r_S c_s \left[\frac{\partial T_S}{\partial t} \right] &= -\nabla \cdot \mathbf{q}_S + r_S + \Pi_S
\end{aligned}
\tag{3}
$$

where the fluid and the solid constituents' temperatures are given, respectively, by T_F and T_S, the partial heat fluxes – analogous to the heat flux vector in Continuum Mechanics – associated with the fluid and the solid constituents by $\mathbf{q}_F$ and $\mathbf{q}_S$ and the external heat supplies to the fluid and the solid constituents are denoted by r_S and r_F. Also r_S represents the solid constituent mass density (the local ratio between the solid constituent mass and the corresponding volume of mixture), $\mathbf{D}_F$ is the symmetrical part of $\nabla \mathbf{v}_F$ tensor and c_s and c_f stand for the solid and the fluid specific heats – measured in a Continuum Mechanics context. Thermal nonequilibrium among the constituents leads to the possible existence of *n* distinct temperature fields at each spatial point for an *n*-constituents' mixture – giving rise to internal heat sources – namely the fields Π_F and Π_S. More specifically, Π_F and Π_S represent, respectively, the fluid constituent and the solid constituent interaction with the remaining constituents of the mixture – expressing an energy transfer per unit time and unit volume. Since the energy sources are internal contributions, it comes that $\sum_{i=1}^{n} \Pi_i = 0$ (Martins-Costa at al, 1993).

The saturation y is defined as the ratio between the fluid fraction j and the porous matrix porosity e:

$$
y = \frac{j}{e} = \frac{r_F}{e r_f} \qquad 0 \le y \le 1 \quad \text{everywhere} \tag{4}
$$

in which r_f is the actual mass density of the fluid – regarded as a single continuum, in contrast to r_F, defined as the fluid constituent mass density.

The momentum source term – accounting for the dynamic interaction among the constituents, in a mixture representing the flow of an incompressible Newtonian fluid through an unsaturated porous matrix is given by the following constitutive relation (Williams, 1978; Saldanha da Gama and Sampaio, 1987):

$$
\mathbf{m}_F = -a y^2 \mathbf{v}_F - b y \nabla y \qquad \text{with} \quad a = \frac{e^2 m_f}{K} \quad b = \frac{e^2 m_f \mathsf{D}}{K} \tag{5}
$$

where m_f represents the fluid viscosity (measured considering a Continuum Mechanics viewpoint), K the porous matrix specific permeability and D a diffusion coefficient – analogous to the usual mass diffusion coefficient.

An analogy with the stress tensor acting on an incompressible Newtonian fluid within a Continuum Mechanics framework probably led Williams (1978) to consider the partial stress tensor acting on the fluid constituent as being proportional to the pressure acting on it and to the gradient of its velocity. A constitutive relation analogous to the usually employed for Cauchy stress tensor with such a behavior was then obtained. A further simplification has been later proposed by Allen (1986), who concluded the normal fluid stresses – resulting from momentum transfer through fluid drag on the porous matrix – were dominant over shear stresses and interphase tractions, leading to the following approximated relation:

$$
\mathbf{T}_F = -e^2 \bar{p} y^2 \mathbf{I} \tag{6}
$$

where $\bar{p}$ is a pressure (assumed constant while the flow is unsaturated) and $\mathbf{I}$ is the identity tensor.

The following constitutive relation for the partial heat flux, analogous to the classical Fourier's law broadly employed in a Continuum Mechanics approach is employed, accounting for all constituents' temperatures, the thermal conductivities as well as the mixture internal structure and kinematics (see Martins-Costa and Saldanha da Gama, 1996 and references therein):

$$
\mathbf{q}_F = -\Lambda k_f e y \nabla T_F \qquad \qquad \mathbf{q}_S = -\Lambda k_s (1 - e) \nabla T_S \tag{7}
$$

in which Λ is a positive valued parameter depending on the porous medium thermal properties and internal structure and k_f and k_s represent the Newtonian fluid and the porous matrix thermal conductivities, measured in a Continuum Mechanics context. It is remarkable that while the partial heat flux for the fluid constituent is proportional to the fluid fraction $j = ey$ the solid constituent one depends on the porous matrix porosity e.

The energy generation function Π_i – an internal energy supply arising from the *i*-constituent thermal interaction with the remaining constituents of the mixture – would be zero at a given point only if thermal equilibrium were assumed. This function must reflect that any *i*-constituent receives energy from its interaction with the *j*-constituents at a higher temperature and provides energy to those at a lower temperature, according to: $\Pi_i = \sum_{j=1}^{n} \hat{R}_{ij}\left(T_j - T_i\right)$ (Costa Mattos et al., 1995; Martins-Costa and Saldanha da Gama, 1996). In the particular mixture considered in the present work, since the gas constituent is assumed inert, thermal interaction is only present between the fluid (liquid) and the solid constituents of the mixture so that $-\Pi_S = \Pi_F$ (Martins-Costa et al., 1993) resulting in $R_{FS} = R_{SF}$ and giving rise to:

$$\Pi_F = -\Pi_S = R_{FS}\left(T_S - T_F\right) \tag{8}$$

in which R_{FS} is a positive-valued factor, which depends on spatial position, on both constituents' thermal properties and on their velocities, accounting for the convective heat transfer.

3. One-dimensional approach

The non-isothermal radial flow in the draining process of a cylindrical shell porous matrix is now considered. A one-dimensional approach is obtained by assuming that all the quantities depend only on the time *t* and on the position *r* and that *v* is the only non-vanishing component of the fluid constituent velocity $\mathbf{v}_F$. Under these assumptions the balance equations (1)-(3) combined with saturation definition (equation 4) and the constitutive relations (5)-(8) give rise to the following nonlinear system:

$$\begin{aligned}
&\frac{\partial y}{\partial t} + \frac{\partial}{\partial r}\left(y\,v\right) = -\frac{y\,v}{r}\\
&r_f e\left[y\,\frac{\partial v}{\partial t} + y\,v\frac{\partial v}{\partial r}\right] = -\frac{\partial}{\partial r}\left(e^2 y^2 \overline{p}\right) - \frac{b r_f^2 e^2}{2}\frac{\partial y^2}{\partial r} - a y^2 v\\
&e y r_f c_f\left[\frac{\partial T_F}{\partial t} + v\frac{\partial T_F}{\partial r}\right] = \Lambda k_f e\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r y\,\frac{\partial T_F}{\partial r}\right)\right] + R_{FS} y\left(T_S - T_F\right)\\
&r_s c_s\left[\frac{\partial T_S}{\partial t}\right] = \Lambda k_s\left(1-e\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_S}{\partial r}\right)\right] + R_{FS} y\left(T_F - T_S\right)
\end{aligned} \tag{9}$$

Redefining a reference pressure p_0 as $p_0 = \overline{p} + \frac{b r_f^2}{2}$ and introducing the following dimensionless quantities

$$\begin{aligned}
&u = v\sqrt{\frac{r_f}{e\,p_0}} \qquad t = \frac{t}{r_*}\sqrt{\frac{e\,p_0}{r_f}} \qquad g = \frac{a\,r_*}{r_f e}\sqrt{\frac{r_f}{e\,p_0}} \qquad x = \frac{r - r_i}{r_e - r_i} \qquad q_F = \frac{T_F}{T_0} \qquad q_S = \frac{T_S}{T_0}\\
&A_F = \frac{\Lambda k_f}{r_f c_f r_*}\sqrt{\frac{r_f}{e\,p_0}} \qquad B_F = \frac{R_{FS} r_*}{e r_f c_f}\sqrt{\frac{r_f}{e\,p_0}} \qquad A_S = \frac{\Lambda k_s\left(1-e\right)}{r_s c_s r_*}\sqrt{\frac{r_f}{e\,p_0}} \qquad B_S = \frac{R_{FS} r_*}{r_s c_s}\sqrt{\frac{r_f}{e\,p_0}}
\end{aligned} \tag{10}$$

in which $r_* = r_e - r_i$ with r_e and r_i standing for the external and internal radii of the cylindrical shell matrix and T_0 is a reference temperature, the nonlinear system (9) may be rewritten in a more convenient form as

$$\begin{aligned}
&\frac{\partial y}{\partial t} + \frac{\partial}{\partial x}\left(y\,u\right) = -\frac{y\,u}{x}\\
&\frac{\partial}{\partial t}\left(y\,u\right) + \frac{\partial}{\partial x}\left(y\,u^2 + y^2\right) = -\frac{y\,u^2}{x} - g y^2 u\\
&y\left[\frac{\partial q_F}{\partial t} + u\frac{\partial q_F}{\partial x}\right] = \frac{A_F}{x}\frac{\partial}{\partial x}\left(x y\,\frac{\partial q_F}{\partial x}\right) + B_F y\left(q_S - q_F\right)\\
&\frac{\partial q_S}{\partial t} = \frac{A_S}{x}\frac{\partial}{\partial x}\left(x\,\frac{\partial q_S}{\partial x}\right) + B_S y\left(q_F - q_S\right)
\end{aligned} \tag{11}$$

4. Numerical procedure

Considering the flow not affected by the thermal problem – the usually employed forced convection assumption, a convenient procedure may be adopted. First the hydrodynamic problem stated in the first two equations of (11) is solved, its approximation being subsequently used as input for the thermal problem stated in the last two equations of (11). The numerical scheme consists in, starting from the fields y , u , q_F and q_S at t_n, obtain the approximations for y and u at a successive time t_{n+1}. These values are, in turn, used as input for the approximation of q_F and q_S at t_{n+1}.

4.1 Hydrodynamic problem

This section presents an adequate scheme to obtain numerical approximations for the non-linear hyperbolic system of partial differential equations described in the first two equations of (11). In order to achieve this goal, two ingredients are combined: Glimm's scheme – a reliable method whose accuracy is mathematically ensured (Glimm, 1965; Chorin, 1976) and an operator splitting technique, accounting for the non-homogeneous portion of the differential equations. This procedure has already been used with success in other nonlinear hyperbolic problems. Examples are wave propagation in fluids, gas flow in pipelines, filling-up of a porous matrix, wave propagation in a damageable elasto-viscoplastic pipe, response of non-linear elastic rods, isothermal and non-isothermal flow of a Newtonian fluid through an unsaturated porous matrix (see Martins-Costa and Saldanha da Gama, 2001 and references therein). Two important features of Glimm's method, which is derived from a theory whose mathematical formulation presents a solid thermodynamic basis – expressed by the entropy condition (Smoller, 1983), deserve a special remark. First if the width of the steps tends to zero the approximation obtained by Glimm's method tends to the exact solution of the problem considering its weak formulation. Another characteristic of this scheme is that it does not dissipate shocks, preserving their magnitude (no diffusion being observed) and position. The admissible deviation from the correct position is of the order of magnitude of the width of each step. It is also remarkable that the problems addressed in all these above mentioned works, due to their hyperbolic nature, do not require boundary conditions, being basically initial value problems (John, 1982).

4.1.1 Glimm's scheme

Assuming, by convenience, $F \equiv y$ and $G \equiv y\,u$ the hydrodynamic problem – essentially an initial value problem subjected to a given data at t_n – may be rewritten as

$$\begin{aligned} &\frac{\partial F}{\partial t}+\frac{\partial G}{\partial x}=-\frac{G}{x} \\ &\frac{\partial G}{\partial t}+\frac{\partial}{\partial x}\left(\frac{G^2}{F}+F^2\right)=-\frac{G^2/F}{x}-g\,FG \end{aligned} \qquad \text{with} \qquad \left.\begin{aligned} F&=\hat{F}_n(x) \\ G&=\hat{G}_n(x) \end{aligned}\right\} \text{ at } \; t=t_n \tag{12}$$

in which $F=\hat{F}_n(x,t)$ and $G=\hat{G}_n(x,t)$.

Since the hyperbolic problem described by equations (12) is non-homogeneous, an operator splitting technique (Martins-Costa and Saldanha da Gama, 2001) is employed. It consists of a decomposition of the operator presented in (12) in such a way that the merely hyperbolic part of the operator is split away from its purely time evolutionary one. The first step is to employ Glimm's scheme to approximate the homogeneous problem associated to (12). Glimm's scheme consists in performing time evolutions by solving the associated Riemann problem – whose complete solution is presented in Martins-Costa and Saldanha da Gama (2001) – between each two consecutive steps. So, in order to obtain a the numerical approximation for the fields F and G at time t_{n+1} – denoted as $\bar{F}_{n+1}$ and $\bar{G}_{n+1}$ – the solution of the Riemann problem associated with homogeneous part of (12) must be known. Essentially, the first step is to approximate the initial condition by piecewise constant functions: $F=\hat{F}_n(x)\approx F_{n_i}=\hat{F}_n(x_i+q_n\Delta x)$ and $G=\hat{G}_n(x)\approx G_{n_i} = \hat{G}_n(x_i+q_n\Delta x)$, for $x_i-\Delta x/2<x<x_i+\Delta x/2$, in which q_n is a number randomly chosen in the open interval ($-1/2,\ +1/2$) and Δx is the width of each step ($\Delta x=x_{i+1}-x_i$).

In the sequence, the associated Riemann problem, defined as:

$$\begin{aligned} &\frac{\partial F}{\partial t}+\frac{\partial G}{\partial x}=0 \\ &\frac{\partial G}{\partial t}+\frac{\partial}{\partial x}\left(\frac{G^2}{F}+F^2\right)=0 \end{aligned} \qquad \text{with} \qquad \begin{aligned} &(F,G)=\left(F_{n_i},G_{n_i}\right) && \text{for } \; t=t_n\,, \quad -\infty<x<x_i+\frac{\Delta x}{2} \\ &(F,G)=\left(F_{n_{i+1}},G_{n_{i+1}}\right) && \text{for } \; t=t_n\,, \quad x_{i+1}-\frac{\Delta x}{2}<x<\infty \end{aligned} \tag{13}$$

is solved for each two consecutive steps, allowing marching from time $t = t_n$ to time $t_{n+1} = t_n + \Delta t$. The approximation for the homogeneous problem associated with (12) at time t_{n+1}, for $x_i < x < x_{i+1}$, is finally reached, being given by $\bar{F}_{n+1} \approx \tilde{F}_{n_i}\left(x,\ t_{n+1}\right)$ and $\bar{G}_{n+1} \approx \tilde{G}_{n_i}\left(x,\ t_{n+1}\right)$. The time step Δt must satisfy the Courant-Friedrichs-Lewy condition (Smoller, 1983):

$$t_{n+1} - t_n \le \frac{\Delta x}{2\left|l\right|_{\max}} \tag{14}$$

preventing interactions among nearby shocks of adjacent Riemann problems and, consequently, assuring uniqueness for the solution. The parameter $\left|l\right|_{\max}$ is the maximum (in absolute value) propagation speed of shocks, considering all the Riemann problems at t_n. It is important to notice that the approximation obtained after each advance in time is no longer a piecewise constant function, thus requiring a new random selection of q_n, in order to approximate the initial condition for the next time step by piecewise constant functions.

4.1.2 Operator splitting technique

Once an initial approximation for the homogeneous problem associated with the first two equations of (12) is reached by employing Glimm's scheme, the numerical approximation for the solution at time t_{n+1} is then obtained by advancing in time with the same time step Δt through the ordinary system. This procedure – which consists in achieving an approximation for the solution $\left(F, G\right)$ at time $t = t_{n+1}$, is repeated until a specified simulation time is reached. The numerical approximation for the solution at the time instant $t = t_{n+1}$ is finally reached by advancing in time to solve the following problem, with the same step $\Delta t = t_{n+1} - t_n$, through equations:

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\frac{G}{x} \\ \frac{\partial G}{\partial t} &= -\frac{G^2/F}{x} - g\,FG \end{aligned} \qquad \left.\begin{aligned} F &= \hat{F}_{n+1}\left(x\right) \\ G &= \hat{G}_{n+1}\left(x\right) \end{aligned}\right\} \text{ at } \ t = t_n \tag{15}$$

The system represented in equation (15) could be rewritten in an appropriated form, by using the chain rule and substituting the first equation in the second one, as

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\frac{G}{x} \\ \frac{\partial G/F}{\partial t} &= -g\,G \end{aligned} \tag{16}$$

Assuming non-zero velocity and saturation, since no splitting would be required if either $G/F = 0$ or $F = 0$, the following ordinary equation is obtained

$$\frac{dF}{dt} = -\frac{1}{xg} \tag{17}$$

Since $1/xg$ may be treated as a constant for a given value of x, equation (17) admits an analytical solution given by

$$F - \frac{1}{xg}\frac{G}{F} = F_0 - \frac{1}{xg}\left(\frac{G}{F}\right)_0 \quad \Rightarrow \quad \frac{G}{F} = xg\left(F - F_0\right) + \left(\frac{G}{F}\right)_0 \tag{18}$$

in which F_0 and $\left(G/F\right)_0$ refer to information obtained at the preceding time instant by employing Glimm's method. Appropriate substitutions give rise to the following analytical solution for the saturation and the velocity:

$$F = \frac{AB\exp(A\Delta t)}{1+g\left[B\exp(A\Delta t)\right]} \qquad \text{with} \qquad A = F_0 g - \frac{(G/F)_0}{x} \qquad B = \frac{F_0}{A - gF_0} \tag{19}$$

$$\frac{G}{F} = x\left\{-A + g\,\frac{AB\exp(A\Delta t)}{1+g\left[B\exp(A\Delta t)\right]}\right\} \qquad \Delta t = t - t_0$$

4.2 Thermal problem

Once that an approximation for the hydrodynamic problem at time $t = t_{n+1}$ has been obtained for the solution (F, G) by employing the procedure described in item 4.1, the following implicit finite difference scheme is employed to approximate the thermal problem – namely to determine q_F and q_S at $t = t_{n+1}$ for all x .

$$F^{n+1}\left[\frac{\partial q_F}{\partial t}\right]^n + G^{n+1}\left[\frac{\partial q_F}{\partial x}\right]^n = \frac{A_F}{x}F^{n+1}\left[\frac{\partial q_F}{\partial x}\right]^n + A_F\left[\frac{\partial F}{\partial x}\right]^n\left[\frac{\partial q_F}{\partial x}\right]^n + A_F F^{n+1}\left[\frac{\partial^2 q_F}{\partial x^2}\right]^n + B_F F^{n+1}\left[q_S - q_F\right]^n$$

$$\left[\frac{\partial q_S}{\partial t}\right]^n = \frac{A_S}{x}\left[\frac{\partial q_S}{\partial x}\right]^n + A_S\left[\frac{\partial^2 q_S}{\partial x^2}\right]^n + B_S F^{n+1}\left[q_F - q_S\right]^n \tag{20}$$

5. Numerical results

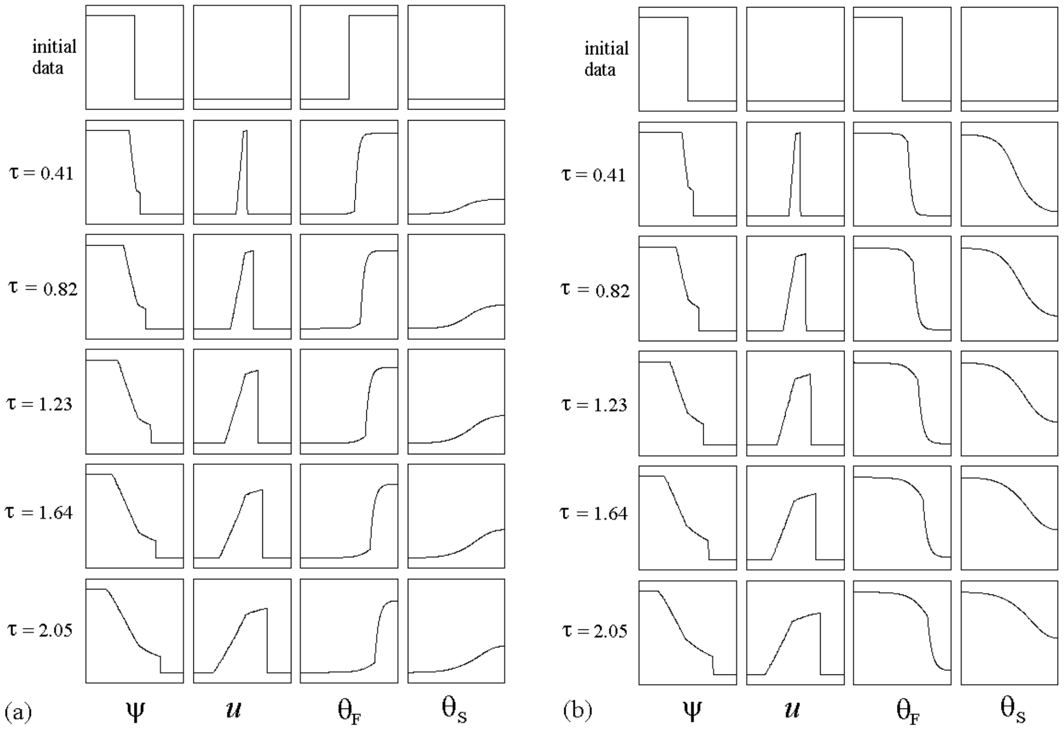

Figure 2. Saturation, fluid constituent velocity and fluid and solid constituents' temperatures variation with radial position with $A_F = 1$, $A_S = 10$, $B_F = 0.1$, $B_S = 10$ and $g = 1$ with (a): $q_F = 0$ at x_i and $q_F = 1$ at x_e and (b): $q_F = 1$ at x_i and $q_F = 0$ at x_e.

Numerical results obtained for the approximation described in the previous section of the forced convection radial flow through the cylindrical shell porous matrix depicted in Fig. (1) are now presented. The nonlinear system presented in equation (11) is subjected to initial data given by distinct step functions for y_0 (from 1.0 to 0.05) and q_{F0} (from 0 to 1.0) and constant values $u_0 = q_{S0} = 0$ as well as to the following boundary conditions

$$
\begin{aligned}
x = 0 \quad &\Rightarrow \quad y = 1 \qquad \frac{\partial q_S}{\partial x} = 0 \qquad q_F = 0 \;\text{ or }\; q_F = 1 \\
x = 1 \quad &\Rightarrow \quad \frac{\partial q_S}{\partial x} = \frac{\partial q_F}{\partial x} = 0
\end{aligned}
\tag{20}
$$

Some selected results considering the influence not only of the diffusive terms' coefficients A_F and A_S but also of the internal source terms' coefficients B_F and B_S are presented in Figs. (2) to (4). Each considered case is presented in a set composed by six lines and four columns of graphs. Each line represents a distinct time instant – the first one being the initial condition, while each column corresponds to the behavior of a distinct variable – namely saturation, fluid constituent velocity and fluid and solid constituents' temperatures. All the depicted diagrams show at the left-hand side ($x = 0 = x_i$) the cylindrical shell porous matrix internal radius, and its external radius ($x = 1 = x_e$) at the right-hand side. Besides, all the qualitative results shown were obtained by employing a convenient normalization, in such a way that the minimum and maximum displayed values correspond to zero and unit values for y , u and q_F . In order that both constituents' temperatures are easily compared, q_S is displayed using the same scale employed for q_F .

All depicted results have been obtained by employing Glimm's scheme with 300 steps for each time advance.

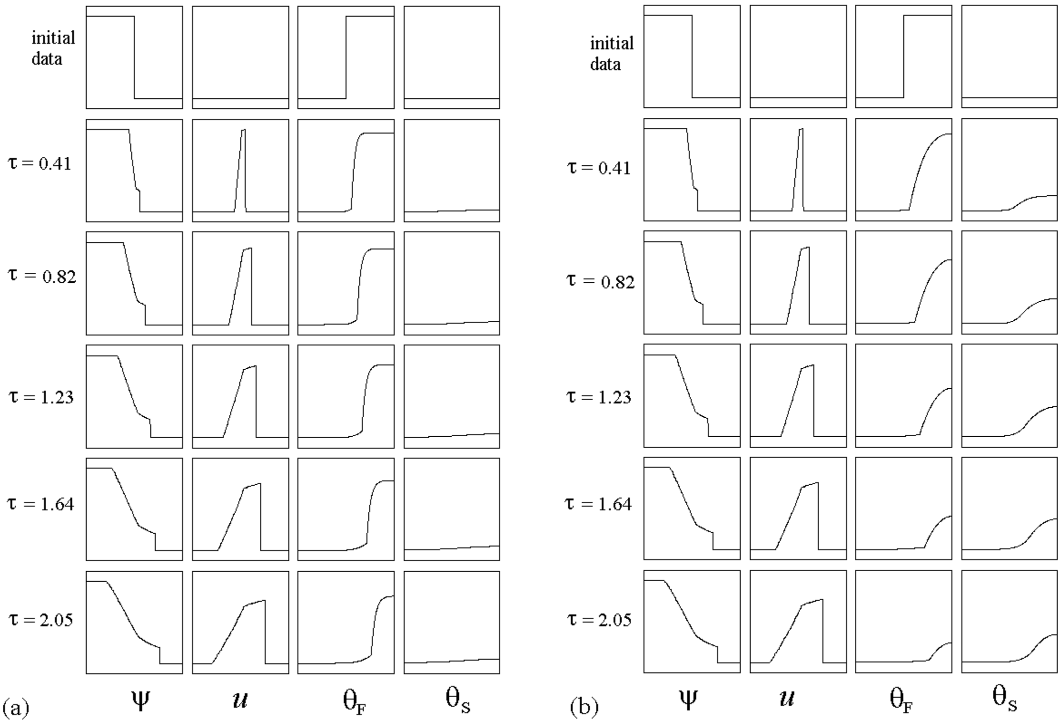

Figure 3. Saturation, fluid constituent velocity and fluid and solid constituents' temperatures variation with radial position using $q_F = 0$ at x_i and $q_F = 1$ at x_e; $g = 1$ with (a): $A_F = 1$, $A_S = 10$, $B_F = 0.1$, $B_S = 1$ and (b): $A_F = 10$, $A_S = 1$, $B_F = 0.1$, $B_S = 10$.

An important feature, present in all depicted results, is that the discontinuities for the variables y , u and $\partial q_F / \partial x$ are in the same spatial position, since the position of the jump for $\partial q_F / \partial x$ must be the same position for the jump of y .

Figure 2 shows two sets of results obtained with the same values of diffusive terms ($A_F = 1$, $A_S = 10$) and internal heat source ($B_F = 0.1$, $B_S = 10$) coefficients considering distinct temperature profiles as initial data. In Fig 2a a step function, characterized by $q_F = 0$ at x_i. and $q_F = 1$ at x_e is assumed while in Fig. 2b $q_F = 1$ at x_i and $q_F = 0$ at x_e. The behavior of saturation and fluid constituent velocity remained unaltered in both cases, the variation being restricted to solid and fluid constituents' temperature profiles. The solid constituent temperature shows a more visible variation with time in Fig. 2a (given by a decay whose intensity decreases with time evolution) than in Fig. 2b – the latter presenting a very discrete increase.

The influence of some coefficients of the energy equations may be observed by comparing Fig. 3 to Fig. 2a. In Fig. 3a, $B_S = 1$ (ten times smaller than its value in Fig. 2a). Comparing these two figures no variation is observed in the fluid constituent temperature, while the solid constituent profile in Fig. 3a remains almost insensitive to time evolution. Comparing Fig. 2a and 3b – in which both constituents diffusive term coefficients have been altered, with $A_F = 10$ (ten times greater than its value in Fig. 2) and $A_S = 1$ (ten times smaller than all previously considered values), it may be observed that the solid constituent temperature behavior is almost unaltered. On the other hand, an important variation is observed in the fluid constituent temperature profiles.

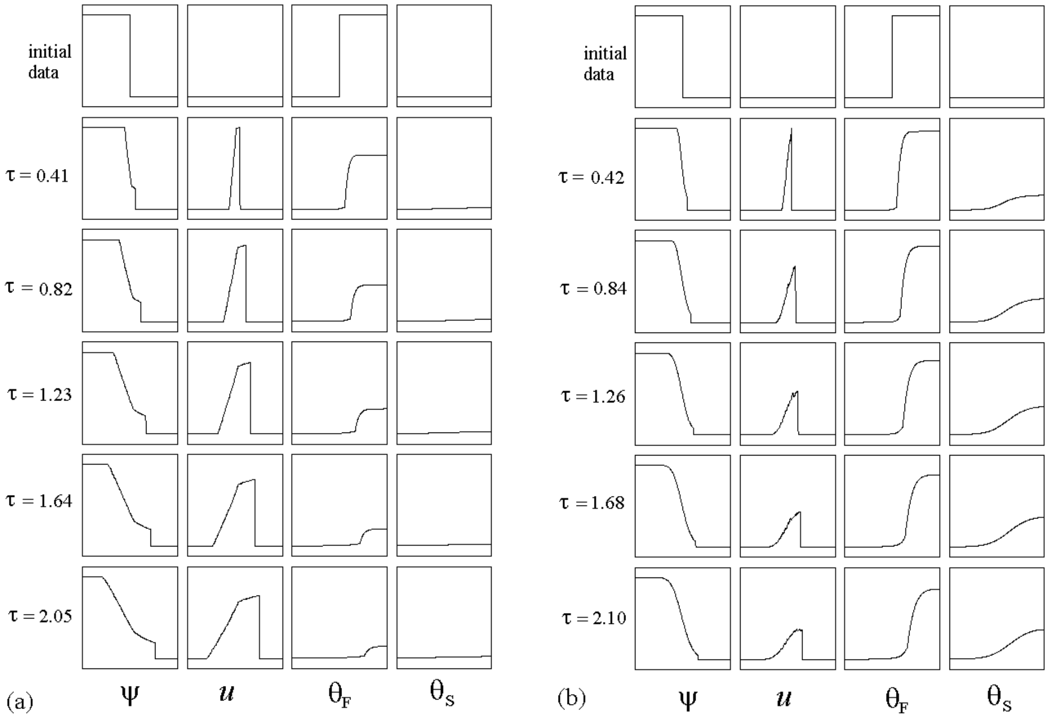

Figure 4. Saturation, fluid constituent velocity and fluid and solid constituents' temperatures variation with radial position using $q_F = 0$ at x_i and $q_F = 1$ at x_e; with (a): $g = 1$, $A_F = 1$, $A_S = 10$, $B_F = 1$, $B_S = 1$ and (b): $g = 10$, $A_F = 1$, $A_S = 10$, $B_F = 0.1$, $B_S = 10$.

The influence of the heat source coefficients in both constituents energy equations is observed by confronting Fig. 4a and Fig. 2a, the former obtained considering $B_F = 1$ and $B_S = 1$, respectively ten times greater and ten times smaller than the values used in the latter figure. In Fig. 4a, the solid constituent temperature profile remains almost unchanged with time evolution and the fluid constituent behavior shows a "damping effect" – when compared to Fig. 2a.

Figure 4b shows the influence of the so-called darcian term coefficient, which has been made ten times greater ($g = 10$) than all the remaining depicted results. The slight distinct values for the depicted time instants are explained by the fact that the time evolution is related to the Courant-Friedrichs-Lewy condition expressed by Eq. (14). Comparing Fig. 4b and Fig. 2a, a barely noticeable change is observed in both q_F and q_S profiles while a strong effect on saturation and fluid constituent velocity profiles is noticed, the latter one even more pronounced – showing a damping effect at its discontinuity. The effect of making the darcian term coefficient ten times smaller ($g = 0.1$) than the value used to obtain Figures 2a to 4a is more pronounced in the fluid constituent velocity profile – with an increase in its wave amplitude. Besides, a slight variation on the saturation and a barely noticeable change at both constituents' temperatures are verified.

6. Final remarks

In this article a non-conventional numerical approach is used to study transport phenomena in an unsaturated porous matrix. Its mathematical representation – namely a nonlinear system, whose numerical approximation is performed by first solving the hydrodynamic problem and later using the obtained solution as input for the thermal problem. The numerical methodology for approximating the hydrodynamic problem – combines Glimm's scheme to an operator splitting technique allowing the accurate approximation of a nonlinear and non-homogeneous system of partial differential equations.

Glimm's method, besides preserving shock waves magnitude and position, is a convenient tool for solving one-dimensional nonlinear problems. It exhibits features such as low storage costs and low computational effort when compared to other numerical procedures to approximate nonlinear problems – the complete simulation requiring circa 2.5 minutes CPU time in a Pentium III, 1200 MHz, 128 Mb RAM.

7. Acknowledgements

The authors M.L. Martins-Costa and R.M. Saldanha da Gama gratefully acknowledge the financial support provided by the agency CNPq through grants 302172/2003-3 and 302462/84-8, respectively.

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